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ON SOME STRONG ZWEIER CONVERGENT SEQUENCE SPACES

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Abstract. In this paper we define three classes of new sequence spaces. We give some relations related to these sequence spaces. We also introduce the concept of $S_Z^{\lambda}-$statistically convergence and obtain some inclusion relations related to these new sequence spaces.

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1. Introduction

Let l_∞, c and c_o be the linear spaces of bounded, convergent and null sequences with complex terms, respectively. Note that l_∞, c and c_o are Banach spaces with the sup-norm

$$\|x\|_\infty = \sup_k |x_k| .$$

A sequence space X with a linear topology is called a K-space if each of maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called FK-space if X is a complete linear metric space and a BK-space is a normed FK-space.

Let Λ denote the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to infinity and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de Vallee-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number l [1] if $t_n(x) \to l$ as $n \to \infty$.

$$[V, \lambda]_o = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\}$$

$$[V,\lambda] = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - l| = 0, \text{for some } l \right\}$$

$$[V,\lambda]_\infty = \left\{ x = (x_k) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty \right\}.$$

The space $[V,\lambda]$ is a BK-space with the norm

$$\|x\|_{[V,\lambda]} = \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| . \tag{1.1}$$

The space $[V,\lambda]_o$ is also BK-space with the same norm.

If $\lambda_n = n$, then (V,λ)-summability and strongly (V,λ)-summability reduce to $(C,1)$ $-$summability and $[C,1]$ $-$summability, respectively.

In [2] , Şengönül introduced sequence spaces Z and Z_o as the set of all sequences such that Z-transforms of them are in the spaces c and c_o, respectively, i.e.,

$$Z = \{x = (x_k) : Zx \in c\} \text{ and } Z_o = \{x = (x_k) : Zx \in c_o\}$$

where $Z = (z_{nk})_{n,k=0}^{\infty}$ denotes by the matrix

$$z_{nk} = \begin{cases} \frac{1}{2}, k \leq n \leq k+1 \\ 0, \quad \text{otherwise} \end{cases} (n,k \in \mathbb{N}).$$

This matrix is called Zweier matrix. The $Z = (z_{nk})_{n,k=0}^{\infty}$ matrix is well-known as a regular matrix [3] .

The purpose of this paper is to introduce and study the concept of $\lambda-$strong Zweier convergence and $\lambda-$statistical Zweier convergence.

2. $\lambda-$strong Zweier convergence

We introduce the sequence spaces $[V_Z,\lambda]_o$, $[V_Z,\lambda]$ and $[V_Z,\lambda]_\infty$ as the set of all sequences such that Z-transforms are in $[V,\lambda]_o$, $[V,\lambda]$ and $[V,\lambda]_\infty$, respectively, that is

$$[V,\lambda]_Z^o = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{1}{2}(x_k + x_{k-1}) \right| = 0 \right\},$$

$$[V,\lambda]_Z = \left\{ x = (x_k) : \ \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{1}{2}(x_k + x_{k-1}) - l \right| = 0, \text{ for some } l \right\}$$

and

$$[V,\lambda]_Z^{\infty} = \left\{ x = (x_k) : \ \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{1}{2}(x_k + x_{k-1}) \right| < \infty \right\}$$

Define the sequence $y = (y_k)$ which will be frequently used throughout the paper, as Z-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \frac{1}{2}(x_k + x_{k-1}) \quad (k \in \mathbb{N}). \tag{2.1}$$

Theorem 2.1. *The sequence spaces* $[V_Z,\lambda]_o, [V_Z,\lambda]$ *and* $[V_Z,\lambda]_\infty$ *are linear spaces over the complex field* $\mathbb{C}$ *which are the BK-spaces with the norm*

$$\|x\|_{[V,\lambda]_Z^o} = \|x\|_{[V,\lambda]_Z} = \|x\|_{[V,\lambda]_Z^\infty} = \|Zx\|_{[V,\lambda]}.$$

Proof. The first part of the theorem is a routine verification and so we omit it. Since the sequence spaces $[V,\lambda]_o$ and $[V,\lambda]$ are BK-spaces with respect to the norm defined (1.1) and the matrix $Z = (z_{nk})_{n,k=0}^{\infty}$ is normal, i.e., $z_{nk} \neq 0$ for $0 \leq k \leq n$ and $z_{nk} = 0$ for $k > n$ for all $n, k \in \mathbb{N}$ and also from Theorem 4.3.2 of Wilansky [4] gives the fact that $[V_Z,\lambda]_o, [V_Z,\lambda]$ and $[V_Z,\lambda]_\infty$ are the BK-spaces.

Theorem 2.2. *The sequence spaces* $[V_Z,\lambda]_o, [V_Z,\lambda]$ *and* $[V_Z,\lambda]_\infty$ *are linearly isomorphic to the sequence spaces* $[V,\lambda]_o, [V,\lambda]$ *and* $[V,\lambda]_\infty$*, respectively, i.e.,* $[V,\lambda]_o \cong [V_Z,\lambda]_o$, $[V,\lambda] \cong [V_Z,\lambda]$ *and* $[V,\lambda]_\infty \cong [V_Z,\lambda]_\infty$.

Proof. We consider only the case $[V,\lambda]_o \cong [V_Z,\lambda]_o$. We should show the existence of a linear bijection between the spaces $[V,\lambda]_o$ and $[V_Z,\lambda]_o$. Consider the transformation Z define, with the notation (2.1), from $[V_Z,\lambda]_o$ to $[V,\lambda]_o$ by

$$\begin{aligned} &Z : [V_Z,\lambda]_o \to [V,\lambda]_o \\ &x \to Zx = y \end{aligned}$$

where the sequence $y = (y_k)$ is given by (1.1). The linearity of transformation Z is clear. Further, it is trivial that $x = 0$ whenever $Zx = 0$ and hence Z is injective. Let $y = (y_k) \in [V,\lambda]_o$ and the sequence $x = (x_k)$ by

$$x_k = 2 \sum_{i=0}^{k} (-1)^{i-k} y_i \ \ (i \in \mathbb{N}).$$

Then

$$\|x\|_{[V,\lambda]_Z^o} = \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{1}{2} (x_k + x_{k-1}) \right|$$

$$= \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{1}{2} \left(2 \sum_{i=0}^{k} (-1)^{i-k} y_i + 2 \sum_{i=0}^{k-1} (-1)^{(i-1)-k} y_i \right) \right|$$

$$= \frac{1}{\lambda_n} \sum_{k \in I_n} |y_k|$$

which says us that $x = (x_k) \in [V_Z, \lambda]_o$. Additionally, we observe that

$$\|x\|_{[V_Z,\lambda]_o} = \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{1}{2} (x_k + x_{k-1}) \right|$$

$$= \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{1}{2} \left(2 \sum_{i=0}^{k} (-1)^{i-k} y_i + 2 \sum_{i=0}^{k} (-1)^{(i-1)-k} y_i \right) \right|$$

$$= \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |y_k| = \|y\|_{[V,\lambda]_o}.$$

Thus, we have $x = (x_k) \in [V_Z, \lambda]_o$ and consequently Z is surjective. Hence, Z is linear bijection which therefore says us that the sequence spaces $[V_Z, \lambda]_o, [V_Z, \lambda]$ and $[V_Z, \lambda]_\infty$ are linearly isomorphic to the sequence spaces $[V, \lambda]_o, [V, \lambda]$ and $[V, \lambda]_\infty$, respectively. This completes the proof.

There is a relation between the sequence space $[V, \lambda]$ and the sequence space $|\sigma_1|$ of strong Cesaro summable sequences defined by

$$|\sigma_1| = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - l| = 0, \text{for some } l \right\}.$$

Clearly, in the special case $\lambda_n = n$ for all $n \in \mathbb{N}$, we have $[V, \lambda] = |\sigma_1|$.

Also, we see that, there are strong connection between the sequence space $[V_Z, \lambda]$ and the sequence space $[w_Z, \lambda]$, which is defined by

$$[w_Z, \lambda] = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{1}{2} (x_k + x_{k-1}) - l \right| = 0, \text{for some } l \right\}.$$

Clearly, in the special case $\lambda_n = n$ for all $n \in \mathbb{N}$, we have $[V_Z, \lambda] = [w_Z, \lambda]$.

3. λ-statistical Zweier convergence

In this section we introduce the concept of S_Z^{λ}-statistical convergence and give some inclusion relations related to this sequence space.

The notion on statistical convergence was introduced by Fast [5] and studied by various authors (see $[6], [7], [8], [9], [10-11])$.

Definition 3.1.[7] *A sequence $x = (x_k)$ is said to be λ-statistically convergent to the number l if for $\varepsilon > 0$,*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \ |x_k - l| \geq \varepsilon\}| = 0.$$

In this case we write $S^{\lambda} - \lim x = l$ or $x_k \to l\left(S^{\lambda}\right)$ and
$S^{\lambda} = \left\{x = (x_k) : \ \text{for some } l, \ S^{\lambda} - \lim x = l\right\}.$

Definition 3.2. *A sequence $x = (x_k)$ is said to be S_Z^{λ}-statistically convergent to the number l if for $\varepsilon > 0$,*

$$\lim_n \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right| = 0.$$

In this case we write $S_Z^{\lambda} - \lim x = l$ or $x_k \to l\left(S_Z^{\lambda}\right)$ and
$S_Z^{\lambda} = \left\{x = (x_k) : \ \text{for some } l, \ S_Z^{\lambda} - \lim x = l\right\}.$

In the case $\lambda_n = n$ we shall write S_Z instead of S_Z^{λ}.

Theorem 3.1. *Let $\lambda = (\lambda_n) \in \Lambda$. $x_k \to l\left([V_Z, \lambda]\right)$ then $x_k \to l\left(S_Z^{\lambda}\right)$.*

Proof. Let $x = (x_k) \in [V_Z, \lambda]$. Then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| =$$

$$\frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon}} \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| < \varepsilon}} \left|\frac{1}{2}(x_k + x_{k-1}) - l\right|$$

$$\geq \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon}} \left|\frac{1}{2}(x_k + x_{k-1}) - l\right|$$

$$\geq \frac{1}{\lambda_n} \sum_{k \in I_n} \varepsilon \geq \frac{\varepsilon}{\lambda_n} \left|\left\{k \in I_n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right|.$$

It follows that $x_k \to l\left(S_Z^{\lambda}\right)$. This completes the proof.

Theorem 3.2. *Let* $\lambda = (\lambda_n) \in \Lambda$. *If* $x = (x_k) \in l_\infty$ *and* $x_k \to l\left(S_Z^{\lambda}\right)$, *then* $x_k \to l\left([V_Z, \lambda]\right)$.

Proof. Suppose that $x = (x_k) \in l_\infty$ and $x_k \to l\left(S_Z^{\lambda}\right)$. Since $\sup\left|\frac{1}{2}(x_k + x_{k-1})\right| < \infty$, there is a constant $A > 0$ such that $\left|\frac{1}{2}(x_k + x_{k-1})\right| < A$ for all $k \in \mathbb{N}$. Therefore we have, for $\varepsilon > 0$

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{1}{2}(x_k + x_{k-1}) - l\right|$$

$$= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ \left|\frac{1}{2}(x_k+x_{k-1})-l\right|\geq\varepsilon}}\left|\frac{1}{2}(x_k + x_{k-1}) - l\right| + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ \left|\frac{1}{2}(x_k+x_{k-1})-l\right|<\varepsilon}}\left|\frac{1}{2}(x_k + x_{k-1}) - l\right|$$

$$\leq \frac{A}{\lambda_n}\left|\left\{k \in I_n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right| + \frac{1}{\lambda_n}\sum_{k\in I_n}\varepsilon$$

$$= \frac{A}{\lambda_n}\left|\left\{k \in I_n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right| + \varepsilon.$$

Taking limit as $\varepsilon \to 0$, the desired result follows.

Corollary 3.3. *Let* $\lambda = (\lambda_n) \in \Lambda$. *Then* $l_\infty \cap [V_Z, \lambda] = l_\infty \cap S_Z^{\lambda}$.

Proof. It follows from Theorem 3.1. and Theorem 3.2.

Theorem 3.4. *Let* $\lambda = (\lambda_n) \in \Lambda$. *If* $\lim_n \inf \frac{\lambda_n}{n} > 0$, *then* $x_k \to l(S_Z)$ *implies* $x_k \to l\left(S_Z^{\lambda}\right)$.

Proof. Given $\varepsilon > 0$, we have

$$\left|\left\{k \leq n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right| \supset \left|\left\{k \in I_n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right|.$$

Therefore

$$\frac{1}{n}\left|\left\{k \leq n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right| \geq \frac{1}{n}\left|\left\{k \in I_n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{\lambda_n}{n}.\frac{1}{\lambda_n}\left|\left\{k \in I_n : \ \left|\frac{1}{2}(x_k + x_{k-1}) - l\right| \geq \varepsilon\right\}\right|.$$

Taking limit as $n \to \infty$ and using $\lim_n \inf \frac{\lambda_n}{n} > 0$, we get that $x_k \to l\left(S_Z^{\lambda}\right)$. This completes the proof.

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