Acta Universitatis Apulensis No. 27/2011
ISSN: 1582-5329 pp. 151-155

MAPPING PROPERTIES OF AN INTEGRAL OPERATOR

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Abstract. In the present paper, we introduce a general integral operator and study mapping properties on some subclasses of analytic univalent functions. Relevant connections of the results presented here with various known results are briefly indicated.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction

Let A denote the class of the functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$.

Further, we denote by S the subclass of A consisting of functions of the form (1) which are also univalent in U.

For $\beta > 1$ and $z \in U$, let

$$M(\beta) = \left\{ f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} < \beta \right\}$$

and

$$N(\beta) = \left\{ f \in A : \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} < \beta \right\}.$$

These classes were extensively studied by Uralegaddi et al. in [15], (see also Owa and Srivastava [9], Porwal and Dixit [13]).

Very recently Dixit and Chandra [4] generalizes these classes by introducing a new subclass $S_k^n(\beta)$ of analytic functions in the unit disk satisfying the condition

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} < \beta, \quad n \in N_0,\ z \in U, 1 < \beta \leq 4/3, \tag{2}$$

where D^n stands for the Salagean-derivative introduced by Salagean in [14] and $f(z)$ of the form

$$f(z) = z + \sum_{j=k+1}^{\infty} a_j z^j. \tag{3}$$

It can be easily seen that $S_1^0(\beta) \equiv M(\beta), \quad S_1^1(\beta) \equiv N(\beta)$.

Further, we denote by $S_1^n(\beta) \equiv S^n(\beta)$.

Breaz [2] studied the mapping properties of the two integral operators on the classes $M(\beta)$ and $N(\beta)$.

In the present paper, we generalized and unified these results by introducing an interesting integral operator as follows

$$F_{m,n,\alpha}(z) = \int_0^z \left(\frac{D^m f_1(t)}{t}\right)^{\alpha_1} \left(\frac{D^m f_2(t)}{t}\right)^{\alpha_2} \cdots \left(\frac{D^m f_n(t)}{t}\right)^{\alpha_n} dt \tag{4}$$

where $f_i(z) \in A$, $\alpha_i > 0$, $\forall i \in \{1, 2, \ldots n\}$ and $m \in N_0$.

Remark 1. For $m = 0, n = 1, \alpha_1 = 1, \alpha_2 = \ldots = \alpha_n = 0$ and $f(z) \in A$, we obtain Alexander integral operator introduced in 1915 in [1]

$$I(z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in U.$$

Remark 2. For $m = 0, n = 1, \alpha_1 = \alpha, \alpha_2 = \ldots = \alpha_n = 0$ and $f(z) \in A$, we obtain the integral operator

$$I_\alpha(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\alpha dt, \quad z \in U,$$

studied in [8], (see also ([5], [7], [12]).

Remark 3. For $m = 1, n = 1, \alpha_1 = 1, \alpha_2 = \ldots = \alpha_n = 0$ and $f(z) \in A$, we obtain the integral operator

$$I(z) = \int_0^z f'(t) dt$$

studied by various authors in ([6], [11]).

Remark 4. For $m = 1, n = 1, \alpha_1 = \alpha, \alpha_2 = \ldots = \alpha_n = 0$ and $f(z) \in A$, we obtain the integral operator

$$I_\alpha(z) = \int_0^z \left[f'(t)\right]^\alpha dt, \quad z \in U$$

studied in [10].

Remark 5. For $m = 0, \alpha_i > 0, i \in \{1, 2, ...n\}$, we obtain the integral operator

$$I_n(z) = \int_0^z \left[\frac{f_1(t)}{t}\right]^{\alpha_1} \ldots \left[\frac{f_n(t)}{t}\right]^{\alpha_n} dt$$

studied in ([2], [3]).

Remark 6. For $m = 1, \alpha_i > 0,\ \forall i \in \{1, 2, ...n\}$, we obtain integral operator

$$I_{\alpha_1,\alpha_2,...\alpha_n}(z) = \int_0^z \left[f_1'(t)\right]^{\alpha_1} \ldots \left[f_n'(t)\right]^{\alpha_n} dt$$

studied in [2].

2. Main Results

We study the condition for the integral operator defined in (4) which map $S^m(\alpha_1)X$ $S^m(\alpha_2)X$ ... $XS^m(\alpha_n)$ into $N(\mu)$.

Theorem 1. *Let $f_i \in S^m(\beta_i)$ for each $i = 1, 2, ...n$ with $\beta_i > 1, m \in N_0$. Then $F_{m,n,\alpha}(z) \in N(\mu)$ where*

$$\mu = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1) \quad \text{and} \quad \alpha_i > 0.$$

Proof. Let

$$F_{m,n,\alpha}(z) = \int_0^z \left[\frac{D^m f_1(t)}{t}\right]^{\alpha_1} \ldots \left[\frac{D^m f_n(t)}{t}\right]^{\alpha_n} dt.$$

Differentiating it, we have

$$F'_{m,n,\alpha}(z) = \left(\frac{D^m f_1(z)}{z}\right)^{\alpha_1} \ldots \left(\frac{D^m f_n(z)}{z}\right)^{\alpha_n}. \tag{5}$$

Logarthmic differentiation of (5) yields

$$\frac{F''_{m,n,\alpha}(z)}{F'_{m,n,\alpha}(z)} = \alpha_1 \left[\frac{(D^m f_1(z))'}{D^m f_1(z)} - \frac{1}{z}\right] + \dots + \alpha_n \left[\frac{(D^m f_n(z))'}{D^m f_n(z)} - \frac{1}{z}\right]$$

or, equivalently

$$\begin{aligned}\mathrm{Re}\left\{1 + \frac{zF''_{m,n}(z)}{F'_{m,n}(z)}\right\} &= \sum_{i=1}^{n} \alpha_i \mathrm{Re}\left\{\frac{z(D^m f_i(z))'}{D^m f_i(z)}\right\} - \sum_{i=1}^{n} \alpha_i + 1 \\ &< \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{i=1}^{n} \alpha_i + 1 \\ &= \sum_{i=1}^{n} \alpha_i(\beta_i - 1) + 1.\end{aligned}$$

Because $\sum_{i=1}^{n} \alpha_i(\beta_i - 1) > 0$, we obtain $F_{m,n,\alpha}(z) \in N(\mu)$, where $\mu = 1 + \sum_{i=1}^{n} \alpha_1(\beta_1 - 1)$.

If we put $m = 0$ in Theorem 1, we obtain the following result obtained by Breaz in [2].

Corollary 1. *Let $f_i \in M(\beta_i)$ with $\beta_i > 1$, for each $i = 1, 2, ..., n$. Then $I_n(z) \in N(\mu)$, where $\mu = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$ and $\alpha_i > 0$, $(\forall i = 1, 2, ...n)$.*

If we put $m = 1$ in Theorem 1, we obtain the following result obtained by Breaz in [2].

Corollary 2. *Let $f_i \in M(\beta_i)$, for each $i = 1, 2, ..., n$ with $\beta_i > 1$. Then $I_{\alpha_1,...,\alpha_n}(z) \in N(\mu)$ with $\mu = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$ and $\alpha_i > 0$, $(\forall i = 1, 2, ...n)$.*

Acknowledgement: The present investigation was supported by the University Grant Commission under grant No. F- 11-12/2006(SA-I).

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