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THE ROMAN DOMINATION NUMBER OF A DIGRAPH

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Abstract. Let $D = (V, A)$ be a finite and simple digraph. A *Roman dominating function* (RDF) on a digraph D is a labeling $f : V(D) \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a in-neighbor with label 2. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF on D. In this paper we present some sharp bounds for $\gamma_R(D)$

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1. Introduction

Let D be a finite and simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The order $n = n(D)$ of a digraph D is the number of its vertices. We write $d^+_D(v)$ for the outdegree of a vertex v and $d^-_D(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then we also write $u \rightarrow v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$, respectively. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v. Consult [10] for the notation and terminology which are not defined here. For a real-valued function $f : V(D) \longrightarrow \mathbf{R}$ the weight of f is $w(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$.

A subset S of vertices of D is a *dominating set* if $N^+[S] = V$. The *domination number* $\gamma(D)$ is the minimum cardinality of a dominating set of D. The domination number was introduced by Lee [7].

A *Roman dominating function* (RDF) on a digraph $D = (V, A)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$

has a in-neighbor u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF on D. A $\gamma_R(D)$*-function* (or γ_R-function) is a Roman dominating function of D with weight $\gamma_R(D)$. The Roman domination for digraphs was introduced by Kamaraj and Jakkammal [6]. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, we have

$$\gamma(D) \le \gamma_R(D) \le 2\gamma(D). \tag{1}$$

The definition of the Roman dominating function for undirected graphs was given multiplicity by Steward [9] and ReVelle and Rosing [8]. Cockayne, Dreyer Jr., Hedetniemi and Hedetniemi [2] as well as Chambers, Kinnersley, Prince and West [1] have given a lot of results on Roman domination.

Our purpose in this paper is to establish some bounds for the Roman domination number of a digraph.

We make use of the following results in this paper.

Proposition A. [7] *Let D be a digraph with order n and minimum indegree $\delta^-(D) \ge 1$. Then,*

$$\gamma(D) \le \frac{2n}{3}.$$

Proposition B. [6] *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(D)$-function of a digraph D. Then*

(a) $\Delta^+(D[V_1]) \le 1$.

(b) If $w \in V_1$, then $N_D^-(w) \cap V_2 = \emptyset$.

(c) If $u \in V_0$, then $|V_1 \cap N_D^+(u)| \le 2$.

(d) V_2 is a $\gamma(D)$-set of the induced subdigraph $D[V_0 \cup V_2]$

(e) Let $H = D[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N^-(v) \cap V_2 \ne \emptyset$, has at least two private neighbors relative to V_2 in the subdigraph H.

Proposition C. [6] *Let D be a digraph with order n. Then*

$$\gamma_R(D) \le n - \Delta^+(D) + 1.$$

2. Bounds on the Roman domination number of digraphs

Our first observation characterize the digraphs which attain the lower bound in (1).

Proposition 1. *Let D be a digraph on n vertices. Then $\gamma(D) = \gamma_R(D)$ if and only if $\Delta^+(D) = 0$.*

Proof. Assume that $\gamma(D) = \gamma_R(D)$. If $f = (V_0, V_1, V_2)$ is a $\gamma_R(D)$-function of D, then the assumption implies that we have equality in $\gamma(D) \le |V_1| + |V_2| \le |V_1| + 2|V_2| = \gamma_R(D)$. This implies that $|V_2| = 0$ and hence we deduce that $|V_0| = 0$. Therefore $\gamma(D) = \gamma_R(D) = |V_1| = |V(D)| = n$. If follows that $\Delta^+(D) = 0$.
Conversely, if $\Delta^+(D) = 0$, then $A(D) = \emptyset$ and so $\gamma(D) = n$. Since $\gamma_R(D) \le n$, the result follows by (1). □

Proposition 2. *If D is a digraph on n vertices, then*

$$\gamma_R(D) \ge \min\{n, \gamma(D) + 1\}.$$

Proof. If $\gamma_R(D) = n$, then we are done. Assume now that $\gamma_R(D) < n$, and suppose on the contrary that $\gamma_R(D) \le \gamma(D)$. If $f = (V_0, V_1, V_2)$ is a $\gamma_R(D)$-function of D, then $V_1 \cup V_2$ is a dominating set of D and thus

$$\begin{aligned} \gamma(D) &\le |V_1| + |V_2| \le |V_1| + 2|V_2| \\ &= \gamma_R(D) \le \gamma(D) \\ &\le |V_1| + |V_2|. \end{aligned}$$

This implies $|V_2| = 0$ and hence $|V_0| = 0$. Therefore we arrive at the contradiction $\gamma_R(D) = |V_1| = n$. □

Proposition 3. *Let D be a digraph on $n \ge 2$ vertices with $\delta^-(D) \ge 1$. Then $\gamma_R(D) = \gamma(D) + 1$ if and only if there is a vertex $v \in V(D)$ with $d^+(v) = n - \gamma(D)$.*

Proof. Assume that D has a vertex v with $d^+(v) = n - \gamma(D)$. Then clearly $f = (V_0, V_1, V_2) = (N^+(v), V(D) - N^+[v], \{v\})$ is an RDF on D of weight $\gamma(D) + 1$. Hence $\gamma_R(D) \le \gamma(D) + 1$, and the result follows by Proposition 2.

Conversely, let $\gamma_R(D) = \gamma(D) + 1$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. Then either (1) $|V_1| = \gamma(D) + 1$ and $|V_2| = 0$ or (2) $|V_1| = \gamma(D) - 1$ and $|V_2| = 1$.

In case (1), since $|V_2| = 0$, we have $|V_0| = 0$. Thus $n = \gamma(D) + 1$. It follows from Proposition A that $n = \gamma(D) + 1 \le \frac{2n}{3} + 1$, a contradiction when $n \ge 4$. If $n = 2$, then the hypothesis $\delta^-(D) \ge 1$ implies that D consists of two vertices x and y such that $x \to y \to x$ and thus $d^+(x) = 1 = 2 - 1 = n - \gamma(D)$. In the case

$n = 3$, let $V(D) = \{x, y, z\}$. The condition $|V_2| = 0$ implies that $\Delta^+(D) \leq 1$. Using $\delta^-(D) \geq 1$, it is straightforward to verify that D is isomorphic to the directed cycle $xyzx$, and we have $d^+(x) = 1 = 3 - 2 = n - \gamma(D)$.

In case (2), let $V_2 = \{v\}$. Obviously $(v, u) \in A(D)$ for each $u \in V_0$. Since v has no out-neighbor in V_1, we obtain $d^+(v) = |V_0| = n - |V_1| + |V_2| = n - \gamma(D)$. □

Proposition 4. *Let D be a digraph on $n \geq 7$ vertices with $\delta^-(D) \geq 1$. Then $\gamma_R(D) = \gamma(D) + 2$ if and only if:*

(i) D does not have a vertex of outdegree $n - \gamma(D)$.

(ii) either D has a vertex of outdegree $n - \gamma(D) - 1$ or D contains two vertices v, w such that $|N^+[v] \cup N^+[w]| = n - \gamma(D) - 2$.

Proof. Let $\gamma_R(D) = \gamma(D) + 2$. It follows from Proposition 3 that D does not have a vertex of outdegree $n - \gamma(D)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. Then either (1) $|V_1| = \gamma(D) + 2$ and $|V_2| = 0$, (2) $|V_1| = \gamma(D)$ and $|V_2| = 1$, or (3) $|V_1| = \gamma(D) - 2$ and $|V_2| = 2$.

In case (1), we have $|V_0| = 0$. Then $V(D) = V_1$. If follows from Proposition A that $n = \gamma(D) + 2 \leq \frac{2n}{3} + 2$ which implies that $n \leq 6$, a contradiction.

In case (2), let $V_2 = \{v\}$. Obviously $(v, u) \in A(D)$ for each $u \in V_0$. Since v has no out-neighbor in V_1, we obtain $d^+(v) = |V_0| = n - |V_1| - |V_2| = n - \gamma(D) - 1$.

In case (3), let $V_2 = \{v, w\}$. Since v and w have no out-neighbor in V_1 and either $(v, u) \in A(D)$ or $(w, u) \in A(D)$ for each $u \in V_0$, it follows that $|N^+[v] \cup N^+[w]| = n - |V_1| = n - (\gamma(D) - 2) = n - \gamma(D) + 2$.

Conversely, assume that D satisfies (i) and (ii). It follows from Proposition 3 and (i) that $\gamma_R(D) \geq \gamma(D) + 2$. If D has a vertex v with $d^+(v) = n - \gamma(D) - 1$, then clearly $f = (N^+(v), V(D) - N^+[v], \{v\})$ is an RDF on D of weight $\gamma(D) + 2$. Hence $\gamma_R(D) \leq \gamma(D) + 2$ and the result follows. If D has two vertices v, w such that $|N^+[v] \cup N^+[w]| = n - \gamma(D) - 2$, then $f = (N^+(v) \cup N^+(w), V(D) - (N^+[v] \cup N^+[w]), \{v, w\})$ is an RDF on D of weight $\gamma(D) + 2$ and the result follows again. This completes the proof. □

Following Cockayne, Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi [2], we will say that a digraph D is a *Roman digraph* if $\gamma_R(D) = 2\gamma(D)$.

Proposition 5. *A digraph D is a Roman digraph if and only if it has a γ_R-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.*

Proof. Let D be a Roman digraph, and let S be a γ-set of D. Then $f = (V(D) - S, \emptyset, S)$ is a Roman dominating function of D such that

$$f(V(D)) = 2|S| = 2\gamma(D) = \gamma_R(D),$$

and therefore f is a γ_R-function with $V_1 = \emptyset$.

Conversely, let $f = (V_0, V_1, V_2)$ be a γ_R-function with $V_1 = \emptyset$ and thus $\gamma_R(D) = 2|V_2|$. Then V_2 is also a dominating set of D, and hence it follows that $2\gamma(D) \leq 2|V_2| = \gamma_R(D)$. Applying (1), we obtain the identity $\gamma_R(D) = 2\gamma(D)$, i.e., D is a Roman digraph. □

Proposition 6. *Let D be a digraph of order n. Then $\gamma_R(D) < n$ if and only if $\Delta^+(D) \geq 2$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of D. The hypothesis $|V_0|+|V_1|+|V_2| = n > \gamma_R(D) = |V_1| + 2|V_2|$ implies $|V_0| \geq |V_2| + 1$. Since each vertex $w \in V_0$ has at least one in-neighbor in V_2, we deduce that

$$\sum_{u \in V_2} d_D^+(u) \geq |V_0| \geq |V_2| + 1.$$

If we suppose on the contrary that $\Delta^+(D) \leq 1$, then we arrive at the contradiction

$$|V_2| \geq \sum_{u \in V_2} d_D^+(u) \geq |V_2| + 1.$$

Conversely, let $\Delta^+(D) \geq 2$. Then Proposition C implies that $\gamma_R(D) \leq n - \Delta^+(D) + 1 < n$, and the proof is complete. □

Corollary 7. *If D is a directed path or directed cycle of order n, then $\gamma_R(D) = n$.*

Next we characterize the digraphs D with the properties that $\gamma_R(D) = 2$, $\gamma_R(D) = 3$, $\gamma_R(D) = 4$ or $\gamma_R(D) = 5$.

Proposition 8. *(1) For a digraph D of order $n \geq 2$, $\gamma_R(D) = 2$ if and only if $\Delta^+(D) = n - 1$ or $n = 2$ and $A(D) = \emptyset$.*

(2) For a digraph D of order $n \geq 3$, $\gamma_R(D) = 3$ if and only if $\Delta^+(D) = n - 2$ or $n = 3$ and $\Delta^+(D) \leq 1$.

(3) For a digraph D of order $n \geq 4$, $\gamma_R(D) = 4$ if and only if $\Delta^+(D) = n - 3$ or $\Delta^+(D) \leq n - 3$ and there are two vertices $u, v \in V(D)$ such that $N_D^+[u] \cup N_D^+[v] = V(D)$ or $n = 4$ and $\Delta^+(D) \leq 1$.

(4) For a digraph D of order $n \geq 5$, $\gamma_R(D) = 5$ if and only if $\Delta^+(D) \leq n-4$ and $|N_D^+[x] \cup N_D^+[y]| \leq |V(D)|-1$ for all pairs of vertices $x, y \in V(D)$. In addition, (i) there are two vertices $u, v \in V(D)$ such that $|N_D^+[u] \cup N_D^+[v]| = |V(D)| - 1$ or (ii) $n = 5$ and $\Delta^+(D) \leq 1$ or (iii) D contains a vertex w with $d^+(w) = n-4$ and the induced subdigraph $D[V(D) - N^+[w]]$ consists of three isolated vertices.

Proof. We omit the proof of (1), because it is clear.

(2) If $\Delta^+(D) = n-2$ or $n = 3$ and $\Delta^+(D) \leq 1$, then it is easy to see that $\gamma_R(D) = 3$.

Conversely, assume that $\gamma_R(D) = 3$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. It follows from (1) that $\Delta(D) \leq n-2$. Now we distinguish two cases.

Case 1. Assume that $V_2 = \emptyset$. Then $|V_1| = 3$ and thus $n = 3$. Therefore Proposition 6 implies that $\Delta^+(D) \leq 1$.

Case 2. Assume that $|V_2| = 1$ and $|V_1| = 1$. If $V_2 = \{v\}$, then we deduce that $d^+(v) = \Delta^+(D) = n-2$.

(3) If $\Delta^+(D) = n-3$ or $\Delta^+(D) \leq n-3$ and there are two vertices $u, v \in V(D)$ such that $N_D^+[u] \cup N_D^+[v] = V(D)$ or $n = 4$ and $\Delta^+(D) \leq 1$, then it is straighforward to verify that $\gamma_R(D) = 4$.

Conversely, assume that $\gamma_R(D) = 4$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. Using (1) and (2), we find that $\Delta(D) \leq n-3$. Now we distinguish three cases.

Case 1. Assume that $V_2 = \emptyset$. Then $|V_1| = 4$ and thus $n = 4$. So Proposition 6 implies that $\Delta^+(D) \leq 1$.

Case 2. Assume that $|V_2| = 1$ and $|V_1| = 2$. If $V_2 = \{v\}$, then we deduce that $d^+(v) = \Delta^+(D) = n-3$.

Case 3. Assume that $|V_2| = 2$. If $V_2 = \{u, v\}$, then we conclude that $N_D^+[u] \cup N_D^+[v] = V(D)$.

(4) The conditions $\Delta^+(D) \leq n-4$ and $|N_D^+[x] \cup N_D^+[y]| \leq |V(D)|-1$ for all pairs of vertices $x, y \in V(D)$ and (3) imply that $\gamma_R(D) \geq 5$. The other three assumptions show that $\gamma_R(D) \leq 5$ and thus we obtain $\gamma_R(D) = 5$.

Conversely, assume that $\gamma_R(D) = 5$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. Using (1), (2) and (3), we see that $\Delta^+(D) \leq n-4$ and $|N_D^+[x] \cup N_D^+[y]| \leq |V(D)|-1$ for all pairs of vertices $x, y \in V(D)$. Again, we distinguish three cases.

Case 1. Assume that $V_2 = \emptyset$. Then $|V_1| = 5$ and thus $n = 5$. Hence Proposition 6 implies (ii) that $\Delta^+(D) \leq 1$.

Case 2. Assume that $|V_2| = 1$ and $|V_1| = 3$. If $V_2 = \{w\}$, then we deduce that $d^+(w) = n-4$. Let $\{a, b, c\} = V(D) - N^+[w]$. If $D[\{a, b, c\}]$ consists of isolated vertices, then we have condition (iii). If $D[\{a, b, c\}]$ contains an arc, say ab, then $|N_D^+[w] \cup N_D^+[a]| = |V(D)| - 1$ and we have shown condition (i).

Case 3. Assume that $|V_2| = 2$ and $|V_1| = 1$. If $V_2 = \{u, v\}$, then it follows that $|N_D^+[u] \cup N_D^+[v]| = |V(D)| - 1$ and condition (i) is proved. □

Theorem 9. *Let D be a digraph of order n and maximum outdegree $\Delta^+(D) \geq 1$. Then*

$$\gamma_R(D) \geq \left\lceil \frac{2n}{1+\Delta^+(D)} \right\rceil + \epsilon$$

with $\epsilon = 0$ when $n \equiv 0, 1 \pmod{(\Delta^+(D)+1)}$ and $\epsilon = 1$ when $n \not\equiv 0, 1 \pmod{(\Delta^+(D)+1)}$.

Proof. Let $n = p(\Delta^+(D)+1)+r$ with integers $p \geq 1$ and $0 \leq r \leq \Delta^+(D)$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. Then $\gamma_R(D) = |V_1| + 2|V_2|$ and $n = |V_0| + |V_1| + |V_2|$. Since each vertex of V_0 has at least one in-neighbor in V_2, we deduce that $|V_0| \leq \Delta^+(D)|V_2|$. Therefore we conclude that

$$\begin{aligned}
(\Delta^+(D)+1)\gamma_R(D) &= (\Delta^+(D)+1)(|V_1| + 2|V_2|) \\
&= (\Delta^+(D)+1)|V_1| + 2|V_2| + 2\Delta^+(D)|V_2| \\
&\geq (\Delta^+(D)+1)|V_1| + 2|V_2| + 2|V_0| \\
&= 2n + (\Delta^+(D)-1)|V_1| \\
&= 2p(\Delta^+(D)+1) + 2r + (\Delta^+(D)-1)|V_1|.
\end{aligned}$$

This inequality chain and the hypothesis that $\Delta^+(D) \geq 1$ lead to the desired bound if $r = 0$ or $r = 1$ or $2 \leq r \leq \Delta^+(D)$ and $V_1 \neq \emptyset$. In the remaining case that $2 \leq r \leq \Delta^+(D)$ and $V_1 = \emptyset$, it follows from $|V_0| \leq \Delta^+(D)|V_2|$ that

$$p(\Delta^+(D)+1) + r = n = |V_0| + |V_2| \leq (\Delta^+(D)+1)|V_2|.$$

Hence the condition $r \geq 2$ leads to $|V_2| \geq p+1$. Therefore we obtain $\gamma_R(D) = 2|V_2| \geq 2(p+1)$, and this completes the proof. □

Theorem 10. *For any digraph D on n vertices,*

$$\gamma_R(D) \leq n \left(\frac{2 + \ln \frac{1+\delta^-(D)}{2}}{1+\delta^-(D)} \right).$$

Proof. Given a digraph D, select a set of vertices A, which each vertex is selected independently with probability p (with p to be defined later). The expected size of of A is np. Let $B = V(D) - N^+[A]$. Obviously, $f = (V(D) - (A \cup B), B, A)$ is an RDF for D.

Now we compute the expected size of B. The probability that v is in B is equal to the probability that v is not in A and that no vertex of A is the in-neighbor of v. This probability is $(1-p)^{1+\deg^-(v)}$. Since $e^{-x} \geq 1-x$ for any $x \geq 0$, and $\deg^-(v) \geq \delta^-(D)$, we conclude that $\Pr(v \in B) \leq e^{-p(1+\delta^-(D))}$. Hence, the expected size of B is at most $ne^{-p(1+\delta^-(D))}$, and the expected weight of f, denoted by $E[f(V(D)]$, is at most $2np + ne^{-p(1+\delta^-(D))}$. The upper bound for $E[f(V(D))]$ is minimized when $p = \dfrac{\ln \frac{1+\delta^-(D)}{2}}{1+\delta^-(D)}$, and substituting this value in for p gives

$$E[f(V(D)] \leq n \left(\frac{2 + \ln \frac{1+\delta^-(D)}{2}}{1+\delta^-(D)} \right).$$

Since the expected weight of $f(V(D))$ is at most value $n \left(\frac{2+\ln \frac{1+\delta^-(D)}{2}}{1+\delta^-(D)} \right)$, there must be some RDF with at most this weight.

The bound is sharp for every orientation of $\frac{n}{2}K_2$. □

A *Roman dominating function* on a graph $G = (V(G), E(G))$ is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $v \in V(G)$ for which $f(v) = 0$ has a neighbor $u \in V(G)$ for which $f(u) = 2$. The *weight* of an Roman dominating function f on G is the value $\omega(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an Roman dominating function on G.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N^+_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following observation is valid.

Observation 11. *If $D(G)$ is the associated digraph of a graph G, then $\gamma(D(G)) = \gamma(G)$ and $\gamma_R(D(G)) = \gamma_R(G)$.*

There are a lot of interesting applications of Obsevation 11, as for example the following three results.

Corollary 12. **([2])** *If G is a connected graph of order $n \geq 2$, then $\gamma_R(G) = \gamma(G)+1$ if and only if there is a vertex $v \in V(G)$ of degree $d_G(v) = n - \gamma(G)$.*

Proof. Since $d_G(v) = d^+_{D(G)}(v)$ for each vertex $v \in V(G) = V(D(G))$ and $n = n(D(G))$, it follows from Proposition 3 that $\gamma_R(D(G)) = \gamma(D(G)) + 1$ if and only if there is a vertex $v \in V(D(G))$ with $d^+_{D(G)}(v) = n(D(G)) - \gamma(D(G))$. Using Observation 11, we obtain the desired result. □

Corollary 13. **([3])** *If G is a graph of order n and maximum $\Delta(G) \geq 1$, then*

$$\gamma_R(G) \geq \left\lceil \frac{2n}{1+\Delta(G)} \right\rceil.$$

Proof. Since $\Delta(G) = \Delta^+(D(G))$ and $n = n(D(G))$, it follows from Theorem 9 and Observation 11 that

$$\gamma_R(G) = \gamma_R(D(G)) \geq \left\lceil \frac{2n(D(G))}{1 + \Delta^+(D(G))} \right\rceil = \left\lceil \frac{2n}{1 + \Delta(G)} \right\rceil.$$

□

Corollary 14. ([2]) *For any graph on n vertices,*

$$\gamma_R(G) \leq n \left(\frac{2 + \ln \frac{1+\delta(G)}{2}}{1 + \delta(G)} \right).$$

Proof. Since $\delta(G) = \delta^-(D(G))$ and $n = n(D(G))$, it follows from Theorem 9 and Observation 11 that

$$\gamma_R(G) = \gamma_R(D(G)) \leq n(D(G)) \left(\frac{2 + \ln \frac{1+\delta^-(D(G))}{2}}{1 + \delta^-(D(G))} \right) = n \left(\frac{2 + \ln \frac{1+\delta(G)}{2}}{1 + \delta(G)} \right).$$

□

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