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A SUBCLASS OF MULTIVALENT UNIFORMLY CONVEX FUNCTIONS ASSOCIATED WITH GENERALIZED SĂLĂGEAN AND RUSCHEWEYH DIFFERENTIAL OPERATORS

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Abstract. In this paper a new subclass of Multivalent uniformly convex functions with negative coefficients defined by a linear combination of generalized Sălăgean and Ruscheweyh differential operators is introduced. Several results concerning coefficient estimates , the result of modified Hadamard product and results for a family of class preserving integral operators are considered. Extreme points and other interesting properties for this class are also indicated.

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1. Introduction and Definitions

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. Also denote by T_p the class of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad (a_k \geq 0, z \in U), \tag{1.2}$$

which are analytic and p-valent in U.
For functions

$$f_j(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,j}\ z^k,\ (a_{k,j} \geq 0),\ (j = 1, 2) \tag{1.3}$$

Hadamard product $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1}\, a_{k,2}\, z^k. \tag{1.4}$$

A function $f(z) \in A_p$ is said to be β-uniformly starlike functions of order α denoted by $\beta - S_p(\alpha)$ if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - p\right|, \tag{1.5}$$

in the class T_p, the modified for some $\alpha(-p \leq \alpha < p)$, $\beta \geq 0$, and is said to be β-uniformly convex of order α denoted by $\beta - K_p(\alpha)$ if it satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta\left|\frac{zf''(z)}{f'(z)} + 1 - p\right|, \tag{1.6}$$

for some $\alpha(-p \leq \alpha < p)$, $\beta \geq 0$ and all $(z \in U)$. The class $0 - S_p(\alpha) = S_p(\alpha)$, and $0 - K_p(\alpha) = K_p(\alpha)$, where $S_p(\alpha)$ and $K_p(\alpha)$ are respectively the well-known classes of starlike and convex functions of order α $(0 \leq \alpha < p)$.
The classes $S_p(\alpha)$ and $K_p(\alpha)$ are introduced by Patil and Thakare [7] while the classes $S(\alpha)$ and $K(\alpha)$ were first studied by Rebortson [8], Schild [12], Silverman [13], and others. The classes $\beta - S_p(\alpha)$ and $\beta - K_p(\alpha)$ were introduced and studied by Goodman [2], Rønning[9], and Minda and Ma [5].
Let

$$S_p^*(\alpha) = S_p(\alpha) \cap T_p, \quad K_p^*(\alpha) = K_p(\alpha) \cap T_p, \tag{1.7}$$

$$\beta - S_p^*(\alpha) = [\beta - S_p(\alpha)] \cap T_p, \quad and \quad \beta - K_p^*(\alpha) = [\beta - K_p(\alpha)] \cap T_p$$

The Sălăgean differential operator [11] can be generalized for a function $f(z) \in A_p$ as follows

$$\begin{aligned} S_{\delta,p}^0 f(z) &= f(z), \\ S_{\delta,p}^1 f(z) &= (1-\delta)f(z) + \delta\frac{zf'(z)}{p} = S_{\delta,p}f(z), \\ &\vdots \\ S_{\delta,p}^n f(z) &= S_{\delta,p}(S_{\delta,p}^{n-1}f(z)). \quad (n \in N, \delta \geq 0, z \in U) \end{aligned} \tag{1.8}$$

The nth Ruscheweyh drivative [1] for a function $f(z) \in A_p$, is defined by

$$R_p^n f(z) = \frac{z^p}{n!}\frac{d^n}{dz^n}\left(z^{n-p}f(z)\right) \quad (n \in N_0 = N \cup \{0\}, z \in U) \tag{1.9}$$

It can be easily seen that the operators S_p^n and R_p^n on the function $f(z) \in A_p$ are given by

$$S_{\delta,p}^n f(z) = z^p + \sum_{k=p+1}^{\infty} \left(1 + (\frac{k}{p} - 1)\delta\right)^n a_k z^k, \tag{1.10}$$

and

$$R_p^n f(z) = z^p + \sum_{k=p+1}^{\infty} C_{n+k-p}^n a_k z^k. \tag{1.11}$$

where $C_{n+k-p}^n = \frac{(n+k-p)!}{n!(k-p)!}$.
Definition 1. *let $n \in N_0$ and $\lambda \geq 0$. Let $D_{\lambda,\delta,p}^n f$ denote the operator defined by $D_{\lambda,\delta,p}^n : A_p \to A_p$*

$$D_{\lambda,\delta,p}^n f\ (z) = (1-\lambda) S_{\delta,p}^n f(z) + \lambda R_p^n f(z) \qquad (z \in U). \tag{1.12}$$

Notice that $D_{\lambda,\delta,p}^n$ is a linear operator and for $f(z) \in A_p$ we have

$$D_{\lambda,\delta,p}^n f\ (z) = z^p + \sum_{k=p+1}^{\infty} \phi_k(n,\lambda,\delta,p) a_k z^k, \tag{1.13}$$

where

$$\phi_k(n,\lambda,\delta,p) = \left[(1-\lambda)\left(1 + (\frac{k}{p} - 1)\delta\right)^n + \lambda C_{n+k-p}^n\right] \tag{1.14}$$

It is clear that $D_{\lambda,\delta,p}^0 f\ (z) = f\ (z)$ and $D_{\lambda,1,p}^1 f\ (z) = \frac{z}{p} f'\ (z)$.When $p = 1$, we get the differential operator studied by Khairnar and More [3].

Definition 2. *For $-p \leq \alpha < p$, $\beta \geq 0$, we let $S_p^n(\alpha,\beta,\lambda,\delta)$ be the subclass of A_p consisting of functions $f(z)$ of the form (1.1) and satisfying the following condition*

$$Re\left\{\frac{z\left(D_{\lambda,\delta,p}^n f\ (z)\right)'}{D_{\lambda,\delta,p}^n f\ (z)} - \alpha\right\} \geq \beta \left|\frac{z\left(D_{\lambda,\delta,p}^n f\ (z)\right)'}{D_{\lambda,\delta,p}^n f\ (z)} - p\right|, \tag{1.15}$$

also let $T_p^n(\alpha,\beta,\lambda,\delta) = S_p^n(\alpha,\beta,\lambda,\delta) \bigcap T_p$.

It may be noted that the class $T_p^n(\alpha,\beta,\lambda,\delta)$ extends the classes of starlike, convex, β-uniformly starlike and β-uniformly convex for suitable choice of $\alpha,\beta,\lambda,\delta$ and n. For example

i) For $n = 0, \lambda = \delta = 1$ the class $T_p^n(\alpha, \beta, \lambda, \delta)$ reduces to the class of β-uniformly starlike functions.
(ii) For $n = 1, \lambda = \delta = 1$ we obtain the class of β-uniformly convex function.
Several other classes studied by various research workers can be obtained from the class $T_p^n(\alpha, \beta, \lambda, \delta)$.

2. Coefficient Estimates

Theorem 1. *A function $f(z)$ defined by (1.2) is in the class $T_p^n(\alpha, \beta, \lambda, \delta)$, $-p \le \alpha < p, \beta \ge 0$ if and only if*

$$\sum_{k=p+1}^{\infty} \{k(1+\beta) - (\alpha + p\beta)\} \ \phi_k(n, \lambda, \delta, p) \ a_k \le (p - \alpha) \ , \tag{2.1}$$

where $\phi_k(n, \lambda, \delta, p)$ is given by (1.14) and the result is sharp.

Proof. Let $f(z) \in T_p^n(\alpha, \beta, \lambda, \delta)$ and z be real then by virtue of (1.13) we have

$$\frac{p - \sum\limits_{k=p+1}^{\infty} k \ \phi_k(n, \lambda, \delta, p) a_k \ z^{k-p}}{1 - \sum\limits_{n=p+1}^{\infty} \phi_k(n, \lambda, \delta, p) a_k \ z^{k-p}} - \alpha \ge \beta \left| \frac{\sum\limits_{k=p+1}^{\infty} (k-p)\phi_k(n, \lambda, \delta, p) a_k \ z^k}{1 - \sum\limits_{n=p+1}^{\infty} \phi_k(n, \lambda, \delta, p) \ a_n \ z^n} \right| .$$

Letting $z \to 1$ along the real axis, we obtain the desire inequality (2.1).
Conversely, assuming that (2.1) holds, then we show that

$$\beta \left| \frac{z\left(D^n_{\lambda,\delta,p} f\ (z)\right)'}{D^n_{\lambda,\delta,p} f\ (z)} \right| - Re \left\{ \frac{z\left(D^n_{\lambda,\delta,p} f\ (z)\right)'}{D^n_{\lambda,\delta,p} f\ (z)} \right\} \le p - \alpha \tag{2.2}$$

We have

$$\beta \left| \frac{z\left(D^n_{\lambda,\delta,p} f\ (z)\right)'}{D^n_{\lambda,\delta,p} f\ (z)} \right| - Re \left\{ \frac{z\left(D^n_{\lambda,\delta,p} f\ (z)\right)'}{D^n_{\lambda,\delta,p} f\ (z)} \right\} \le (1+\beta) \left| \frac{z\left(D^n_{\lambda,\delta,p} f\ (z)\right)'}{D^n_{\lambda,\delta,p} f\ (z)} \right|$$

$$\le \frac{(1+\beta) \sum\limits_{k=p+1}^{\infty} (k-p) \ \phi_k(n, \lambda, \delta, p) \ a_k}{1 - \sum\limits_{n=p+1}^{\infty} \phi_k(n, \lambda, \delta, p) \ a_k} .$$

This expression is bounded above by $(p-\alpha)$ if

$$\sum_{k=p+1}^{\infty} \{k(1+\beta)-(\alpha+p\beta)\}\, \phi_k(n,\lambda,\delta,p) a_k \leq (p-\alpha) \ ,$$

The equality in (2.1) is attained for the function

$$f(z)=z^p-\frac{(p-\alpha)}{\{k(1+\beta)-(\alpha+p\beta)\}\, \phi_k(n,\lambda,\delta,p)}\, z^k \ . \qquad k\geq p+1 \tag{2.3}$$

This completes the proof of the theorem.

Corollary 1. *Let the function $f(z)$ defined by (1.2) be in the class $T_p^n(\alpha,\beta,\lambda,\delta)$, $-p\leq\alpha<p$, $\beta\geq 0$, then*

$$a_k \leq \frac{(p-\alpha)}{\{k(1+\beta)-(\alpha+p\beta)\}\, \phi_k(n,\lambda,\delta,p)} \ , \quad k\geq p+1.$$

3. Results Involving Modified Hadamard Product

Theorem 2. *For $n\in N_0$, $\lambda,\delta\geq 0$, $-p\leq\alpha<p$ and $\beta\geq 0$ let $f_1(z)\in T_p^n(\alpha,\beta,\lambda,\delta)$ and $f_2(z)\in T_p^n(\gamma,\beta,\lambda,\delta)$. Then $f_1 * f_2(z)\in T_p^n(\sigma,\beta,\lambda,\delta)$, where*

$$\sigma=p-\frac{(1+\beta)(p-\alpha)(p-\gamma)}{(p+1+\beta-\alpha)(p+1+\beta-\gamma)\ \left[(1-\lambda)\left(1+\frac{\delta}{p}\right)^n+\lambda(n+1)\right]-(p-\alpha)(p-\gamma)} \tag{3.1}$$

and the result is sharp.

Proof. To prove the theorem it is sufficient to assert that

$$\sum_{k=p+1}^{\infty} \frac{\{k(1+\beta)-(\sigma+p\beta)\}}{p-\sigma}\phi_k(n,\lambda,\delta,p) a_{k,1}\ a_{k,2} \leq 1 \ , \tag{3.2}$$

where $\phi_k(n,\lambda,\delta,p)$ is defined in (1.14) and σ is defined in (3.1). Now by virtue of Cauchy-Schwarz inequality and Theorem 1, it follows that

$$\sum_{k=p+1}^{\infty} \frac{\{k(1+\beta)-(\alpha+p\beta)\}^{1/2}\,\{k(1+\beta)-(\gamma+p\beta)\}^{1/2}}{\sqrt{(p-\alpha)(p-\gamma)}}\phi_k(n,\lambda,\delta,p)\,\sqrt{a_{n,1}a_{n,2}} \leq 1 \ ,$$

(3.3)

Hence (3.2) is true if

$$\frac{\{k(1+\beta)-(\sigma+p\beta)\}}{p-\sigma}\phi_k(n,\lambda,\delta,p)\ a_{n,1}\ a_{n,2}$$

$$\leq \frac{\{k(1+\beta)-(\alpha+p\beta)\}^{1/2}\{n(1+\beta)-(\gamma+p\beta)\}^{1/2}}{\sqrt{(p-\alpha)(p-\gamma)}}\phi_k(n,\lambda,\delta,p)\ \sqrt{a_{n,1}a_{n,2}}$$

or equivalently

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\{k(1+\beta)-(\alpha+p\beta)\}^{1/2}\{n(1+\beta)-(\gamma+p\beta)\}^{1/2}}{\sqrt{(p-\alpha)(p-\gamma)}} \times \frac{p-\sigma}{\{k(1+\beta)-(\sigma+\beta)\}} \tag{3.4}$$

By virtue of (3.3), (3.2) is true if

$$\frac{\sqrt{(p-\alpha)(p-\gamma)}}{\{k(1+\beta)-(\alpha+p\beta)\}^{1/2}\{k(1+\beta)-(\gamma+p\beta)\}^{1/2}\phi_k(n,\lambda,\delta,p)}$$

$$\leq \frac{\{k(1+\beta)-(\alpha+p\beta)\}^{1/2}\{n(1+\beta)-(\gamma+p\beta)\}^{1/2}}{\sqrt{(p-\alpha)(p-\gamma)}} \times \frac{p-\sigma}{\{k(1+\beta)-(\sigma+p\beta)\}}$$

which yields

$$\sigma \leq p - \frac{(k-p)(\beta+1)(p-\alpha)(p-\gamma)}{\{k(1+\beta)-(\alpha+p\beta)\}\{k(1+\beta)-(\gamma+p\beta)\}\phi_k(n,\lambda,\delta,p)-(p-\alpha)(p-\gamma)}. \tag{3.5}$$

Under the stated conditions in the theorem, we observe that the function $\phi_k(n,\lambda,\delta,p)$ is a decreasing for k $(k \geq p+1)$, and thus (3.5) is satisfied if σ is given by (3.1). Finally the result is sharp for

$$f_1(z) = z^p - \frac{(p-\alpha)}{(p+1+\beta-\alpha)\left[(1-\lambda)\left(1+\frac{\delta}{p}\right)^n+\lambda(n+1)\right]}\ z^{p+1},$$

$$f_2(z) = z^p - \frac{(p-\gamma)}{(p+1+\beta-\gamma)\left[(1-\lambda)\left(1+\frac{\delta}{p}\right)^n+\lambda(n+1)\right]}\ z^{p+1}.$$

Theorem 3. *Under the conditions stated in Theorem 2, let the functions $f_j(z)$ $(j = 1, 2)$ defined by (1.3) be in the class $T_p^n(\alpha, \beta, \lambda, \delta)$. Then $f_1 * f_2(z) \in T_p^n(\sigma, \beta, \lambda, \delta)$, where*

$$\sigma = p - \frac{(1+\beta)(p-\alpha)^2}{(p+1+\beta-\alpha)^2\left[(1-\lambda)\left(1+\frac{\delta}{p}\right)^n + \lambda(n+1)\right] - (p-\alpha)^2}. \tag{3.6}$$

Proof. The result follows by setting $\alpha = \gamma$ in Theorem 3 .

Theorem 4. *Under the conditions stated in Theorem 2, let the functions $f_j(z)$ $(j = 1, 2)$ defined by (1.3) be in the class $T_p^n(\alpha, \beta, \lambda, \delta)$. Then*

$$h(z) = z^p - \sum_{k=p+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2)\, z^p \tag{3.7}$$

is in the class $T_p^n(\sigma, \beta, \lambda, \delta)$, where

$$\sigma = p - \frac{2(1+\beta)(p-\alpha)^2}{(p+1+\beta-\alpha)^2\left[(1-\lambda)\left(1+\frac{\delta}{p}\right)^n + \lambda(n+1)\right] - 2(p-\alpha)^2}. \tag{3.8}$$

Proof. In view of Theorem 1, it is sufficient to prove that

$$\sum_{k=p+1}^{\infty} \frac{\{k(1+\beta) - (\sigma + p\beta)\}}{p-\sigma} \phi_k(n, \lambda, \delta, p)(\, a_{n,1}^2 + a_{n,2}^2) \leq 1\,, \tag{3.9}$$

where $\phi_k(n, \lambda, \delta, p)$ is defined in (1.14) and σ is defined in (3.8).
as $f_j(z) \in T_p^n(\alpha, \beta, \lambda, \delta)(j = 1, 2)$, Theorem 1 yields

$$\sum_{k=p+1}^{\infty} \left[\frac{\{k(1+\beta) - (\alpha + p\beta)\}\, \phi_k(n, \lambda, \delta, p)}{(p-\alpha)}\right]^2 a_{k,j}^2$$

$$\leq \sum_{k=p+1}^{\infty} \left[\frac{\{k(1+\beta) - (\alpha + p\beta)\}\, \phi_k(n, \lambda, \delta, p)}{(p-\alpha)}\, a_{k,j}\right]^2 \leq 1$$

hence

$$\sum_{k=p+1}^{\infty} \frac{1}{2} \left[\frac{\{k(1+\beta) - (\alpha + p\beta)\}\, \phi_k(n, \lambda, \delta, p)}{(p-\alpha)}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1 \tag{3.10}$$

(3.9) is true if

$$\frac{\{k(1+\beta)-(\sigma+p\beta)\}}{p-\sigma}\phi_k(n,\lambda,\delta,p)(\ a_{n,1}^2+a_{n,2}^2)$$

$$\leq \frac{1}{2}\left[\frac{\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)}{(p-\alpha)}\right]^2\ (a_{n,1}^2+a_{n,2}^2)\ ,$$

that is, if

$$\sigma \leq p-\ \frac{2(k-p)(1+\beta)(p-\alpha)^2}{[n(1+\beta)-(\alpha+p\beta)]^2\,\phi_k(n,\lambda,\delta,p)-2(p-\alpha)^2}\ . \tag{3.11}$$

Under the stated conditions in the theorem, we observe that the function $\phi_k(n,\lambda,\delta,p)$ is a decreasing for $k(k\geq p+1)$, and thus (3.11) is satisfied if σ is given by (3.8).

4. Family of Class Preserving Integral Operators

In this section, we discuss some class preserving integral operators. We recall here the Komatu operator [6] defined by

$$H(z)=P^d_{c,p}f(z)=\frac{(c+p)^d}{\Gamma(d)z^c}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{d-1} f(t)dt \tag{4.1}$$

where $d>0, c>-p$ and $z\in U$.
Also we recall the generalized Jung-Kim-Srivastava integral operator [4] defined by

$$I(z)=Q^d_{c,p}f(z)=\frac{\Gamma(d+c+p)}{\Gamma(c+p)\Gamma(d)}\frac{1}{z^c}\int_0^z t^{c-1}\left(1-\frac{t}{z}\right)^{d-1} f(t)dt \quad . \tag{4.2}$$

Theorem 5. *If* $f(z)\in T^n_p(\alpha,\beta,\lambda,\delta)$, *then* $H(z)\in T^n_p(\alpha,\beta,\lambda,\delta)$.

Proof. Let the function $f(z)\in T^n_p(\delta,\beta,\lambda,\delta)$ be defined by (1.2). It can be easily verified that

$$H(z)=z^p-\sum_{k=p+1}^{\infty}\left(\frac{c+p}{c+k+p}\right)^d a_k z^k \quad (a_k\geq 0, p\in N) \tag{4.3}$$

Now $H(z)\in T^n_p(\alpha,\beta,\lambda,\delta)$ if

$$\sum_{k=p+1}^{\infty}\frac{[\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)]}{(p-\alpha)}\left(\frac{c+p}{c+k+p}\right)^d a_k\ \leq 1 \tag{4.4}$$

Now as $\frac{c+p}{c+k+p} \leq 1$ for $k \in N$, so it is clear that

$$\sum_{k=p+1}^{\infty} \frac{[\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)]}{(p-\alpha)} \left(\frac{c+p}{c+k+p}\right)^d a_k$$

$$\leq \sum_{k=p+1}^{\infty} \frac{[\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)]}{(p-\alpha)} a_k \leq 1$$

Therefore $H(z) \in T_p^n(\alpha,\beta,\lambda,\delta)$.

Theorem 6. *Let $d > 0, c > -p$ and $f(z) \in T_p^n(\alpha,\beta,\lambda,\delta)$. Then $H(z)$ defined by (4.1) is $p-$valent in the disk $|z| < R_1$, where*

$$R_1 = \inf_k \left\{ \frac{p[k(1+\beta)-(\alpha+p\beta)](c+k+p)^d \phi_k(n,\lambda,\delta,p)}{k(c+p)^d(p-\alpha)} \right\}^{\frac{1}{k}} \tag{4.5}$$

Proof. In order to prove the assertion , it is enough to show that

$$\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq p \tag{4.6}$$

Now, in view of (4.3), we get

$$\left| \frac{H'(z)}{z^{p-1}} - p \right| = \left| - \sum_{k=p+1}^{\infty} k \left(\frac{c+p}{c+k+p}\right)^d a_k \; z^k \right| \leq \sum_{k=p+1}^{\infty} k \left(\frac{c+p}{c+k+p}\right)^d a_k \; |z|^k$$

This expression is bounded by p if

$$\sum_{k=p+1}^{\infty} \frac{k}{p} \left(\frac{c+p}{c+k+p}\right)^d a_k \; |z|^k \leq 1 \tag{4.7}$$

Given that $f(z) \in T_p^n(\alpha,\beta,\lambda,\delta)$, so in view of Theorem 1 , we have

$$\sum_{k=p+1}^{\infty} \frac{[\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)]}{(p-\alpha)} a_k \leq 1$$

Thus, (4.7) holds if

$$k \left(\frac{c+p}{c+k+p}\right)^d a_k \; |z|^k \leq \frac{p[\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)]}{(p-\alpha)},$$

that is

$$|z| \leq \left\{ \frac{p[k(1+\beta)-(\alpha+p\beta)](c+k+p)^d \phi_k(n,\lambda,\delta,p)}{k(c+p)^d(p-\alpha)} \right\}^{\frac{1}{k}}$$

The result follows by setting $|z| = R_1$.
Following similar steps as in the proofs of Theorem 5 and Theorem 6, we can state the following two theorems concerning the generalized Jung-Kim-Srivastava integral operator $I(z)$.

Theorem 7. *If* $f(z) \in T_p^n(\alpha, \beta, \lambda, \delta)$, *then* $I(z) \in T_p^n(\alpha, \beta, \lambda, \delta)$.

Theorem 8. *Let* $d > 0, c > -p$ *and* $f(z) \in T_p^n(\alpha, \beta, \lambda, \delta)$. *Then* $I(z)$ *defined by (4.2) is* $p-$*valent in the disk* $|z| < R_2$, *where*

$$R_2 = \inf_k \left\{ \frac{p\left\{k(1+\beta) - (\alpha + p\beta)\right\} \phi_k(n, \lambda, \delta, p)(p+c+d)_k}{k(p-\alpha)(p+c)_k} \right\}^{\frac{1}{k}}. \tag{4.8}$$

5. Extreme Points of the Class $T_p^n(\alpha, \beta, \lambda, \delta)$

Theorem 9. *Let* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{(p-\alpha)}{\left\{k(1+\beta) - (\alpha + p\beta)\right\} \phi_k(n, \lambda, \delta, p)} z^k, \quad (k \geq p+1). \tag{5.1}$$

Then $f(z) \in T_p^n(\alpha, \beta, \lambda, \delta)$ *if and only if it can be expressed in the form*

$$f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z), \tag{5.2}$$

where $\lambda_k \geq 0$ *and* $\sum_{k=p}^{\infty} \lambda_k = 1$, *and* $\phi_k(n, \lambda, \delta, p)$ *is given in (1.14).*

Proof. Let (5.2) holds, then by (5.1) we have

$$f(z) = \lambda_p z^p - \sum_{k=p+1}^{\infty} \frac{(p-\alpha)}{\left\{k(1+\beta) - (\alpha + p\beta)\right\} \phi_k(n, \lambda, \delta, p)} \lambda_k z^k.$$

Now

$$\sum_{k=p+1}^{\infty} \left\{k(1+\beta) - (\alpha + p\beta)\right\} \phi_k(n, \lambda, \delta, p)\ a_k$$

$$= \sum_{k=p+1}^{\infty} \left\{k(1+\beta) - (\alpha + p\beta)\right\} \phi_k(n, \lambda, \delta, p) \times \frac{(p-\alpha)}{\left\{k(1+\beta) - (\alpha + p\beta)\right\} \phi_k(n, \lambda, \delta, p)} \lambda_k$$

$$= (p-\alpha)\sum_{k=p+1}^{\infty}\lambda_k \leq (p-\alpha)\sum_{k=p}^{\infty}\lambda_k \leq p-\alpha.$$

Hence by Theorem 1, $f(z) \in T_p^n(\alpha,\beta,\lambda,\delta)$.
Conversely, suppose $f(z) \in T_p^n(\alpha,\beta,\lambda,\delta)$. Since

$$a_k \leq \frac{(p-\alpha)}{\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)}\ , \quad (k \geq p+1)$$

setting $\lambda_k = \frac{\{k(1+\beta)-(\alpha+p\beta)\}\phi_k(n,\lambda,\delta,p)}{(p-\alpha)}\ a_k$ and $\lambda_p = 1-\sum\limits_{k=p+1}^{\infty}\lambda_k$, we get (5.2). This completes the proof of the theorem.

6. Closure Properties

Theorem 10. *Let the functions $f_j(z)$ defined by (1.3) be in the class $T_p^n(\alpha,\beta,\lambda,\delta)$. Then the function $h(z)$ defined by*

$$h(z) = z^p - \sum_{k=p+1}^{\infty} d_k z^k$$

belongs to $T_p^n(\alpha,\beta,\lambda,\delta)$, where

$$d_k = \frac{1}{m}\sum_{j=1}^{m} a_{k,j}, \qquad (a_{k,j} \geq 0).$$

Proof. Since $f_j(z) \in T_p^n(\delta,\beta,\lambda,\delta)$, it follows from Theorem 1 that

$$\sum_{k=p+1}^{\infty}\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)\ a_{k,j} \leq (p-\alpha)\ , \tag{6.1}$$

where $\phi_k(n,\lambda,\delta,p)$ is given in (1.14). Therefore

$$\sum_{k=p+1}^{\infty}\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)\ d_k$$

$$= \sum_{k=p+1}^{\infty}\{k(1+\beta)-(\alpha+p\beta)\}\,\phi_k(n,\lambda,\delta,p)\left(\frac{1}{m}\sum_{j=1}^{m} a_{kj}\right) \leq p-\alpha,$$

by (6.1), which yields that $h(z) \in T_p^n(\delta,\beta,\lambda,\delta)$.

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