Acta Universitatis Apulensis No. 26/2011
ISSN: 1582-5329 pp. 143-148

# UNIVALENCE CRITERIA FOR A FAMILY OF INTEGRAL OPERATORS

S. Porwal, K.K. Dixit, M.Darus

Abstract. The purpose of the present paper is to extend the univalent condition of some integral operators. Relevant connections of the results presented here with various known results are briefly indicated. Further, we improve some recent results of ([4,8,9]).

*Key words:* Analytic, Univalent, Integral Operators.

2000 *Mathematics Subject Classification*: 30C45

## 1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. We denote by $S$ the subclass of $A$ consisting of functions which are also univalent in $U$. Ahlfors [1] and Becker [2] obtained the following univalence criterion.

**Lemma 1** *Let $c$ be a complex number, $|c| \leq 1,\ c \neq -1$.*

If $f(z) = z + a_2 z^2 + \ldots$ is a regular function in $U$ and

$$\left| c|z|^2 + (1 - |z|^2) \frac{z f''(z)}{f'(z)} \right| \leq 1, \tag{2}$$

for all $z \in U$ then the function $f(z)$ is regular and univalent in $U$. Further, Pescar [7] determine

**Lemma 2** *Let $\alpha$ be a complex number Re $\alpha > 0$ and $c$ a complex number, $|c| \leq 1,\ c \neq -1$. Suppose also that the function $f(z)$ given by (1) is analytic in $U$. If*

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \qquad (z \in U), \tag{3}$$

*then the function $F_\alpha(z)$ defined by $F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t) dt\right)^{1/\alpha} = z + \dots$ is analytic and univalent in $U$.*
*Another univalence condition given by Ozaki and Nunokawa [6] as follows*

**Lemma 3** *Let $f \in A$ satisfy the following inequality*

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq 1, \qquad (z \in U). \tag{4}$$

*Then $f$ is univalent in $U$. The following celebrated result is a basic tool in our investigation.*
**Schwarz Lemma [5]** *Let the analytic function $f(z)$ be regular in the disk $U_R = \{z \in C : |z| < R\}$, with $|f(z)| < M$, $M$ is fixed. If $f(0) = 0$ with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \qquad (z \in U_R), \tag{5}$$

*the equality (in (5) for $z \neq 0$) can hold only if $f(z) = e^{i\theta} \frac{Mz^m}{R^m}$, where $\theta$ is constant.*

## 2. Main result

In this section, we state the main univalent condition involving the general integral operator given by

$$G_{m,n,\alpha}(z)$$

$$= \left\{ [(m+n)(\alpha-1)+1] \int_0^z t^{m(\alpha-1)} \prod_{i=1}^n [g_i(t)]^{\alpha-1} \prod_{j=1}^m [h_j'(t)]^{\alpha-1} dt \right\}^{\frac{1}{(m+n)(\alpha-1)+1}} \tag{6}$$

and by specializing the parameters involved in our main result we obtain a number of known univalent conditions. It is worthy to note that our result not only generalizes but also improves some recent results of ([4,8,9]).

**Theorem 1** *Suppose that each of the functions $g_i(z) \in A$ $(i = 1, 2, \dots n)$ satisfies the inequality (4) and $h_j(z) \in A$, $(j = 1, 2, \dots m)$ satisfies the ineuqality $\left|\frac{h_j''(z)}{h_j'(z)}\right| \leq 1, \quad z \in U, \quad (j = 1, 2, \dots m)$. Also, let $M$ be a fixed positive real numbver and $c$ be a complex number. If $\alpha \in \left[\frac{(2M+1)n+m}{(2M+1)n+m+1}, \frac{(2M+1)n+m}{(2M+1)n+m-1}\right]$, $|c| \leq 1 - \frac{|\alpha-1|}{\alpha}\{(2M+1)n+m\}, \quad c \neq -1$ and $|g_i(z)| \leq M \quad (z \in U;\ i = 1, 2, \dots n)$, then the function $G_{m,n,\alpha}(z)$ defined by (6) is in the class $S$.*

*Proof.* From (6), we have

$$G_{m,n,\alpha}(z)$$

$$= \left\{ [(m+n)(\alpha-1)+1] \int_0^z t^{m(\alpha-1} \prod_{i=1}^{n} [g_i(t)]^{\alpha-1} \prod_{j=1}^{m} [h'_j(t)]^{\alpha-1} dt \right\}^{1/(m+n)(\alpha-1)+1}.$$

Let us consider the function

$$f(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{g_i(t)}{t} \right)^{\alpha-1} \prod_{j=1}^{m} \left( h'_j(t) \right)^{\alpha-1} dt. \tag{7}$$

The function $f$ is regular in $U$. From (7) we have

$$f'(z) = \prod_{i=1}^{n} \left( \frac{g_i(z)}{z} \right)^{\alpha-1} \prod_{j=1}^{m} \left( h'_j(z) \right)^{\alpha-1}. \tag{8}$$

Logarithmic differentiation of (8) yields $\frac{f''(z)}{f'(z)} = (\alpha-1) \left[ \sum_{i=1}^{n} \left( \frac{g'_i(z)}{g_i(z)} - \frac{1}{z} \right) + \sum_{j=1}^{m} \frac{h''_j(z)}{h'_j(z)} \right]$.
Now, for all $z \in U$, we have

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right|$$

$$= \left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \left( \frac{\alpha-1}{\alpha} \right) \left\{ \sum_{i=1}^{n} \left( \frac{zg'_i(z)}{g_i(z)} - 1 \right) + \sum_{j=1}^{m} \frac{zh''_j(z)}{h'_j(z)} \right\} \right|$$

$$\leq |c| + \frac{|\alpha-1|}{\alpha} \left\{ \sum_{i=1}^{n} \left( \left| \frac{z^2 g'_i(z)}{[g_i(z)]^2} \right| \left| \frac{g_i(z)}{z} \right| + 1 \right) + \sum_{j=1}^{m} \left( |z| \left| \frac{h''_j(z)}{h'_j(z)} \right| \right) \right\}. \tag{9}$$

We have $g_i(z) = 0$ at $z = 0$ and $|g_i(z)| \leq M$ and by the Schwarz-Lemma, we obtain $|g_i(z)| \leq M|z|, \quad (i = 1, 2, \ldots n)$, using (9) we have $\left| c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq |c| + \frac{|\alpha-1|}{\alpha} \{(2M+1)n+m\}$. For $\alpha \in \left[ \frac{(2M+1)n+m}{(2M+1)n+m+1}, \frac{(2M+1)n+m}{(2M+1)n+m-1} \right]$, we have $|c| \leq 1 - \frac{|\alpha-1|}{\alpha}\{(2M+1)n+m\} \leq 1$ and, hence, we obtain $\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \quad z \in U$. Finally, by applying Lemma 2, we conclude that the function $G_{m,n,\alpha}(z)$ defined by (6) is in the class $S$. This evidently completes the proof of Theorem 1.
If we put $m = 0$ the operator $G_{m,n,\alpha}(z)$ reduces to the operator

$$G_{n,\alpha}(z) = \left\{ [n(\alpha-1)+1] \int_0^z \prod_{i=1}^{n} [g_i(t)]^{\alpha-1} \right\}^{1/n(\alpha-1)+1} \tag{10}$$

studied by Breaz et al. [4], (See also [3]).
Thus from Theorem 1, we have

**Corollary 2** *Suppose that each of the functions* $g_i(z) \in A,\ (i = 1, 2, \ldots n)$ *satisfies the inequality* (4). *Also, let* $M$ *be a fixed positive real number and* $c$ *be a complex number.*
*If* $\alpha \in R \left(\alpha \in \left[\frac{(2M+1)n}{(2M+1)n+1},\ \frac{(2M+1)n}{(2M+1)n-1}\right]\right)$ *and* $|c| \leq 1 - \frac{|\alpha-1|}{\alpha}\{(2M+1)n\}$ *where* $|g_i(z)| \leq M\ (z \in U,\ i = 1, 2, \ldots n)$, *then the function* $G_{n,\alpha}(z)$ *defined by* (10) *is in the class* $S$.

**Remark 1** We easily see that the Corollary 2 improves the result of [4, Theorem 4] because our result holds for $\alpha \in \left[\frac{(2M+1)n}{(2M+1)n+1},\ \frac{(2M+1)n}{(2M+1)n-1}\right]$ whereas Breaz et al.[4] result holds only for $\alpha \in \left[1,\ \frac{(2M+1)n}{(2M+1)n-1}\right]$.
If we put $m = 0,\ M = n = 1$ in Theorem 1, we have

**Corollary 3** *Let the function* $g \in A$ *satisfy the inequality* (4). *Also, let*

$$\alpha \in R, \left(\alpha \in \left[\frac{3}{4},\ \frac{3}{2}\right]\right) \quad \text{and } c \in C.$$

*If* $|c| \leq 1 - \dfrac{3|\alpha-1|}{\alpha}$ *and* $|g(z)| \leq 1,\ (z \in U)$ *then the function* $G_\alpha(z)$ *defined by*

$$G_\alpha(z) = \left(\alpha \int_0^z [g(t)]^{\alpha-1} dt\right)^{1/\alpha}$$

*is in the class* $S$.

Remark 2 We easily see that Corollary 3 improves corresponding result due to Pescar [8], because our result holds for $\alpha \in \left[\frac{3}{4},\ \frac{3}{2}\right]$ whereas Pescar result holds only for $\alpha \in \left[1, \frac{3}{2}\right]$.
If we put $m = 0,\ n = 1$ in Theorem 1, we obtain the following result due to Pescar [9].

**Corollary 4** *Let the fucntion* $g \in A$ *satisfy* (4), $M$ *be a fixed positive real number and* $c$ *be a complex number. If*

$$\alpha \in \left[\frac{2M+1}{2M+2},\ \frac{2M+1}{2M}\right], |c| \leq 1 - \frac{|\alpha-1|}{\alpha}(2M+1),\ c \neq -1$$

*and* $|g(z)| \leq M,\ (z \in U)$, *then the function*

$$G_\alpha(z) = \left[\alpha \int_0^z [g(t)]^{\alpha-1} dt\right]^{1/\alpha}$$

*is in the class* $S$.
*If we put* $n = 0,\ m = 1$ *in Theorem 1. We thus obtain the following interesting consequence of Theorem 1.*

**Corollary 5** *Let* $f \in A$ *satisfy the inequality* $\left|\frac{f''(z)}{f'(z)}\right| \leq 1, \quad z \in U$. *Also let* $\alpha$ *be a real number,* $\alpha \geq \frac{1}{2}$, *and* $c$ *complex number,* $|c| \leq 1 - \frac{|\alpha-1|}{\alpha}$, *then the function*

$$G_\alpha(z) = \left[\alpha \int_0^z (tf'(t))^{\alpha-1} dt\right]^{1/\alpha}$$

*is in the class* $S$.

**Remark 3** We easily see that Corollary 5 improves Theorem 1 of [9] because our result holds for $\alpha \geq 1/2$ whereas Pescar [9] result holds only for $\alpha \geq 1$.
Note also, related work regarding the integral operator can also be found in ([10]-[12]).
**Acknowledgement:** The present investigation was supported by the University Grant Commission under grant No.F- 11-12/2006(SA-I). The third author was supported by MOHE: UKM-ST-06-FRGS0244-2010.

## References

[1] L.V. Ahlfors, *Sufficient conditions for quasiconformal extension*, Proc. Conf. Univ. of Maryland, Ann. of Math. Studies 79 (1973), 23-29.

[2] J. Becker, *Lownersche differential gleichung and Schlichteits-Kriterion*, Math. Ann., 202 (4) (1973), 321-335.

[3] D. Breaz and N. Breaz, *Univalence of an integral operator*, Mathematica (Cluj) 47(70) (2005), 35-38.

[4] D. Breaz, N. Breaz and H.M. Srivastava, *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett., 22 (2009), 41-44.

[5] O. Mayer, *The functions theory of one variable complex*, Bucuresti, 1981.

[6] S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc., 33(2) (1972), 392-394.

[7] V. Pescar, *A new generalization of Ahlfors's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc. (Second Series), 19 (1996), 53-54.

[8] V. Pescar, *On the univalence of some integral operators*, J. Indian Acad. Math., 27 (2005), 239-243.

[9] V. Pescar, *On the univalence of some integral operators*, General Mathematics, 14 (2) (2006), 77-84.

[10] D. Breaz, S. Owa and N. Breaz, *A new integral univalent operator*, Acta Universitatis Apulensis, No 16/2008, pp. 11-16.

[11] A.Mohammed, M. Darus and D. Breaz, *Some properties for certain integral operators*, Acta Universitatis Apulensis, No. 23/2010, pp. 79-89.

[12] A. Mohammed and M. Darus, *A new integral operator for meromorphic functions*, Acta Universitatis Apulensis, No. 24/2010, pp. 231-238.

S. Porwal, K.K. Dixit
Department of Mathematics
Janta College, Bakewar, Etawah, (U.P.) India-206124
E-mail: *saurabh.840@rediffmail.com*

M. Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
E-mail: *maslina@ukm.my*(Corresponding author)