Acta Universitatis Apulensis No. 25/2011
ISSN: 1582-5329 pp. 341-347

# ON LORENTZIAN SASAKIAN MANIFOLDS PROPER SEMI-INVARIANT SUBMANIFOLD IN LORENTZIAN $\alpha-$SASAKIAN MANIFOLDS

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Abstract. Recently, Yıldız and Murathan introduced the notion of Lorentzian $\alpha-$Sasakian manifolds and studied its properties. The aim of the present paper is to study the integrability of the distribution and give some results on proper semi-invariant submanifold of Lorentzian $\alpha-$ Sasakian manifold

*Keywords:* $\alpha-$Sasakian manifold, Lorentzian $\alpha-$Sasakian manifold, Proper semi-Invariant, Levi-Civita connection.

2000 *Mathematics Subject Classification.* 53B15, 53C35, 53C25

## 1. Introduction

The notion of Lorentzian $\alpha-$Sasakian manifolds was introduced by Yıldız and Murathan [8]. In [4], Matsumoto studied several properties of Lorentzian para contact structure. In this paper, we show that integrability condition of the distribution in proper semi-invariant submanifold of Lorentzian $\alpha-$Sasakian Manifolds. Also we give some intereting results concerning distributions.

## 2. Preliminaries

Let $\widetilde{M}$ be an $(2n+1)-$dimensional differentiable manifold of differentiability class $C^\infty$ endowed with a $C^\infty$-vector valued linear function $\phi$, a $C^\infty$ vector field $\xi$, 1-form $\eta$ and Lorentzian metric $g$ of type $(0,2)$ such that for each $p \in \widetilde{M}$, the tensor $g_p : T_p\widetilde{M} \times T_p\widetilde{M} \longrightarrow R$ is a non-degenerate inner product of signature $(-,+,+,...,+)$ where $T_p\widetilde{M}$ denotes the tangent vector space of $\widetilde{M}$ at $p$ and $R$ is the real number space, which satisfies

$$\widetilde{\pi}(\xi) = -1, \tag{0.1}$$

$$\phi^2 = I + \eta \otimes \xi \tag{0.2}$$

$$\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X,Y) + \eta(X)\eta(Y), \tag{0.3}$$

$$\widetilde{g}(X, \xi) = \eta(X), \tag{0.4}$$

for all vector fields $X,Y$ tangent to $\widetilde{M}$. Such structure $(\phi,\xi,\eta,\widetilde{g})$ is termed as Lorentzian para contact [4].

In a Lorentzian para contact structure the following holds:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{0.5}$$

A Lorentzian para contact manifold is called Lorentzian $\alpha-$Sasakian ($L\alpha-$Sasakian) manifold if

$$(\widetilde{\nabla}_X \phi)(Y) = \alpha(\widetilde{g}(X,Y)\xi + \eta(Y)X) \tag{0.6}$$

Also a Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$ satisfies

$$\widetilde{\nabla}_X \xi = \phi X, \tag{0.7}$$

$$(\widetilde{\nabla}_X \eta)(Y) = \alpha\widetilde{g}(X,\phi Y) \tag{0.8}$$

where $\widetilde{\nabla}$ denotes the operator of covarient differentiation with respect to the Lorentzian metric $\widetilde{g}$ and $\alpha$ is constant.

Let us put

$$\Phi(X,Y) = \alpha\widetilde{g}(X,\phi Y) \tag{0.9}$$

then the tensor field $\Phi$ is symmetric $(0,2)$-tensor field. Thus, we have

$$\Phi(X,Y) = \Phi(Y,X) \tag{0.10}$$

$$\Phi(X,Y) = (\widetilde{\nabla}_X \eta)(Y) \tag{0.11}$$

The submanifold $M$ of the Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$ is said to be

semi-invariant if it is endowed with the pair of orthogonal distribution $(D,D^{\perp})$ satisfying the conditions

$(i)$ $TM = D \oplus D^{\perp} \oplus (\xi)$,

$(ii)$ The distribution $D$ is invariant under $\phi$, that is,

$$\phi D_x = D_x, \text{ for each } x \in M$$

$(iii)$ The distribution $D^{\perp}$ is anti-invariant under $\phi$,that is,

$$\phi D_x^{\perp} \subset T_x M^{\perp}, \text{ for each } x \in M$$

We sat that $M$ is *a proper semi-invariant submanifold* if both the distribution $D$ and $D^{\perp}$ are non-zero. For any vector bundle $H$ on $M$ [resp., $\widetilde{M}$], we denote by $\Gamma(H)$ the module of all differentiable section of $H$ neighbourhood co-ordinate on $M$ [resp., $\widetilde{M}$].

The projection morphisms of $TM$ to $D$ and $D^{\perp}$ are denoted by $P$ and $Q$ respectively. Then we have

$$X = PX + QX + \eta(X)\xi \tag{0.12}$$

and

$$\phi N = BN + CN \tag{0.13}$$

where $BN$ and $CN$ denote the tangential and normal component of $\phi N$, respectively.

The equations of Gauss and Weingarten for the immersion of $M$ in $\widetilde{M}$ are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{0.14}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{0.15}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, where $\nabla$ is the Levi-Civita connection on $M$, $\nabla^{\perp}$ is the linear connection induced by $\widetilde{\nabla}$ on the normal bundle $TM^{\perp}$, $h$ is the second fundamental form of $M$ and $A_N$ is the fundamental tensor of Weingarten with respect to the normal section $N$. By using (0.14) and (0.15), we get

$$g(h(X,Y),N) = g(A_N X, Y) \tag{0.16}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

A submanifold $M$ is said to be

$(i)$ totally geodesic in $\widetilde{M}$ if

$$h = 0 \text{ or equivalently } A_N = 0 \tag{0.17}$$

for any $N \in TM^{\perp}$.

$(ii)$ totally umbilical if

$$h(X,Y) = g(X,Y)F \tag{0.18}$$

where $F$ is mean curvature vector.

$(iii)$ minimal in $\widetilde{M}$ if the mean curvature vector $F$ vanishes [2].

## 3. Basic Lemmas

We define

$$k(X,Y) = \nabla_X \phi PY - A_{\phi QY} X \tag{0.19}$$

for $X, Y \in \Gamma(TM)$. Then we have the following lemma:

Let $M$ be a semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then we have

$$P(k(X,Y)) = \alpha(g(X,Y)P\xi - \eta(Y)PX)) \tag{0.20}$$

$$Q(k(X,Y)) = Bh(X,Y) + \alpha(g(X,Y)Q\xi - \eta(Y)QX)) \tag{0.21}$$

$$h(X, \phi PY) + \nabla^{\perp}_{X} \phi QY = \phi Q \nabla_X Y + Ch(X, Y) \tag{0.22}$$

$$\eta(k(X, Y)) = 0 \tag{0.23}$$

for all $X, Y \in \Gamma(TM)$.

*Proof.* Appliying (0.12), (0.13), (0.14) and (0.15) in (0.6), we obtain (0.20), (0.21), (0.22) and (0.23) respectively. □

Let $M$ be a totally umbilical semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then we have

$$\nabla_X \xi = 0, \; h(X, \xi) = \eta(X) \;\; \text{for any} \;\; \in \Gamma(D^{\perp}) \tag{0.24}$$

$$\nabla_Y \xi = \phi Y, \; h(Y, \xi) = \eta(Y) \;\; \text{for any} \;\; \in \Gamma(D) \tag{0.25}$$

$$\nabla_\xi \xi = 0, \; h(\xi, \xi) = -1 \tag{0.26}$$

*Proof.* In consequence of (0.7) and (0.12), we have

$$\phi PX + \phi QX = \nabla_X \xi + h(X, \xi) \tag{0.27}$$

Thus, (0.24) – (0.26) follows from (0.27) and (0.1). □

Let $M$ be a semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then we have

$$\nabla_\xi W \in \Gamma(D^{\perp}), \; \text{for} \; W \in \Gamma(D^{\perp})$$

and

$$\nabla_\xi V \in \Gamma(D), \; \text{for} \; V \in \Gamma(D)$$

*Proof.* The proof is trivial. □

Let $M$ be a semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then we get

$$[X, \xi] \in \Gamma(D^{\perp}), \; \text{for} \; X \in \Gamma(D^{\perp}) \tag{0.28}$$

and

$$[Y, \xi] \in \Gamma(D), \; \text{for} \; Y \in \Gamma(D) \tag{0.29}$$

*Proof.* ¿From lemma 3.3, The proof is obvious. □

## 4. Integrability of distribution on a proper semi-invariant submanifold in a Lorentzian $\alpha-$Sasakian manifold

Let $M$ be a proper semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then the distribution $D^{\perp}$ is intagrable.

*Proof.* Using (0.7), we get

$$\begin{aligned} g([X,Y],\xi) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \\ &= -g(Y, \nabla_X \xi) + g(X, \nabla_Y \xi) \\ &= 0 \end{aligned}$$

where $\nabla_X \xi = 0$ for all $X, Y \in \Gamma(D^{\perp})$. □

Let $M$ be a proper semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then we have

$$A_{\phi X} Y - A_{\phi Y} X = 0 \tag{0.30}$$

for all $X, Y \in \Gamma(D^{\perp})$.

*Proof.* Let $X, Y \in \Gamma(D^{\perp})$. Then $\phi X, \phi Y \in \Gamma(TM^{\perp})$. By using (0.4), (0.15) and (0.7) we have

$$\eta(A_{\phi X} Y) = -g(\nabla_Y \phi X, \xi) = g(\phi X, \nabla_Y \xi) = g(\phi X, \phi Y) = g(X, Y) \tag{0.31}$$

Similarly,

$$\eta(A_{\phi Y} X) = -g(\nabla_X \phi Y, \xi) = g(\phi Y, \nabla_X \xi) = g(\phi Y, \phi X) = g(Y, X) \tag{0.32}$$

for all $X, Y \in \Gamma(D^{\perp})$. From (0.31) and (0.32), we obtain (0.30). □

Let $M$ be a totally umbilical proper semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then

$$\nabla_X \phi Y - \nabla_Y \phi X = \eta(X) Y - \eta(Y) X \tag{0.33}$$

*Proof.* ¿From (0.14),, we get

$$\widetilde{\nabla}_X \phi Y - \widetilde{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + h(Y, \phi X) - h(X, \phi Y). \tag{0.34}$$

Then using (0.6) in (0.34), (0.33) is obtained. □

Let $M$ be a totally umbilical proper semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$. Then we have

$$\phi X = \nabla_X \xi + Fg(X, \xi), \ \xi \in TM \tag{0.35}$$

$$\phi X = \nabla_X \xi, \ \xi \in TM^{\perp} \tag{0.36}$$

$$\phi X = -A_{\xi} X + \nabla^{\perp}_X \xi \tag{0.37}$$

$$\frac{1}{\alpha}\Phi(X, Y) = -g(X, Y)\eta(F) \tag{0.38}$$

*Proof.* ¿From (0.7) and (0.14), we get (0.35). Also, using (0.7) and (0.15) we obtain (0.36) and (0.37). Again, taking scalar product of equation (0.37) with $Y$ we get (0.38). □

Let $M$ be a totally umbilical proper semi-invariant submanifold in Lorentzian $\alpha-$Sasakian manifold $\widetilde{M}$ such that the structure vector field $\xi$ $\xi$ is tangent to $M$. Then if $M$ is totally geodesic if and only if it is minimal.

*Proof.* Let $M$ be totally geodesic. Using (0.3), (0.5) and (0.35) in (0.18), we get

$$0 = h(\xi, \xi) = g(\xi, \xi)F = -F. \tag{0.39}$$

which proves the assertion of proposition. □

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