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UNSTEADY FLOW OF A DUSTY FLUID THROUGH A CHANNEL HAVING TRIANGULAR CROSS-SECTION IN FRENET FRAME FIELD SYSTEM

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Abstract. An Unsteady motion of a viscous liquid with uniform distribution of dust particles under the influence of time dependent pressure gradient through a channel having triangular cross-section has been considered. The intrinsic decomposition of flow equations are carried out in Frenet frame field System. The governing equations of the flow are solved using Laplace transform and variable separable method. The skin friction at the boundaries are calculated. Finally the conclusions are given on basis of the velocity profiles drawn for different values of time t and number density N.

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1. Introduction

In recent years many researchers in fluid dynamics has been diverted towards study of the influence of dust particles on the motion of fluids. The presence of dust particles in fluids has certain influence on the motion of the fluids and such situation arise for instance, in the movement of dust-laden air in, combustion, polymer technology, and electrostatic precipitation. Other important application involving dust particles are fluidization, purification of crude oil, centrifugal separation of matter from fluid, petroleum industry and in the engineering problems concerned with atmospheric fallout, dust collection, nuclear reactor cooling, powder technology, performance of solid fuel rocket nozzles and paint spraying etc.

Saffman [16] investigated the effect of stability of the laminar flow of a dusty gas in which the dust particles are uniformly distributed. Further, Michael and Miller [13] investigated the motion of dusty gas with uniform distribution of the dust particles occupied in the semi-infinite space above a rigid plane boundary.

Chernyshov [6] obtained the exact solution for nonsteady two-dimensional problem of the motion of an incompressible viscous fluid in a rigid tube whose cross-section is a regular triangle. Rukmangadachari [15] has studied the solutions of dusty viscous flow through a cylinder of triangular cross-section. N.C.Ghosh, B.C.Ghosh & R.S.R.Gorla [10] have studied unsteady motion of a dusty viscoelastic Maxwell-type conducting fluid under arbitrary pressure gradient through a long uniform tube of rectangular cross section. Attia [1] studied the unsteady Hartmann flow of a dusty viscous incompressible electrically conducting fluid under the influence of an exponentially decreasing pressure gradient is studied without neglecting the ion slip. Chamkha [7] has obtained the analytical solution for unsteady flow of an electrically conducting dusty-gas in a channel due to an oscillating pressure gradient. Balasubramanian & Chen [5] have given mathematical model for unsteady MHD flow and heat transfer of dusty fluid between two cylinders with variable physical properties.

Frenet frames are a central construction in modern differential geometry, in which structure is described with respect to an object of interest rather than with respect to external coordinate systems. Some researchers like Kanwal [12], Truesdell [17], Indrasena [11], Purushotham [14], Bagewadi and Gireesha [2], [3] have applied differential geometry techniques to study the fluid flow. Further, the authors [2], [3] have studied two-dimensional dusty fluid flow in Frenet frame field system, which is one of the moving frame. Recently the authors [8], [9] have studied the flow of unsteady dusty fluid in different regions under varying time dependent pressure gradients. The present work deals with the study of flow of an unsteady dusty fluid through a channel having triangular cross-section under the influence of time dependent pressure gradient in frenet frame field system. Initially it is assumed that the fluid and dust particles are to be at rest. The analytical expressions are obtained for velocities of both fluid and dust particles using Laplace transform technique and variable separable method. Further the skin friction at the boundary is calculated. The velocity profiles of both fluid and dust phase are shown graphically for different time t and number density N.

2.Equations of Motion

The equations of motion of unsteady viscous incompressible fluid with uniform distribution of dust particles are given by [16]:

$$
\begin{aligned}
&For\ \ fluid\ \ phase \\
\nabla \cdot \overrightarrow{u} &= 0, \qquad \text{(Continuity)} \qquad (1) \\
\frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u} \cdot \nabla)\overrightarrow{u} &= -\frac{1}{\rho}\nabla p + \nu \nabla^2 \overrightarrow{u} \qquad (2)
\end{aligned}
$$

$$+ \quad \frac{kN}{\rho}(\overrightarrow{v} - \overrightarrow{u}) \quad \text{(Linear Momentum)}$$

For dust phase

$$\nabla \cdot \overrightarrow{v} = 0, \qquad \text{(Continuity)} \tag{3}$$

$$\frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v} \cdot \nabla)\overrightarrow{v} = \frac{k}{m}(\overrightarrow{u} - \overrightarrow{v}) \qquad \text{(Linear Momentum)} \tag{4}$$

We have following nomenclature: We have following nomenclature:

$\overrightarrow{u}-$velocity of the fluid phase, $\overrightarrow{v}-$velocity of dust phase, $\rho-$density of the gas, $p-$pressure of the fluid, $N-$number density of dust particles, $\nu-$kinematic viscosity, $k = 6\pi a\mu-$Stoke's resistance (drag coefficient), $a-$spherical radius of dust particle, $m-$mass of the dust particle, $\mu-$the co-efficient of viscosity of fluid particles, $t-$time.

3.Frenet Frame Field System

Let $\overrightarrow{s}, \overrightarrow{n}, \overrightarrow{b}$ be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruences formed by fluid phase velocity and dusty phase velocity lines respectively as shown in the figure-1.

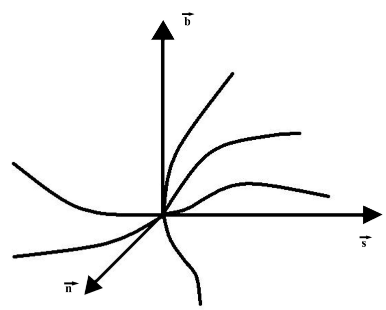

Figure-1: Frenet Frame Field System

Geometrical relations are given by Frenet formulae [4]

$$
\begin{aligned}
&i) \quad \frac{\partial \overrightarrow{s}}{\partial s} = k_s \overrightarrow{n}, \ \frac{\partial \overrightarrow{n}}{\partial s} = \tau_s \overrightarrow{b} - k_s \overrightarrow{s}, \ \frac{\partial \overrightarrow{b}}{\partial s} = -\tau_s \overrightarrow{n} \\
&ii) \quad \frac{\partial \overrightarrow{n}}{\partial n} = k'_n \overrightarrow{s}, \ \frac{\partial \overrightarrow{b}}{\partial n} = -\sigma'_n \overrightarrow{s}, \ \frac{\partial \overrightarrow{s}}{\partial n} = \sigma'_n \overrightarrow{b} - k'_n \overrightarrow{n} \\
&iii) \quad \frac{\partial \overrightarrow{b}}{\partial b} = k''_b \overrightarrow{s}, \ \frac{\partial \overrightarrow{n}}{\partial b} = -\sigma''_b \overrightarrow{s}, \ \frac{\partial \overrightarrow{s}}{\partial b} = \sigma''_b \overrightarrow{n} - k''_b \overrightarrow{b} \\
&iv) \quad \nabla.\overrightarrow{s} = \theta_{ns} + \theta_{bs}; \ \nabla.\overrightarrow{n} = \theta_{bn} - k_s; \ \nabla.\overrightarrow{b} = \theta_{nb}
\end{aligned} \tag{5}
$$

where $\partial/\partial s$, $\partial/\partial n$ and $\partial/\partial b$ are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, tangential, principal normal and binormal.

The functions (k_s, k'_n, k''_b) and $(\tau_s, \sigma'_n, \sigma''_b)$ are the curvatures and torsions of the above curves and θ_{ns} and θ_{bs} are normal deformations of these spatial curves along their principal normal and binormal respectively.

4.Formulation and Solution of the Problem

The present discussion considers a laminar flow of viscous incompressible, dusty fluid through a rectangular channel. The flow is due to the influence of constant pressure gradient varying with time. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size. The number density of the dust particles is taken as a constant throughout the flow. As Figure-2 shows, the axis of the channel is along binormal direction and the velocity components of both fluid and dust particles are respectively given by:

$$\overrightarrow{u} = u_b(s,n,t)\overrightarrow{b}, \quad \overrightarrow{v} = v_b(s,n,t)\overrightarrow{b} \tag{6}$$

where (u_s, u_n, u_b) and (v_s, v_n, v_b) are velocity components of fluid and dust particles respectively.

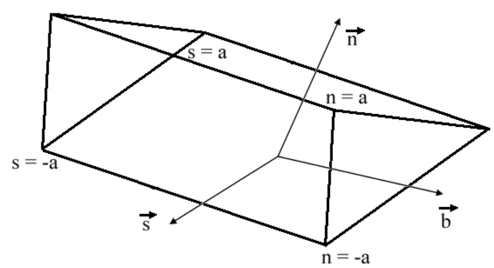

Figure 2: The Geometry of the Problem.

By virtue of system of equations (5) the intrinsic decomposition of equations (2) and (4) using equation (6) one can get,

$$\begin{aligned}
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial s} + \nu\left(\tau_s k_s u_b - 2\sigma'_n \frac{\partial u_b}{\partial n}\right) \\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial n} + \nu\left(\sigma'_n k'_n u_b - 2\tau_s \frac{\partial u_b}{\partial s}\right) \\
\frac{\partial u_b}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial b} + \nu\left[\frac{\partial^2 u_b}{\partial s^2} + \frac{\partial^2 u_b}{\partial n^2} - C_r u_b\right] + \frac{kN}{\rho}(v_b - u_b) \qquad (7) \\
\frac{\partial v_b}{\partial t} &= \frac{k}{m}(u_b - v_b) \qquad (8) \\
v_b^2 k''_b &= 0 \qquad (9)
\end{aligned}$$

where $C_r = (\tau_s^2 + \sigma_n'^2 + k''^2{}_b)$ is called curvature number [3].

From equation (9) we see that $v_b^2 k_b'' = 0$ which implies either $v_b = 0$ or $k_b'' = 0$. The choice $v_b = 0$ is impossible, since if it happens then $u_b = 0$, which shows that the flow doesn't exist. Hence $k_b'' = 0$, it suggests that the curvature of the streamline along binormal direction is zero. Thus no radial flow exists.

Since we have assumed that the constant pressure gradient to be imposed on the system for $t > 0$, we can write

$$-\frac{1}{\rho}\frac{\partial p}{\partial b} = c_o \ \text{(a constant)} \tag{10}$$

Equation (7) and (8) are to be solved subject to the initial and boundary conditions:

$$\left\{\begin{array}{lll} \text{Initial condition;} & \text{at } t = 0;\ u_b = 0, v_b = 0 \\ \text{Boundary condition;} & \text{for } t > 0; u_b = a_1\ e^{iw_1 t} + a_2\ e^{iw_2 t} \text{ at } \ s = \pm a, \\ & \text{and} \qquad u_b = 0 \ \text{ at } \ n = a \ \ \& \ \ n = -a \end{array}\right\} \tag{11}$$

Let U_b and V_b are given by

$$U_b = \int_0^\infty e^{-xt} u_b dt \quad \text{and} \quad V_b = \int_0^\infty e^{-xt} v_b dt \tag{12}$$

denote the Laplace transforms of u_b and v_b respectively.

Using relations (10) and (12) in equations (7), (8) and (11) one can obtain the following:

$$xU_b = \frac{c_o}{x} + \nu\left(\frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - C_r U_b\right) + \frac{l}{\tau}(V_b - U_b) \tag{13}$$

$$V_b = \frac{U_b}{(1 + x\tau)} \tag{14}$$

$$U_b = \frac{a_1}{x - iw_1} + \frac{a_2}{x - iw_2} \quad \text{at} \quad s = a \ \ \& \ \ s = -a \tag{15}$$
$$\text{and} \ \ U_b = 0 \ \ \text{at} \ \ n = a \ \ \& \ \ n = -a,$$

where $l = \frac{mN}{\rho}$ and $\tau = \frac{m}{k}$.

From equations (13) and (14) we obtain, the following equation

$$\frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - Q^2 U_b + R = 0 \tag{16}$$

where

$$Q^2 = \left(C_r + \frac{x}{\nu} + \frac{xl}{\nu(1 + x\tau)}\right) \quad \text{and} \quad \mathrm{R} = \frac{\mathrm{c}_0}{\nu\mathrm{x}}$$

To solve equation (13) we assume the solution in the following form [18]

$$U_b(s,n) = w_1(s,n) + w_2(s) \tag{17}$$

Substitution of $U_b(s,n)$ in equation (13) yields

$$\frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_2}{\partial s^2} + \frac{\partial^2 w_1}{\partial n^2} - Q^2(w_1 + w_2) + R = 0$$

so that if w_2 satisfies

$$\frac{\partial^2 w_2}{\partial s^2} - Q^2 w_2 + R = 0$$

then

$$\frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_1}{\partial n^2} - Q^2 w_1 = 0 \tag{18}$$

In similar manner if $U_b(s,n)$ is inserted in no slip boundary conditions, one can obtain

$$\left\{ \begin{array}{l} U_b(a,n) = w_1(a,n) + w_2(a) = \frac{a_1}{x-iw_1} + \frac{a_2}{x+iw_2}, \\ U_b(-a,n) = w_1(-a,n) + w_2(-a) = \frac{a_1}{x-iw_1} + \frac{a_2}{x+iw_2}, \\ U_b(s,a) = w_1(s,a) + w_2(s) = 0, \;\; U_b(s,-a) = w_1(s,-a) + w_2(s) = 0 \end{array} \right\}$$

By solving the problem

$$\frac{\partial^2 w_2}{\partial s^2} - Q^2 w_2 + R = 0, \;\; w_2(a) = \frac{a_1}{x-iw_1} + \frac{a_2}{x+iw_2}, \;\; w_2(-a) = \frac{a_1}{x-iw_1} + \frac{a_2}{x+iw_2}$$

we obtain the solution in the form

$$w_2(s) = \frac{\cosh(Qs)}{\cosh(Qa)} \left(\frac{a_1}{x-iw_1} + \frac{a_2}{x+iw_2} \right) - \frac{R}{Q^2} \left(\frac{\cosh(Qs) - \cosh(Qa)}{\cosh(Qa)} \right) \tag{19}$$

Using variable separable method, the solution of the problem (18) with the conditions

$$w_1(a,n) = 0, \quad w_1(-a,n) = 0, \quad w_1(s,a) = -w_2(s), \quad w_1(s,-a) = -w_2(s)$$

is obtained in the form

$$\begin{aligned} w_1(s,n) &= -\frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \sin\left(\frac{r_1 \pi}{a} s\right) \left(\frac{a_1}{x-iw_1} + \frac{a_2}{x+iw_2} \right) \\ &\times \frac{\cosh(An)\left[1 - (-1)^{r_1} \cosh(Qa)\right]}{A^2 \cosh(Qa) \cosh(Aa)} \\ &+ \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \sin\left(\frac{r_1 \pi}{a} s\right) \frac{R}{Q^2} \frac{\left[1 - (-1)^{r_1} \cosh(Qa)\right]}{A^2 \cosh(Qa) \cosh(Aa)} \cosh(An) \\ &- \frac{2}{\pi} \sum_{r_1=0}^{\infty} \frac{1}{r_1} \sin\left(\frac{r_1 \pi}{a} s\right) \frac{R}{Q^2} \frac{\left[1 - (-1)^{r_1} \cosh(An)\right]}{\cosh(Aa)} \end{aligned} \tag{20}$$

where $A = \sqrt{\frac{Q^2a^2+r_1^2\pi^2}{a^2}}$

Now by substituting (19) and (20) in (17) we have

$$\begin{aligned} U_b(s,n) = & - \frac{2\pi}{a^2}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{a}s\right)\left(\frac{a_1}{x-iw_1}+\frac{a_2}{x+iw_2}\right) \\ & \times \frac{\cosh(An)\left[1-(-1)^{r_1}\cosh(Qa)\right]}{A^2\cosh(Qa)\cosh(Aa)} \\ & + \frac{2\pi}{a^2}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{a}s\right)\frac{R}{Q^2}\frac{\cosh(An)\left[1-(-1)^{r_1}\cosh(Qa)\right]}{A^2\cosh(Qa)\cosh(Aa)} \\ & - \frac{2}{\pi}\sum_{r_1=0}^{\infty}\frac{1}{r_1}\sin\left(\frac{r_1\pi}{a}s\right)\frac{R}{Q^2}\frac{\left[1-(-1)^{r_1}\cosh(An)\right]}{\cosh(Aa)} \\ & + \frac{\cosh(Qs)}{\cosh(Qa)}\left(\frac{a_1}{x-iw_1}+\frac{a_2}{x+iw_2}\right) \\ & - \frac{R}{Q^2}\left(\frac{\cosh(Qs)-\cosh(Qa)}{\cosh(Qa)}\right) \end{aligned}$$

Using U_b in equation (14) one can see that

$$\begin{aligned} V_b(s,n) = & - \frac{2\pi}{a^2}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{a}s\right)\left(\frac{a_1}{x-iw_1}+\frac{a_2}{x+iw_2}\right) \\ & \times \frac{\left[1-(-1)^{r_1}\cosh(Qa)\right]}{A^2\cosh(Qa)\cosh(Aa)}\frac{\cosh(An)}{1+x\tau} \\ & + \frac{2\pi}{a^2}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1\pi}{a}s\right)\frac{R}{Q^2}\frac{\left[1-(-1)^{r_1}\cosh(Qa)\right]}{A^2\cosh(Qa)\cosh(Aa)}\frac{\cosh(An)}{1+x\tau} \\ & - \frac{2}{\pi}\sum_{r_1=0}^{\infty}\frac{1}{r_1}\sin\left(\frac{r_1\pi}{a}s\right)\frac{R}{Q^2}\frac{\left[1-(-1)^{r_1}\cosh(An)\right]}{\cosh(Aa)(1+x\tau)} \\ & + \frac{\cosh(Qs)}{\cosh(Qa)(1+x\tau)}\left(\frac{a_1}{x-iw_1}+\frac{a_2}{x+iw_2}\right) \\ & - \frac{R}{Q^2}\left(\frac{\cosh(Qs)-\cosh(Qa)}{\cosh(Qa)(1+x\tau)}\right) \end{aligned}$$

By taking inverse Laplace transformation to U_b and V_b, we obtain u_b and v_b as follows:

$$u_b(s,n,t) = \frac{-2\pi}{a^2}\sum_{r_1=0}^{\infty} r_1\sin\left[\frac{r_1\pi s}{a}\right]\left\{a_1\left[\frac{(\phi_1\cos(w_1t)-\phi_2\sin(w_1t))}{(y_3^2+z_3^2)(c_1^2+c_2^2)(c_3^2+C_4^2)}\right.\right.$$

$$
\begin{aligned}
+ & \quad \frac{i(\phi_1 \sin(w_1 t) + \phi_2 cos w_1 t)}{(y_3^2 + z_3^2)(c_1^2 + c_2^2)(c_3^2 + C_4^2)}\Bigg] - a_1(-1)^{r_1}\left[\frac{(\phi_3 \cos(w_1 t) - \phi_4 \sin(w_1 t))}{(y_3^2 + z_3^2)(c_3^2 + C_4^2)}\right. \\
+ & \quad \frac{i(\phi_3 \sin(w_1 t) + \phi_4 \cos(w_1 t))}{(y_3^2 + z_3^2)(c_3^2 + C_4^2)}\Bigg] + \frac{a_1[1 - (-1)^{r_1}\cos(Q_1 a)]}{\cos(Q_1 a)(x_1 - x_2)} \\
\times & \quad \left[\frac{e^{x_1 t}(x_1 + i w_1)}{x_1^2 + w_1^2} - \frac{e^{x_2 t}(x_2 + i w_1)}{x_2^2 + w_1^2}\right] \\
+ & \quad \frac{a_1 \nu \pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\frac{\cosh(A_1 n)}{A_1^2\cosh(A_1 a)} \\
\times & \quad \left[\frac{e^{x_3 t}(x_3 + i w_1)(1 + x_3\tau)^2)}{(x_3^2 + w_1^2)[(l + (x_3\tau + 1)^2)]} + \frac{e^{x_4 t}(x_4 + i w_1)(1 + x_4\tau)^2)}{(x_4^2 + w_1^2)[(l + (x_4\tau + 1)^2)]}\right] \\
- & \quad \frac{4\nu a_1}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[1 - (-1)^{r_2}\cos(Q_2 a)]}{\cos(Q_2 a)}\cos\left[\frac{(2r_2+1)\pi n}{2a}\right] \\
\times & \quad \left[\frac{e^{x_5 t}(x_5 + i w_1)(1 + x_5\tau)^2}{(x_5^2 + w_1^2)[(l + (x_5\tau + 1)^2)]} + \frac{e^{x_6 t}(x_6 + i w_1)(1 + x_6\tau)^2}{(x_6^2 + w_1^2)[(l + (x_6\tau + 1)^2)]}\right] \\
+ & \quad a_2\left[\frac{(\phi_5)\cos(w_2 t) - \phi_6 \sin(w_2 t)}{(y_7^2 + z_7^2)(c_5^2 + c_6^2)(c_7^2 + C_8^2)} + \frac{i(\phi_6)\cos(w_2 t) - \phi_5 \sin(w_2 t)}{(y_7^2 + z_7^2)(c_5^2 + c_6^2)(c_7^2 + C_8^2)}\right] \\
- & \quad a_2(-1)^{r_1}\left[\frac{(\phi_7) cos w_2 t + \phi_8 \sin(w_2 t)}{(y_7^2 + z_7^2)(c_7^2 + C_8^2)} + \frac{i(\phi_8)\cos(w_2 t) - \phi_7 \sin(w_2 t)}{(y_7^2 + z_7^2)(c_7^2 + C_8^2)}\right] \\
+ & \quad \frac{a_2[1 - (-1)^{r_1}\cos(Q_1 a)]}{\cos(Q_1 a)(x_1 - x_2)}\left[\frac{e^{x_1 t}(x_1 - i w_2)}{x_1^2 + w_2^2} - \frac{e^{x_2 t}(x_2 - i w_2)}{x_2^2 + w_2^2}\right] \\
+ & \quad \frac{a_2 \nu \pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\frac{\cosh(A_1 n)}{A_1^2\cosh(A_1 a)}\left[\frac{e^{x_3 t}(x_3 - i w_2)(1 + x_3\tau)^2)}{(x_3^2 + w_2^2)[(l + (x_3\tau + 1)^2)]}\right. \\
+ & \quad \frac{e^{x_4 t}(x_4 - i w_2)(1 + x_4\tau)^2)}{(x_4^2 + w_2^2)[(l + (x_4\tau + 1)^2)]}\Bigg] - \frac{4\nu a_2}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[1 - (-1)^{r_1}\cos(Q_2 a)]}{\cos(Q_2 a)} \\
\times & \quad \left[\frac{e^{x_5 t}(x_5 - i w_2)(1 + x_5\tau)^2}{(x_5^2 + w_2^2)[(l + (x_5\tau + 1)^2)]} + \frac{e^{x_6 t}(x_6 - i w_2)(1 + x_6\tau)^2}{(x_6^2 + w_2^2)[(l + (x_6\tau + 1)^2)]}\right] \\
\times & \quad \cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\Bigg\} \\
+ & \quad \frac{2\pi}{a^2}\sum_{r_1=0}^{\infty}\sin\left(\frac{r_1 \pi s}{a}\right)\left\{\frac{c_0\,[1 - (-1)^{r_1}\cosh(Xa)]\cosh(Yn)}{\nu X^2 Y^2\cosh(Xa)\cosh(Ya)}\right. \\
+ & \quad \frac{c_0}{\nu}\frac{[1 - (-1)^{r_1}]}{Q_1^2\cosh(Q_1 a)}\frac{\cosh(Q_1 n)}{(x_7 - x_8)}\left[\frac{e^{x_7 t}}{x_7} - \frac{e^{x_8 t}}{x_8}\right] + \frac{c_0}{\nu}\frac{[1 - (-1)^{r_1}\cos(Q_1 a)]}{Q_1^2\cosh(Q_1 a)(x_1 - x_2)}
\end{aligned}
$$

$$
\begin{aligned}
\times &\quad \left[\frac{e^{x_1 t}}{x_1} - \frac{e^{x_2 t}}{x_2}\right] - \frac{4c_o}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{\cosh(A_1 n)}{A_1^2\cosh(A_1 a)}\left[\frac{e^{x_3 t}(1+x_3\tau)^2}{x_3[l+(1+x_3\tau)^2]}\right.\\
+ &\quad \left.\frac{e^{x_4 t}(1+x_4\tau)^2}{x_4[l+(1+x_4\tau)^2]}\right] - \frac{4c_0}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[1-(-1)^{r_1}\cos(Q_2 a)]}{Q_2^2\cos(Q_2 a)}\\
\times &\quad \cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5 t}(1+x_5\tau)^2}{x_5[l+(1+x_5\tau)^2]} + \frac{e^{x_6 t}(1+x_6\tau)^2}{x_6[l+(1+x_6\tau)^2]}\right]\Bigg\}\\
- &\quad \frac{2}{\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)_1^r]}{r_1}\sin\left[\frac{r_1\pi s}{a}\right]\left\{\frac{c_0}{\nu X^2}\frac{\cosh(Yn)}{\cosh(Ya)}\right.\\
\times &\quad +\frac{c_0}{\nu}\frac{\cosh(Q_1 n)}{\cosh(Q_1 a)(x_7-x_8)}\left[\frac{e^{x_7 t}}{x_7} - \frac{e^{x_8 t}}{x_8}\right] + \frac{c_0\pi}{a^2}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}(2r_2+1)}{Q^2}\\
\times &\quad \cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5 t}(1+x_5\tau)^2}{x_5[l+(1+x_5\tau)^2]} + \frac{e^{x_6 t}(1+x_6\tau)^2}{x_6[l+(1+x_6\tau)^2]}\right]\Bigg\}\\
+ &\quad a_1\left[\frac{(\phi_9\cos w_1 t - \phi_{10}\sin w_1 t)}{(c_1^2+c_2^2)} + \frac{i(\phi_{10}\cos w_1 t + \phi_9\sin w_1 t)}{(c_1^2+c_2^2)}\right]\\
+ &\quad \frac{a_1\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cosh\left[\frac{(2r_2+1)\pi s}{2a}\right]\\
\times &\quad \left[\frac{e^{x_3 t}(x_3+iw_1)(1+x_3\tau)^2)}{(x_3^2+w_1^2)[(l+(x_3\tau+1)^2)]} + \frac{e^{x_4 t}(x_4+iw_1)(1+x_4\tau)^2)}{(x_4^2+w_1^2)[(l+(x_4\tau+1)^2)]}\right]\\
+ &\quad a_2\left[\frac{(\phi_{11}\cos w_2 t + \phi_{12}\sin w_2 t) + i(\phi_{12}\cos w_2 t - \phi_{11}\sin w_2 t)}{(c_5^2+c_6^2)}\right]\\
+ &\quad \frac{a_2\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cosh\left[\frac{(2r_2+1)\pi s}{2a}\right]\\
\times &\quad \left[\frac{e^{x_3 t}(x_3-iw_2)(1+x_3\tau)^2)}{(x_3^2+w_2^2)[(l+(x_3\tau+1)^2)]} + \frac{e^{x_4 t}(x_4-iw_2)(1+x_4\tau)^2)}{(x_4^2+w_2^2)[(l+(x_4\tau+1)^2)]}\right]\\
- &\quad \frac{c_0}{\nu X^2}\left[\frac{\cosh(Xs) - cosh(Xa)}{\cosh(Xa)}\right]\\
+ &\quad \frac{4c_0}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\cosh\left[\frac{(2r_2+1)\pi s}{2a}\right]\left[\frac{e^{x_3 t}(x_3-iw_2)(1+x_3\tau)^2)}{(x_3^2+w_2^2)[(l+(x_3\tau+1)^2)]}\right.\\
+ &\quad \left.\frac{e^{x_4 t}(x_4-iw_2)(1+x_4\tau)^2}{(x_4^2+w_2^2)[(l+(x_4\tau+1)^2)]}\right]
\end{aligned}
$$

$$
\begin{aligned}
v_b(s,n,t) &= \frac{-2\pi}{a^2}\sum_{r_1}^{\infty} r_1 \sin\left[\frac{r_1\pi s}{a}\right]\left\{a_1\left[\frac{(\phi_{13}\cos(w_1t)-\phi_{14}\sin(w_1t))}{(1+w_1^2\tau^2)(y_3^2+z_3^2)(c_1^2+c_2^2)(c_3^2+C_4^2)}\right.\right.\\
&+ \left.\frac{i(\phi_{14}\cos w_1t+\phi_{13}\sin w_1t)}{(1+w_1^2\tau^2)(y_3^2+z_3^2)(c_1^2+c_2^2)(c_3^2+C_4^2)}\right]\\
&- a_1(-1)^{r_1}\left[\frac{(\phi_{15}\cos(w_1t)-\phi_{16}sinw_1t)+i(\phi_{15}\sin(w_1t)+\phi_{16}\cos(w_1t))}{(1+w_1^2\tau^2)(y_3^2+z_3^2)(c_3^2+C_4^2)}\right]\\
&+ \frac{a_1[1-(-1)^{r_1}\cos(Q_1a)]}{\cos(Q_1a)(x_1-x_2)}\left[\frac{e^{x_1t}(x_1+iw_1)}{(x_1^2+w_1^2)(1+x_1\tau)}-\frac{e^{x_2t}(x_2+iw_1)}{(1+x_2\tau)(x_2^2+w_1^2)}\right]\\
&+ \frac{a_1\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\frac{\cosh(A_1n)}{A_1^2cosh(A_1a)}\left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\tau))}{(x_3^2+w_1^2)[(l+(x_3\tau+1)^2)]}\right.\\
&+ \left.\frac{e^{x_4t}(x_4+iw_1)(1+x_4\tau))}{(x_4^2+w_1^2)[(l+(x_4\tau+1)^2)]}\right]-\frac{4\nu a_1}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\\
&\times \frac{[1-(-1)^{r_1}\cos(Q_2a)]}{\cos(Q_2a)}\cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5t}(x_5+iw_1)(1+x_5\tau)}{(x_5^2+w_1^2)[(l+(x_5\tau+1)^2)]}\right.\\
&+ \left.\frac{e^{x_6t}(x_6+iw_1)(1+x_6\tau)}{(x_6^2+w_1^2)[(l+(x_6\tau+1)^2)]}\right]\\
&+ a_2\left[\frac{(\phi_{17}\cos w_2t+\phi_{18}\sin w_2t)+i(\phi_{18}\cos w_2t-\phi_{17}\sin w_2t)}{(1+w_2^2\tau^2)(y_7^2+z_7^2)(c_5^2+c_6^2)(c_7^2+C_8^2)}\right]\\
&- a_2(-1)^{r_1}\left[\frac{(\phi_{19}\cos w_2t+\phi_{20}\sin w_2t)+i(\phi_{20}\cos w_2t-\phi_{19}sinw_2t)}{(1+w_2^2\tau^2)(y_7^2+z_7^2)(c_7^2+C_8^2)}\right]\\
&+ \frac{a_2[1-(-1)^{r_1}\cos(Q_1a)]}{\cos(Q_1a)(x_1-x_2)}\left[\frac{e^{x_1t}(x_1-iw_2)}{(1+x_1\tau)(x_1^2+w_2^2)}-\frac{e^{x_2t}(x_2-iw_2)}{(1+x_2\tau)(x_2^2+w_2^2)}\right]\\
&+ \frac{a_2\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\frac{\cosh(A_1n)}{A_1^2cosh(A_1a)}\left[\frac{e^{x_3t}(x_3-iw_2)(1+x_3\tau))}{(x_3^2+w_2^2)[(l+(x_3\tau+1)^2)]}\right.\\
&+ \left.\frac{e^{x_4t}(x_4-iw_2)(1+x_4\tau))}{(x_4^2+w_2^2)[(l+(x_4\tau+1)^2)]}\right]-\frac{4\nu a_2}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\\
&\times \frac{[1-(-1)^{r_1}\cos(Q_2a)]}{\cos(Q_2a)}\cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5t}(x_5-iw_2)(1+x_5\tau)}{(x_5^2+w_2^2)[(l+(x_5\tau+1)^2)]}\right.\\
&+ \left.\left.\frac{e^{x_6t}(x_6-iw_2)(1+x_6\tau)}{(x_6^2+w_2^2)[(l+(x_6\tau+1)^2)]}\right]\right\}+\frac{2\pi}{a^2}\sum_{r_1=0}^{\infty}\sin\left[\frac{r_1\pi s}{a}\right]\\
&\times \left\{\frac{c_0\,[1-(-1)^{r_1}\cosh(Xa)]\cosh(Yn)}{\nu X^2Y^2\cosh(Xa)\cosh(Ya)}+\frac{c_0}{\nu}\frac{[1-(-1)^{r_1}]}{Q_1^2\cosh(Q_1a)}\frac{\cosh(Q_1n)}{(x_7-x_8)}\right.
\end{aligned}
$$

$$
\begin{aligned}
\times \quad & \left[\frac{e^{x_7 t}}{x_7(1+x_7\tau)} - \frac{e^{x_8 t}}{x_8(1+x_8\tau)}\right] + \frac{c_0}{\nu}\frac{[1-(-1)^{r_1}\cos(Q_1 a)]}{Q_1^2\cosh(Q_1 a)(x_1-x_2)} \\
\times \quad & \left[\frac{e^{x_1 t}}{x_1(1+x_1\tau)} - \frac{e^{x_2 t}}{x_2(1+x_2\tau)}\right] - \frac{4c_o}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{\cosh(A_1 n)}{A_1^2\cosh(A_1 a)} \\
\times \quad & \left[\frac{e^{x_3 t}(1+x_3\tau)}{x_3[l+(1+x_3\tau)^2]} + \frac{e^{x_4 t}(1+x_4\tau)}{x_4[l+(1+x_4\tau)^2]}\right] - \frac{4c_0}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)} \\
\times \quad & \frac{[1-(-1)_1^r\cos Q_2 a]}{Q_2^2\cos(Q_2 a)} cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5 t}(1+x_5\tau)}{x_5[l+(1+x_5\tau)^2]}\right. \\
+ \quad & \left.\left.\frac{e^{x_6 t}(1+x_6\tau)}{x_6[l+(1+x_6\tau)^2]}\right]\right\} - \frac{2}{\pi}\sum_{r_1=0}^{\infty}\frac{[1-(-1)_1^r]}{r_1}\sin\left[\frac{r_1\pi s}{a}\right] \\
\times \quad & \left\{\frac{c_0}{\nu X^2}\frac{\cosh(Yn)}{\cosh(Ya)} + \frac{c_0}{\nu}\frac{\cosh(Q_1 n)}{\cosh(Q_1 a)(x_7-x_8)}\left[\frac{e^{x_7 t}}{x_7(1+x_7\tau)}\right.\right. \\
- \quad & \left.\frac{e^{x_8 t}}{x_8(1+x_8\tau)}\right] + \frac{c_0\pi}{a^2}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}(2r_2+1)}{Q^2}\cos\left[\frac{(2r_2+1)\pi n}{2a}\right] \\
\times \quad & \left.\left[\frac{e^{x_5 t}(1+x_5\tau)}{x_5[l+(1+x_5\tau)^2]} + \frac{e^{x_6 t}(1+x_6\tau)}{x_6[l+(l+x_6\tau)^2]}\right]\right\} \\
+ \quad & a_1\left[\frac{(\phi_{21}\cos w_1 t - \phi_{22}\sin w_1 t) + i(\phi_{22}\cos w_1 t + \phi_{21}\sin w_1 t)}{(1+w_1^2\tau^2)(c_1^2+c_2^2)}\right] \\
+ \quad & \frac{a_1\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cosh\left[\frac{(2r_2+1)\pi s}{2a}\right] \\
\times \quad & \left[\frac{e^{x_3 t}(x_3+iw_1)(1+x_3\tau)}{(x_3^2+w_1^2)[(l+(x_3\tau+1)^2)]} + \frac{e^{x_4 t}(x_4+iw_1)(1+x_4\tau)}{(x_4^2+w_1^2)[(l+(x_4\tau+1)^2)]}\right] \\
+ \quad & a_2\left[\frac{(\phi_{23}\cos w_2 t + \phi_{24}\sin w_2 t)}{(c_5^2+c_6^2)(1+w_2^2\tau^2)} + \frac{i(\phi_{24}\cos w_2 t - \phi_{23}\sin w_2 t)}{(c_5^2+c_6^2)(1+w_2^2\tau^2)}\right] \\
+ \quad & \frac{a_2\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\cosh\left[\frac{(2r_2+1)\pi s}{2a}\right] \\
\times \quad & \left[\frac{e^{x_3 t}(x_3-iw_2)(1+x_3\tau)}{(x_3^2+w_2^2)[(l+(x_3\tau+1)^2)]} + \frac{e^{x_4 t}(x_4-iw_2)(1+x_4\tau)}{(x_4^2+w_2^2)[(l+(x_4\tau+1)^2)]}\right] \\
- \quad & \frac{c_0}{\nu X^2}\left[\frac{\cosh(Xs) - cosh(Xa)}{\cosh(Xa)}\right] - \frac{4c_0}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\cos\left[\frac{(2r_2+1)\pi s}{2a}\right] \\
\times \quad & \left[\frac{e^{x_3 t}(x_3-iw_2)(1+x_3\tau)}{(x_3^2+w_2^2)[(l+(x_3\tau+1)^2)]} + \frac{e^{x_4 t}(x_4-iw_2)(1+x_4\tau)}{(x_4^2+w_2^2)[(l+(x_4\tau+1)^2)]}\right]
\end{aligned}
$$

Shearing Stress (Skin Friction):
The Shear stress at the boundaries $s = a,\ \ s = -a$ and $n = a,\ \ n = -a$ are given by

$$\begin{aligned}
D_{an} &= \frac{2\pi^2\mu}{a^3}\sum_{r_1=0}^{\infty} r_1^2 \Big\{ a_1 \left[\frac{(\phi_1\cos(w_1t) - \phi_2\sin(w_1t)) + i(\phi_1\sin(w_1t) + \phi_2 cos w_1 t)}{(y_3^2+z_3^2)(c_1^2+c_2^2)(c_3^2+C_4^2)}\right] \\
&- a_1(-1)^{r_1}\left[\frac{(\phi_3\cos(w_1t) - \phi_4\sin(w_1t)) + i(\phi_3\sin(w_1t) + \phi_4\cos(w_1t))}{(y_3^2+z_3^2)(c_3^2+C_4^2)}\right] \\
&+ \frac{a_1[1-(-1)^{r_1}\cos(Q_1a)]}{\cos(Q_1a)(x_1-x_2)}\left[\frac{e^{x_1t}(x_1+iw_1)}{x_1^2+w_1^2} - \frac{e^{x_2t}(x_2+iw_1)}{x_2^2+w_1^2}\right] \\
&+ \frac{a_1\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{(2r_2+1)\cosh(A_1n)}{A_1^2\cosh(A_1a)}\left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\tau)^2}{(x_3^2+w_1^2)[(l+(x_3\tau+1)^2)]}\right. \\
&+ \left.\frac{e^{x_4t}(x_4+iw_1)(1+x_4\tau)^2}{(x_4^2+w_1^2)[(l+(x_4\tau+1)^2)]}\right] - \frac{4\nu a_1}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[1-(-1)^{r_1}\cos(Q_2a)]}{\cos(Q_2a)} \\
&\times \cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5t}(x_5+iw_1)(1+x_5\tau)^2}{(x_5^2+w_1^2)[(l+(x_5\tau+1)^2)]}\right. \\
&+ \left.\frac{e^{x_6t}(x_6+iw_1)(1+x_6\tau)^2}{(x_6^2+w_1^2)[(l+(x_6\tau+1)^2)]}\right] \\
&+ a_2\left[\frac{(\phi_5\cos(w_2t) - \phi_6\sin(w_2t) + i(\phi_6\cos(w_2t) - \phi_5\sin(w_2t)}{(y_7^2+z_7^2)(c_5^2+c_6^2)(c_7^2+C_8^2)}\right] \\
&- a_2(-1)^{r_1}\left[\frac{(\phi_7 cos w_2 t + \phi_8\sin(w_2t) + i(\phi_8\cos(w_2t) - \phi_7\sin(w_2t)}{(y_7^2+z_7^2)(c_7^2+C_8^2)}\right] \\
&+ \frac{a_2[1-(-1)^{r_1}\cos(Q_1a)]}{\cos(Q_1a)(x_1-x_2)}\left[\frac{e^{x_1t}(x_1-iw_2)}{x_1^2+w_2^2} - \frac{e^{x_2t}(x_2-iw_2)}{x_2^2+w_2^2}\right] \\
&+ \frac{a_2\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{(2r_2+1)\cosh(A_1n)}{A_1^2\cosh(A_1a)}\left[\frac{e^{x_3t}(x_3-iw_2)(1+x_3\tau)^2}{(x_3^2+w_2^2)[(l+(x_3\tau+1)^2)]}\right. \\
&+ \left.\frac{e^{x_4t}(x_4-iw_2)(1+x_4\tau)^2}{(x_4^2+w_2^2)[(l+(x_4\tau+1)^2)]}\right] - \frac{4\nu a_2}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{[1-(-1)^{r_1}\cos(Q_2a)]}{\cos(Q_2a)} \\
&\times \cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5t}(x_5-iw_2)(1+x_5\tau)^2}{(x_5^2+w_2^2)[(l+(x_5\tau+1)^2)]}\right. \\
&+ \left.\frac{e^{x_6t}(x_6-iw_2)(1+x_6\tau)^2}{(x_6^2+w_2^2)[(l+(x_6\tau+1)^2)]}\right]\Big\} - \frac{2\pi^2\mu}{a^3}\sum_{r_1}^{\infty}(-1)^{r_1}r_1^2
\end{aligned}$$

$$
\begin{aligned}
\times & \quad \left\{ \frac{\frac{c_0}{\nu X^2}\left[1-(-1)^{r_1} cosh(Xa)\right]\cosh(Yn)}{Y^2\cosh(Xa)\cosh(Ya)} + \frac{c_0}{\nu}\frac{\left[1-(-1)^{r_1}\right]}{Q_1^2\cosh(Q_1 a)}\frac{\cosh(Q_1 n}{(x_7-x_8)} \right. \\
\times & \quad \left[\frac{e^{x_7 t}}{x_7} - \frac{e^{x_8 t}}{x_8}\right] + \frac{c_0}{\nu}\frac{\left[1-(-1)^{r_1}\cos(Q_1 a)\right]}{Q_1^2\cosh(Q_1 a)(x_1-x_2)}\left[\frac{e^{x_1 t}}{x_1} - \frac{e^{x_2 t}}{x_2}\right] \\
- & \quad \frac{4c_o}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{\cosh(A_1 n)}{A_1^2\cosh(A_1 a)}\left[\frac{e^{x_3 t}(1+x_3\tau)^2}{x_3[l+(1+x_3\tau)^2]} + \frac{e^{x_4 t}(1+x_4\tau)^2}{x_4[l+(1+x_4\tau)^2]}\right] \\
- & \quad \frac{4c_0}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{\left[1-(-1)_1^{r}\cos(Q_2 a)\right]}{Q_2^2\cos(Q_2 a)}\cos\left[\frac{(2r_2+1)\pi n}{2a}\right] \\
\times & \quad \left. \left[\frac{e^{x_5 t}(1+x_5\tau)^2}{x_5[l+(1+x_5\tau)^2]} + \frac{e^{x_6 t}(1+x_6\tau)^2}{x_6[l+(1+x_6\tau)^2]}\right]\right\} - \frac{2\mu}{a}\sum_{r_1=0}^{\infty}\left[(-1)_1^{r}-1\right] \\
\times & \quad \left\{ \frac{c_0}{\nu X^2}\frac{\cosh(Yn)}{\cosh(Ya)} + \frac{c_0}{\nu}\frac{\cosh(Q_1 n)}{\cosh(Q_1 a)(x_7-x_8)}\left[\frac{e^{x_7 t}}{x_7} - \frac{e^{x_8 t}}{x_8}\right] \right. \\
+ & \quad \frac{c_0\pi}{a^2}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}(2r_2+1)}{Q^2}\cos\left[\frac{(2r_2+1)\pi n}{2a}\right]\left[\frac{e^{x_5 t}(1+x_5\tau)^2}{x_5[l+(1+x_5\tau)^2]}\right. \\
+ & \quad \left.\left.\frac{e^{x_6 t}(1+x_6\tau)^2}{x_6[l+(1+x_6\tau)^2]}\right]\right\} - \frac{a_1\mu}{c_1^2+c_2^2}\left[\cos w_1 t(m_{25}H_{13} - m_{26}G_{13} + m_{27}G_{13}\right. \\
+ & \quad m_{28}H_{13}) - \sin w_1 t(m_{27}H_{13} - m_{28}G_{13} - m_{25}G_{13} - m_{26}H_{13}) \\
+ & \quad i\cos w_1 t(m_{27}H_{13} - m_{28}G_{13} - m_{25}G_{13} - m_{26}H_{13}) \\
+ & \quad \left. i\sin w_1 t(m_{25}H_{13} - m_{26}G_{13} + m_{27}G_{13} + m_{28}H_{13})\right] \\
+ & \quad \frac{a_1\pi^2\nu\mu}{2a^3}\sum_{r_2=0}^{\infty}(2r_2+1)^2\left[\frac{e^{x_3 t}(x_3+iw_1)(1+x_3\tau)^2)}{(x_3^2+w_1^2)[(l+(x_3\tau+1)^2)]}\right. \\
+ & \quad \left.\frac{e^{x_4 t}(x_4+iw_1)(1+x_4\tau)^2}{(x_4^2+w_1^2)[(l+(x_4\tau+1)^2)]}\right] + \frac{a_2\mu}{c_5^2+c_6^2}\left[\cos w_2 t(m_{29}H_{14} - m_{30}G_{14}\right. \\
+ & \quad m_{31}G_{14} + m_{32}H_{14}) + \sin w_2 t(m_{29}G_{14} + m_{30}H_{14} - m_{31}H_{14} - m_{32}G_{14}) \\
+ & \quad i\cos w_2 t(m_{29}G_{14} + m_{30}H_{14} - m_{31}H_{14} - m_{32}G_{14}) \\
- & \quad \left.\sin w_2 t(m_{29}H_{14} - m_{30}G_{14} + m_{31}G_{14} + m_{32}H_{14})\right] - \frac{c_o\mu\sinh(Xa)}{\nu X\cosh(Xa)} \\
+ & \quad \frac{a_2\nu\pi^2\mu}{2a^3}\sum_{r_2=0}^{\infty}(2r_2+1)^2\left[\frac{e^{x_3 t}(1+x_3\tau)^2(x_3+iw_1)}{x_3[l+(1+x_3\tau)^2]}\right. \\
+ & \quad \left.\frac{e^{x_4 t}(1+x_4\tau)^2(x_4+iw_1)}{x_4[l+(1+x_4\tau)^2]}\right]
\end{aligned}
$$

$$
\begin{aligned}
& - && \frac{2c_o\mu}{a}\left[\frac{e^{x_3t}(1+x_3\tau)^2}{x_3[l+(1+x_3\tau)^2]}+\frac{e^{x_4t}(1+x_4\tau)^2}{x_4[l+(1+x_4\tau)^2]}\right] \\
D_{-an} & = && -D_{an} \\
D_{sa} & = && \frac{-2\pi\mu}{a^2}\sum_{r_1=0}^{\infty} r_1\sin\left[\frac{r_1\pi s}{a}\right]\left\{\frac{a_1}{k_1}\left[\cos w_1t(m_1H_{11}-m_2G_{11}-m_3G_{11}\right.\right. \\
& - && \left. m_4H_{11})p_1(m_1G_{11}+m_2H_{11}+m_3H_{11}-m_4G_{11})p_2\right] \\
& - && \sin w_1t\left[(m_1H_{11}-m_2G_{11}-m_3G_{11}-m_4H_{11})p_2-(m_1G_{11}+m_2H_{11}\right. \\
& + && \left. m_3H_{11}-m_4G_{11})p_1\right]+i\sin w_1t\left[(m_1H_{11}-m_2G_{11}-m_3G_{11}-m_4H_{11})p_1\right. \\
& + && \left.(m_1G_{11}+m_2H_{11}+m_3H_{11}-m_4G_{11})p_2\right]+i\cos w_1t\left[(m_1H_{11}-m_2G_{11}\right. \\
& - && \left. m_3G_{11}-m_4H_{11})p_2-(m_1G_{11}+m_2H_{11}+m_3H_{11}-m_4G_{11})p_1\right] \\
& - && \frac{a_1(-1)^{r_1}}{k_2}\left[\cos w_1t(m_5H_{11}-m_6G_{11}-m_7G_{11}-m_8G_{11}+m_9G_{11}\right. \\
& + && m_{10}H_{11}-m_{12}G_{11})-\sin w_1t(m_9H_{11}-m_{10}G_{11}-m_{11}G_{11}-m_{12}G_{11} \\
& - && m_5G_{11}-m_6H_{11}-m_7H_{11}-m_8G_{11})+i\sin w_1t(m_5H_{11}-m_6G_{11} \\
& - && m_7G_{11}-m_8G_{11}+m_9G_{11}+m_{10}H_{11}-m_{12}G_{11})+i\cos w_t(m_9H_{11} \\
& - && \left. m_{10}G_{11}-m_{11}G_{11}-m_{12}G_{11}-m_5G_{11}-m_6H_{11}-m_7H_{11}-m_8G_{11})\right] \\
& + && \frac{a_1\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\frac{\sinh(A_1a)}{A_1\cosh(A_1a)} \\
& \times && \left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\tau)^2)}{(x_3^2+w_1^2)[(l+(x_3\tau+1)^2)]}+\frac{e^{x_4t}(x_4+iw_1)(1+x_4\tau)^2)}{(x_4^2+w_1^2)[(l+(x_4\tau+1)^2)]}\right] \\
& - && \frac{2\nu a_1}{a}\sum_{r_2=0}^{\infty}\frac{[1-(-1)^{r_1}\cos(Q_2a)]}{\cos(Q_2a)}\left[\frac{e^{x_5t}(1+x_5\tau)^2}{x_5[l+(1+x_5\tau)^2]}+\frac{e^{x_6t}(1+x_6\tau)^2}{x_6[l+(1+x_6\tau)^2]}\right] \\
& + && \frac{a_3}{k_3}\left\{\left[\cos w_1t(m_{13}H_{12}-m_{14}G_{12}-m_{15}G_{12}-m_{16}H_{12})p_3+(m_{13}G_{12}\right.\right. \\
& + && \left. m_{14}H_{12}+m_{15}H_{12}-m_{16}G_{12})p_4\right]+\sin w_2t\,[(m_{13}G_{12}+m_{14}H_{12}+m_{15}H_{12} \\
& - && m_{16}G_{12})p_3-(m_{13}H_{12}-m_{14}G_{12}-m_{15}G_{12}-m_{16}H_{12})p_4] \\
& + && i\cos w_2t\left[(m_{13}G_{12}+m_{14}H_{12}+m_{15}H_{12}-m_{16}G_{12})p_3-(m_{13}H_{12}-m_{14}G_{12}\right]
\end{aligned}
$$

$$
\begin{aligned}
- & \quad m_{15}G_{12} - m_{16}H_{12})p_4 \; - i\sin w_2 t\,[(m_{13}H_{12} - m_{14}G_{12} - m_{15}G_{12} \\
+ & \quad m_{16}H_{12})p_3 + (m_{13}G_{12} + m_{14}H_{12} + m_{15}H_{12} - m_{16}G_{12})p_4]\} \\
- & \quad \frac{a_2(-1)^{r_1}}{k_4}\,[\cos w_2 t(m_{17}H_{12} - m_{18}G_{12} - m_{19}G_{12} \\
- & \quad m_{20}H_{12} - m_{21}G_{12} - m_{22}H_{12} + m_{23}H_{12} - m_{24}G_{12}) \\
+ & \quad \sin w_2 t(m_{17}G_{12} - m_{18}H_{12} - m_{19}H_{12} - m_{20}G_{12} - m_{21}H_{12} - m_{22}G_{12} \\
+ & \quad m_{23}G_{12} - m_{24}H_{12}) + i\cos w_2 t(m_{17}G_{12} - m_{18}H_{12} - m_{19}H_{12} - m_{20}G_{12} \\
- & \quad m_{21}H_{12} - m_{22}G_{12} + m_{23}G_{12} - m_{24}H_{12}) \\
- & \quad i\sin w_2 t(m_{17}H_{12} - m_{18}G_{12} - m_{19}G_{12} - m_{20}H_{12} - m_{21}G_{12} - m_{22}H_{12}] \\
+ & \quad m_{23}H_{12} - m_{24}G_{12}) + \frac{a_2\nu\pi}{a^2}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}(2r_2+1)\cosh(A_1 a)}{A_1\cosh(A_1 a)} \\
\times & \quad \left[\frac{e^{x_3 t}(x_3 + iw_1)(1 + x_3\tau)^2)}{(x_3^2 + w_1^2)[(l + (x_3\tau + 1)^2)]} + \frac{e^{x_4 t}(x_4 + iw_1)(1 + x_4\tau)^2)}{(x_4^2 + w_1^2)[(l + (x_4\tau + 1)^2)]}\right] \\
+ & \quad \frac{2\nu a_2}{a}\sum_{r_2=0}^{\infty}\frac{[1 - (-1)^{r_1}\cos(Q_2 a)]}{\cos(Q_2 a)} \\
\times & \quad \left[\frac{e^{x_5 t}(x_5 - iw_2)(1 + x_5\tau)^2}{(x_5^2 + w_2^2)[(l + (x_5\tau + 1)^2)]} + \frac{e^{x_6 t}(x_6 - iw_2)(1 + x_6\tau)^2}{(x_6^2 + w_2^2)[(l + (x_6\tau + 1)^2)]}\right]\Bigg\} \\
+ & \quad \frac{2\pi\mu}{a^2}\sum_{r_1=0}^{\infty} r_1\sin(\frac{r_1\pi s}{a})\Bigg\{\frac{c_0}{\nu X^2}\frac{[1 - (-1)^{r_1}\cosh Xa]\sinh Ya}{Y\cosh Xa\cosh Ya} + \frac{c_0}{\nu}[1 - (-1)^{r_1}] \\
\times & \quad \frac{\sinh Q_1 a}{\cos Q_1 a(x_7 - x_8)}\left[\frac{e^{x_7 t}}{x_7} - \frac{e^{x_8 t}}{x_8}\right] - \frac{4c_o}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{\sinh(A_1 a)}{A_1\cosh(A_1 a)} \\
\times & \quad \left[\frac{e^{x_3 t}(1 + x_3\tau)^2}{x_3[l + (1 + x_3\tau)^2]} + \frac{e^{x_4 t}(1 + x_4\tau)^2}{x_4[l + (1 + x_4\tau)^2]}\right] \\
- & \quad \frac{2c_0}{a}\sum_{r_2=0}^{\infty}(-1)^{r_2}\frac{[1 - (-1)^{r_1}\cos(Q_2 a)]}{Q_2^2\cos(Q_2 a)}\left[\frac{e^{x_5 t}(1 + x_5\tau)^2}{x_5[l + (1 + x_5\tau)^2]}\right. \\
+ & \quad \left.\frac{e^{x_6 t}(1 + x_6\tau)^2}{x_6[l + (1 + x_6\tau)^2]}\right]\Bigg\} - \frac{2\mu}{\pi}\sum_{r_1=0}^{\infty}\frac{[1 - (-1)_1^r}{r_1}\sin(\frac{r_1\pi}{a}) \\
\times & \quad \left\{\frac{c_0 Y\sinh(Ya)}{\nu X^2\cosh(Ya)} + \frac{c_0}{\nu}Q_1\frac{\sinh Q_1 a}{\cos Q_1 a(x_7 - x_8)}\left[\frac{e^{x_7 t}}{x_7} - \frac{e^{x_8 t}}{x_8}\right]\right. \\
- & \quad \frac{c_0\pi^2}{a^3}\sum_{r_2=0}^{\infty}\frac{(2r_2+1)^2}{Q_2^2}\left[\frac{e^{x_5 t}(1 + x_5\tau)^2}{x_5[l + (1 + x_5\tau)^2]} + \frac{e^{x_6 t}(1 + x_6\tau)^2}{x_6[l + (1 + x_6\tau)^2]}\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{2\pi^2\mu}{a^3}\sum_{r_1=0}^{\infty} r_1^2\cos(\frac{r_1\pi s}{a}) \\
&\times \quad \left\{\left[\frac{(G_1\cos(w_1t) - H_1\sin(w_1t) + i(G_1\sin(w_1t) + H_1 cos w_1 t)}{(y_3^2+z_3^2)(c_1^2+c_2^2)(c_3^2+C_4^2)}\right] a_1\right. \\
&- \quad a_1(-1)^{r_1}\left[\frac{(G_2\cos(w_1t) - H_2\sin(w_1t) + i(G_2\sin(w_1t) + H_2\cos(w_1t)}{(y_3^2+z_3^2)(c_3^2+C_4^2)}\right] \\
&+ \quad \frac{a_1[1-(-1)^{r_1}\cos(Q_1a)]}{\cos(Q_1a)(x_1-x_2)}\left[\frac{e^{x_1t}(x_1+iw_1)}{x_1^2+w_1^2} - \frac{e^{x_2t}(x_2+iw_1)}{x_2^2+w_1^2}\right] \\
&+ \quad \frac{a_1\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\frac{\cosh(A_1a)}{A_1^2\cosh(A_1a)}\left[\frac{e^{x_3t}(x_3+iw_1)(1+x_3\tau)^2}{(x_3^2+w_1^2)[(l+(x_3\tau+1)^2)]}\right. \\
&+ \quad \left.\frac{e^{x_4t}(x_4+iw_1)(1+x_4\tau)^2}{(x_4^2+w_1^2)[(l+(x_4\tau+1)^2)]}\right] \\
&+ \quad a_2\left[\frac{(G_3\cos(w_2t) - H_3\sin(w_2t) + i(H_3\cos(w_2t) - G_3\sin(w_2t)}{(y_7^2+z_7^2)(c_5^2+c_6^2)(c_7^2+C_8^2)}\right] \\
&- \quad a_2(-1)^{r_1}\left[\frac{(G_4 cos w_2 t + H_4\sin(w_2t) + i(H_4\cos(w_2t) - G_4\sin(w_2t)}{(y_7^2+z_7^2)(c_7^2+C_8^2)}\right] \\
&+ \quad \frac{a_2[1-(-1)^{r_1}\cos(Q_1a)]}{\cos(Q_1a)(x_1-x_2)}\left[\frac{e^{x_1t}(x_1-iw_2)}{x_1^2+w_2^2} - \frac{e^{x_2t}(x_2-iw_2)}{x_2^2+w_2^2}\right] \\
&+ \quad \frac{a_2\nu\pi}{a^2}\sum_{r_2=0}^{\infty}(-1)^{r_2}(2r_2+1)\frac{\cosh(A_1a)}{A_1^2\cosh(A_1a)}\left[\frac{e^{x_3t}(x_3-iw_2)(1+x_3\tau)^2)}{(x_3^2+w_2^2)[(l+(x_3\tau+1)^2)]}\right. \\
&+ \quad \left.\left.\frac{e^{x_4t}(x_4-iw_2)(1+x_4\tau)^2}{(x_4^2+w_2^2)[(l+(x_4\tau+1)^2)]}\right]\right\} - \frac{2\pi^2\mu}{a^3}\sum_{r_1}^{\infty} r_1^2\cos\left[\frac{r_1\pi s}{a}\right] \\
&\times \quad \left\{\frac{c_0\,[1-(-1)^{r_1} cosh(Xa)]\cosh(Ya)}{\nu X^2Y^2\cosh(Xa)\cosh(Ya)} + \frac{c_0}{\nu}\frac{[1-(-1)^{r_1}]}{Q_1^2\cosh(Q_1a)}\frac{\cosh(Q_1a}{(x_7-x_8)}\right. \\
&\times \quad \left[\frac{e^{x_7t}}{x_7} - \frac{e^{x_8t}}{x_8}\right] + \frac{c_0}{\nu}\frac{[1-(-1)^{r_1}\cos(Q_1a)]}{Q_1^2\cosh(Q_1a)(x_1-x_2)}\left[\frac{e^{x_1t}}{x_1} - \frac{e^{x_2t}}{x_2}\right] \\
&- \quad \frac{4c_o}{\pi}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_2}}{(2r_2+1)}\frac{\cosh(A_1a)}{A_1^2\cosh(A_1a)}\left[\frac{e^{x_3t}(1+x_3\tau)^2}{x_3[l+(1+x_3\tau)^2]}\right. \\
&+ \quad \left.\left.\frac{e^{x_4t}(1+x_4\tau)^2}{x_4[l+(1+x_4\tau)^2]}\right]\right\} - \frac{2\mu}{a}\sum_{r_1=0}^{\infty}[(-1)_1^r - 1]\cos\left[\frac{r_1\pi s}{a}\right] \\
&\times \quad \left\{\frac{c_0}{\nu X^2} + \frac{c_0}{\nu}\frac{1}{(x_7-x_8)}\left[\frac{e^{x_7t}}{x_7} - \frac{e^{x_8t}}{x_8}\right]\right\} - \frac{a_1\mu}{c_1^2+c_2^2}\,[G_5\cos w_1t - H_5\sin w_1t
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad i(H_5 \cos w_1 t + G_5 \sin w_1 t)] + \frac{a_1 \pi^2 \nu \mu}{2a^3} \sum_{r_2=0}^{\infty} (2r_2+1)^2 \\
&\times \quad \left[\frac{e^{x_3 t}(x_3 + iw_1)(1 + x_3\tau)^2)}{(x_3^2 + w_1^2)[(l + (x_3\tau + 1)^2)]} + \frac{e^{x_4 t}(x_4 + iw_1)(1 + x_4\tau)^2}{(x_4^2 + w_1^2)[(l + (x_4\tau + 1)^2)]} \right] \\
&+ \quad \frac{a_2 \mu}{c_5^2 + c_6^2} [G_6 \cos w_2 t + H_6 \sin w_2 t + i(H_6 \cos w_2 t - G_6 \sin w_2 t)] \\
&+ \quad \frac{a_2 \nu \pi^2 \mu}{2a^3} \sum_{r_2=0}^{\infty} (2r_2+1)^2 \sin\left[\frac{(2r_2+1)\pi s}{2a}\right] \left[\frac{e^{x_3 t}(1 + x_3\tau)^2 (x_3 + iw_1)}{x_3[l + (1 + x_3\tau)^2]} \right. \\
&+ \quad \left. \frac{e^{x_4 t}(1 + x_4\tau)^2 (x_4 + iw_1)}{x_4[l + (1 + x_4\tau)^2]} \right] - \frac{c_o \mu \sinh(Xa)}{\nu X \cosh(Xa)} - \frac{2c_o\mu}{a} \left[\frac{e^{x_3 t}(1 + x_3\tau)^2}{x_3[l + (1 + x_3\tau)^2]} \right. \\
&+ \quad \left. \frac{e^{x_4 t}(1 + x_4\tau)^2}{x_4[l + (1 + x_4\tau)^2]} \right]
\end{aligned}
$$

$$D_{-sa} = -D(sa)$$

where

$$
\begin{aligned}
y_1 &= \frac{c_r \nu + c_r \nu w_1^2 \tau^2 + w_1^2 l \tau}{\nu(1 + w_1^2\tau^2)}, \quad z_1 = \frac{w_1 + w_1 l + w_1^3 \tau^2}{\nu(1 + w_1^2\tau^2)}, \quad y_2 = \sqrt{\frac{y_1 + \sqrt{y_1^2 + z_1^2}}{2}}, \\
z_2 &= \sqrt{\frac{-y_1 + \sqrt{y_1^2 + z_1^2}}{2}}, \quad z_3 = \frac{w_1 a^2 + w_1 l a^2 + r_1^2 \pi^2 \nu w_1 \tau + w_1^3 a^2 \tau}{\nu a^2 (1 + w_1^2 \tau^2)}, \\
y_3 &= \frac{c_r a^2 \nu + r_1^2 \pi^2 \nu + c_r \nu a^2 \tau^2 w_1^2 + w_1^2 l a^2 \tau + r_1^2 \pi^2 \nu w_1^2 \tau^2}{\nu a^2 (1 + w_1^2 \tau^2)}, \\
y_4 &= \sqrt{\frac{y_3 + \sqrt{y_3^2 + z_3^2}}{2}}, \quad z_4 = \sqrt{\frac{-y_3 + \sqrt{y_3^2 + z_3^2}}{2}}, \quad c_1 = \cosh(y_2 a)\cos(z_2 a), \\
c_2 &= \sinh(y_2 a)\sin(z_2 a), \quad c_3 = \cosh(y_4 a)\cos(z_4 a), \quad c_4 = \sinh(y_4 a)\sin(z_4 a), \\
d_1 &= \cosh(y_4 n)\cos(z_4 n), \quad d_2 = \sin(y_4 n)\sin(z_4 n), \quad e_1 = d_1 y_3 - d_2 z_3, \\
e_2 &= d_2 y_3 + d_1 z_3, \quad e_3 = c_1 c_3 - c_2 c_4, \quad e_4 = c_2 c_3 + c_1 c_4, \quad \phi_1 = e_1 e_3 + e_2 e_4, \\
\phi_2 &= e_1 e_4 - e_2 e_3, \quad \phi_3 = e_1 c_3 + e_2 c_4, \quad \phi_4 = e_1 c_4 - e_2 c_3, \quad b_1 = a^2 \tau, \\
b_2 &= c_r a^2 \nu \tau + a^2 + l a^2 + r_1^2 \pi^2 \nu \tau, \quad b_3 = c_r a^2 \nu + r_1^2 \pi^2 \nu, \quad x_1 = \frac{-b_2 + \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}, \\
x_2 &= \frac{-b_2 - \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}, \quad Q_1^2 = \frac{r_1^2 \pi^2}{a^2}, \quad Q_1 = \frac{r_1 \pi}{a}, \quad b_4 = 4a^2 \tau,
\end{aligned}
$$

$$
\begin{aligned}
b_5 &= 4a^2c_r\nu\tau + 4a^2 + 4a^2l + (2r_2+1)\pi^2\nu\tau, \quad b_6 = 4c_ra^2\nu + (2r_2+1)^2\pi^2\nu,\\
x_3 &= \frac{-b_5+\sqrt{b_5^2-4b_4b_6}}{2b_4}, \quad x_4 = \frac{-b_5-\sqrt{b_5^2-4b_4b_6}}{2b_4}, \quad b_7 = 4a^2\tau,\\
A_1 &= \sqrt{\frac{r_1^2\pi^2}{a^2} - \frac{(2r_2+1)^2\pi^2}{4a^2}}, \quad b_9 = (2r_2+1)^2\pi^2\nu + 4c_ra^2\nu + 4r_1^2\pi^2\nu,\\
b_8 &= 4c_ra^2\nu\tau + 4a^2 + 4a^2l + 4r_1^2\pi^2\nu\tau + (2r_2+1)^2\pi^2\nu\tau,\\
x_5 &= \frac{-b_8+\sqrt{b_8^2-4b_7b_9}}{2b_7}, \quad x_6 = \frac{-b_8-\sqrt{b_8^2-4b_7b_9}}{2b_7}, \quad z_5 = \frac{w_2+w_2l+w_2^3\tau^2}{\nu(1+w_2^2\tau^2)},\\
y_5 &= \frac{c_r\nu + c_r\nu w_2^2\tau^2 + w_2^2l\tau}{\nu(1+w_2^2\tau^2)}, \quad Q_2 = \sqrt{\frac{r_1^2\pi^2}{a^2} + \frac{(2r_2+1)^2\pi^2}{4a^2}},\\
y_6 &= \sqrt{\frac{y_5+\sqrt{y_5^2+z_5^2}}{2}}, \quad z_6 = \sqrt{\frac{-y_5+\sqrt{y_5^2+z_5^2}}{2}}, \quad y_8 = \sqrt{\frac{y_7+\sqrt{y_7^2+z_7^2}}{2}}\\
y_7 &= \frac{c_r\nu a^2 + r_1^2\pi^2\nu + c_r\nu w_2^2\tau^2a^2 + w_2^2la^2\tau + r_1^2\pi^2\nu w_2^2\tau^2}{\nu a^2(1+w_2^2\tau^2)},\\
z_8 &= \sqrt{\frac{-y_7+\sqrt{y_7^2+z_7^2}}{2}}, \quad c_5 = \cosh(y_6a)\cos(z_6a), \quad c_6 = \sinh(y_6a)\sin(z_6a),\\
c_7 &= \cosh(y_8a)\cos(z_8a), \quad c_8 = \sinh(y_8a)\sin(z_8a), \quad d_3 = \cosh(y_8n)\cos(z_8n),\\
d_4 &= \sinh(y_8n)\sin(z_8n), \quad e_5 = d_3y_7 - d_4z_7, \quad e_6 = d_4y_7 + d_3z_7,\\
e_7 &= c_5c_7 - c_6c_8, \quad e_8 = c_6c_7 + c_5c_8, \quad \phi_5 = e_5e_7 + e_8e_6, \quad \phi_6 = e_6e_7 - e_5e_8,\\
\phi_7 &= e_5c_7 + e_6c_8, \quad \phi_8 = e_6c_7 - e_5c_8, \quad X = \sqrt{C_r}, \quad Y = \sqrt{c_r + \frac{r_1^2\pi}{a^2}}, \quad b_{10} = \tau,\\
b_{11} &= c_r\nu\tau + 1 + l, \quad b_{12} = c_r\nu, \quad x_7 = \frac{-b_{11}+\sqrt{b_{11}^2-4b_{10}b_{12}}}{2b_{10}},\\
x_8 &= \frac{-b_{11}-\sqrt{b_{11}^2-4b_{10}b_{12}}}{2b_{10}}, \quad d_5 = \cosh(y_2s)\cos(z_2s), \quad d_6 = \sinh(y_2s)\sin(z_2s),\\
\phi_9 &= d_5c_1 + d_6c_2, \quad \phi_{10} = c_2d_5 - c_1d_6, \quad d_7 = \cosh(y_6s)\cos(z_6s),\\
d_8 &= \sinh(y_6s)\sin(z_6s), \quad \phi_{11} = d_7c_5 + d_8c_6, \quad \phi_{12} = d_8c_5 - d_7c_6,\\
\phi_{13} &= \phi_1 + w_1\tau\phi_2, \quad \phi_{14} = \phi_2 - w_1\tau\phi_1, \quad \phi_{15} = \phi_3 + w_1\tau\phi_4, \quad \phi_{16} = \phi_4 - w_1\tau\phi_3,\\
\phi_{17} &= \phi_5 - \phi_6w_2\tau, \quad \phi_{18} = \phi_6 + \phi_5w_2\tau, \quad \phi_{19} = \phi_7 - \phi_8w_2\tau, \quad \phi_{20} = \phi_8 + \phi_7w_2\tau,\\
\phi_{21} &= \phi_9 + \phi_{10}w_1\tau, \quad \phi_{22} = \phi_{10} - \phi_9w_1\tau, \quad \phi_{23} = \phi_{11} - \phi_{12}w_2\tau,\\
\phi_{24} &= \phi_{12} + \phi_{11}w_2\tau, \quad m_1 = y_3y_4, \quad m_2 = y_3z_4, \quad m_3 = z_3y_4, \quad m_4 = z_3z_4, \quad m_5 = y_3c_3,
\end{aligned}
$$

$$\begin{aligned}
m_6 &= z_3c_3, \quad m_7 = y_3c_4, \quad m_8 = z_3c_4, \quad m_9 = c_4y_3y_4, \quad m_{10} = c_4y_3z_4,\\
m_{11} &= c_4z_3y_4, \quad m_{12} = c_4z_3z_4, \quad m_{13} = y_7y_8, \quad m_{14} = y_7z_8, \quad m_{15} = z_7y_8,\\
m_{16} &= z_7z_8, \quad m_{17} = c_7y_7y_8, \quad m_{18} = c_7y_7z_8, \quad m_{19} = c_7z_7y_8, \quad m_{20} = c_7z_7z_8,\\
m_{21} &= c_8y_7y_8, \quad m_{22} = c_8y_7z_8, \quad m_{23} = c_8z_7y_8, \quad m_{24} = c_8z_7z_8, \quad m_{25} = c_1y_2,\\
m_{26} &= c_1z_2, \quad m_{27} = c_2y_2, \quad m_{28} = c_2z_2, \quad m_{29} = c_5y_6, \quad m_{30} = c_5z_6, \quad m_{31} = c_6y_6,\\
m_{32} &= c_6z_6, \quad p_1 = c_1c_2 - c_2c_4, \quad p_2 = c_2c_3 + c_1c_4, \quad p_3 = c_5c_7 - c_6c_8,\\
p_4 &= c_6c_7 + c_5c_8, \quad H_{11} = \cosh(y_4a)\cos(z_4a), \quad H_{12} = \sinh(y_8a)\cos(z_8a),\\
H_{13} &= \sinh(y_2a)\cos(z_2a), \quad H_{14} = \sinh(y_6s)\cos(z_6s), \quad G_{11} = \cosh(y_4a)\sin(z_4a),\\
G_{12} &= \cosh(y_8a)\sin(z_8a), \quad G_{13} = \cosh(y_2s)\sin(z_2s), \quad G_{14} = \cosh(y_6s)\sin(z_6s),\\
G_1 &= c_3e_3y_3 - c_4e_3z_3 + c_4e_4y_3 + c_3e_4z_3, \quad H_1 = c_3e_4y_3 - c_4e_4z_3 - c_4e_3y_3 - c_3e_3z_3,\\
G_2 &= c_3^2y_3 - c_4c_3z_3 + c_4^2y_3 + c_3c_4z_3, \quad H_2 = c_4c_3^2 - c_4^2z_3 - c_3c_4y_3 - c_3c_4z_3,\\
G_3 &= c_7e_7y_7 - c_8e_7z_7 + c_8e_8y_7 + c_7e_8z_7, \quad H_3 = c_8e_7y_7 + c_7e_7z_7 - c_7e_8y_7 + c_8e_8z_7,\\
G_4 &= c_4c_7y_7 - c_4c_8z_7 + c_4c_8y_7 + c_7c_8z_7, \quad H_4 = c_7c_8y_7 + c_7^2z_7 - c_7c_8y_7 + c_8^2z_7,\\
G_5 &= \phi_9^1, \quad H_5 = \phi_{10}^1, \quad G_6 = \phi_{11}^1, \quad H_6 = \phi_{12}^1.
\end{aligned}$$

5. Conclusion

Figures 3 and 5 show the velocity profiles for the fluid and dust particles, which are paraboloid in nature. From these it is observed that the flow of fluid particles is parallel to that of dust. Also the velocity of both fluid and dust particles, which are nearer to the axis of flow, move with the greater velocity. One can observe that the appreciable effect of Number density of the dust particles of both fluid and dust i.e., as N increases velocities of both the phases decreases. Further observation shows that if the dust is very fine i.e., mass of the dust particles is negligibly small then the relaxation time of dust particle decreases and ultimately as $\tau \to 0$ the velocities of fluid and dust particles will be the same. Also it is evident from the graphs that as time t increases the velocity of both the phases decreases ultimately for large time t the velocity becomes zero.

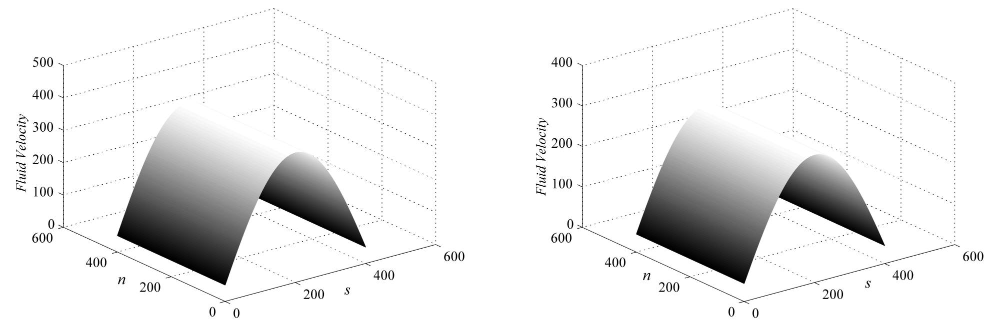

Figure-3: Variation of fluid velocity with s and n (for $t = 0.1$ & $t = 0.2$)

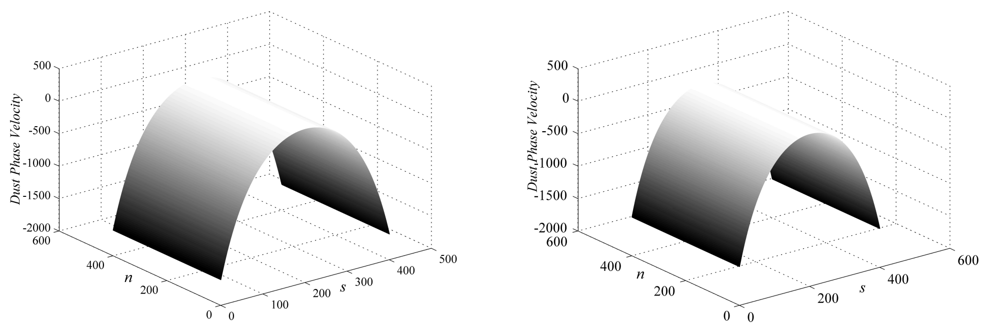

Figure-4: Variation of dust velocity with s and n (for $t = 0.1$ & $t = 0.2$)

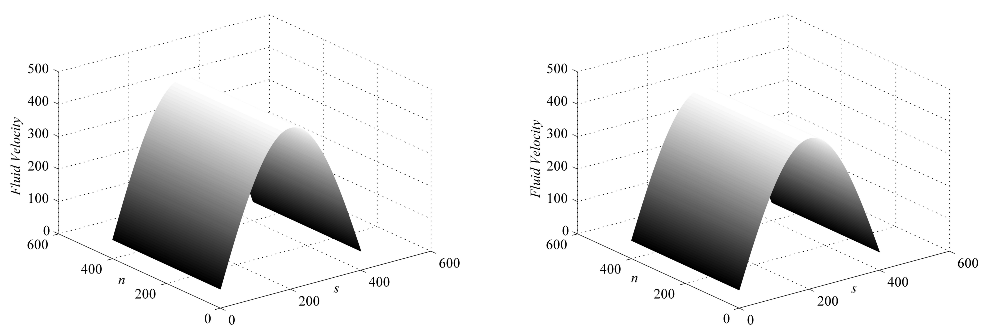

Figure-5: Variation of fluid velocity with s and n (for $N = 0.25$ & $N = 0.5$)

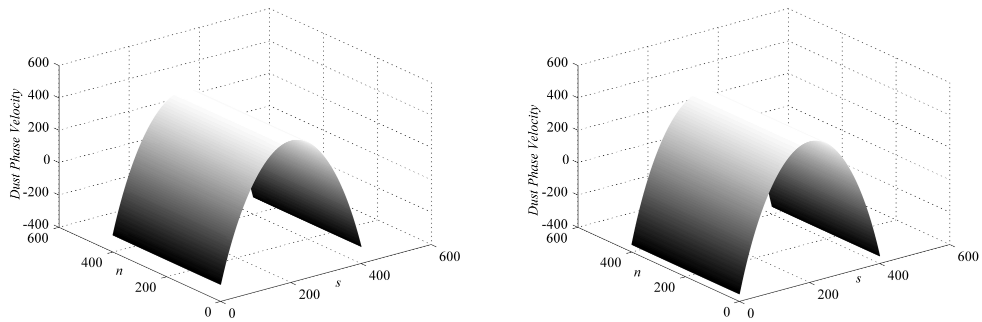

Figure-6: Variation of dust velocity with s and n (for $N = 0.25$ & $N = 0.5$)

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