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QUADRATIC FORMS, ELLIPTIC CURVES AND INTEGER SEQUENCES

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Abstract. In this work, we consider some properties of quadratic form $F(x, y) = 2x^2+3xy+y^2$. We show that this form is universal. Later we determine the number of rational points on elliptic curves related to F. In the last section, we define an integer sequence $A = A_n(P,Q)$ with parameters P and Q associated with F and derive some algebraic identities on it.

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1. Preliminaries

In this section we give some preliminaries on binary quadratic forms. Recall that a real binary quadratic form (or just a form) F is a polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2 \tag{1}$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. F is an integral form if and only if $a, b, c \in \mathbf{Z}$ and is indefinite if and only if $\Delta(F) > 0$. An indefinite definite form $F = (a, b, c)$ of discriminant Δ is said to be reduced if $\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}$. Most properties of quadratic forms can be giving by the aid of extended modular group $\overline{\Gamma}$ (see [9]). Gauss (1777-1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$\begin{aligned} F(x, y) \quad &= \quad a(r^2 + brs + cs^2)x^2 + (2art + bru + bts + 2csu)xy \\ &\quad\ +(at^2 + btu + cu^2)y^2 \end{aligned} \tag{2}$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} = [r; s; t; u] \in \overline{\Gamma}$, that is, gF is gotten from F by making the substitution $x \to rx + tu, y \to sx + uy$. Moreover, $\Delta(F) = \Delta(gF)$ for all $g \in \overline{\Gamma}$, that is, the action of $\overline{\Gamma}$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \overline{\Gamma}$. Let F and G be two forms. If there exists a $g \in \overline{\Gamma}$ such that $gF = G$, then F and G are called equivalent. If $\det g = 1$, then F and G are called properly equivalent and if $\det g = -1$, then F and G are called improperly equivalent. A quadratic form F is called ambiguous if it is improperly equivalent to itself. An element $g \in \overline{\Gamma}$ is called an automorphism of F if $gF = F$. If $\det g = 1$, then g is called a proper automorphism of F and if $\det g = -1$, then g is called an improper automorphism of F. Let $Aut(F)^+$ denote the set of proper automorphisms of F and let $Aut(F)^-$ denote the set of improper automorphisms of F (for further details on binary quadratic forms see [1, 2, 5, 8]).

Representation of integers (or primes) by binary quadratic forms has an important role on the theory of numbers and many authors. We considered this problem in [10, 11, 12, 13, 14, 15]. In fact, this problem intimately connected to reciprocity laws. The major problem of the theory of quadratic forms was: Given a quadratic form F, find all integers n that can be represented by F, that is, for which

$$F(x, y) = ax^2 + bxy + cy^2 = n. \tag{3}$$

This problem was studied for specific quadratic forms by Fermat, and intensively investigated by Euler. Fermat considered the representation of integers as sums of two squares. It was, however, Gauss in the Disquisitions [6] who made the fundamental breakthrough and developed a comprehensive and beautiful theory of binary quadratic forms. Most important was his definition of the composition of two forms and his proof that the (equivalence classes of) forms with a given discriminant Δ form a commutative group under this composition. A form F is called universal if it represents all integers (see [3, 4]).

2. Quadratic Forms and Elliptic Curves

In this section, we will consider some properties of quadratic form $F = (2, 3, 1)$ and then consider the number of rational points on elliptic curves E_F associated with F.

Theorem 2.1. *The form $F = (2, 3, 1)$ is universal.*

Proof. Let n be any integer. Then the quadratic equation

$$F(x, y) = 2x^2 + 3xy + y^2 = n$$

has a solution for $(x, y) = (1 - n, n - 2)$. Indeed,

$$F(1 - n, n - 2) = 2(1 - n)^2 + 3(1 - n)(n - 2) + (n - 2)^2 = n.$$

So F is universal.

Now we can give the following theorem concerning the automorphisms of F.

Theorem 2.2. *For the universal for $F = (2, 3, 1)$, we have*

$$Aut(F)^+ = \{\pm[1; 0; 0; 1]\} \quad and \quad Aut(F)^- = \{\pm[1; -3; 0; -1]\}.$$

Proof. Let $F = (2, 3, 1)$. Then by (2), the system of equations

$$\begin{aligned} 2r^2 + 3rs + s^2 &= 2 \\ 4rt + 3ru + 3ts + 2su &= 3 \\ 2t^2 + 3tu + u^2 &= 1 \end{aligned}$$

has a solution for $r = 1, s = 0, t = 0, u = 1$ and $r = -1, s = 0, t = 0, u = -1$, that is, $gF = F$ for $g = \pm[1; 0; 0; 1]$. Note that $\det(g) = 1$. So $Aut(F)^+ = \{\pm[1; 0; 0; 1]\}$. Also this system of equations has a solution for $r = 1, s = -3, t = 0, u = -1$ and $r = -1, s = 3, t = 0, u = 1$, that is, $gF = F$ for $g = \pm[1; -3; 0; -1]$. Note that $\det(g) = -1$. Hence $Aut(F)^- = \{\pm[1; -3; 0; -1]\}$.

From above theorem we can give the following corollary.

Corollary 2.3. *The universal form $F = (2, 3, 1)$ is ambiguous.*

Proof. Recall that a form is ambiguous if it is improperly equivalent to itself, that is, there exists at least one element $g \in \overline{\Gamma}$ such that $gF = F$. We show in above theorem that the sets of improper automorphism of F is non-empty. So it is ambiguous.

Now we generalize our definitions to finite fields $\mathbf{F}_p$ for a prime $p \geq 5$. A binary quadratic form F^p over $\mathbf{F}_p$ is a form in two variables x and y of the type

$$F^p = F^p(x, y) = ax^2 + bxy + cy^2, \tag{4}$$

where $a, b, c \in \mathbf{F}_p$. We denote F^p briefly by $F^p = (a, b, c)$. The discriminant of F^p is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta^p = \Delta^p(F^p)$. Let

$$\overline{\Gamma}^p = \{g^p = [r; s; t; u] : r, s, t, u \in \mathbf{F}_p \;\; and \;\; ru - st \equiv \pm 1 (mod\ p)\}.$$

Then we can see $\overline{\Gamma}^p$ as the extended modular group for $\mathbf{F}_p$. Let F^p and G^p be two forms over $\mathbf{F}_p$. If there exists a $g^p \in \overline{\Gamma}^p$ such that $g^p F^p = G^p$, then F^p and G^p are called equivalent. If $\det g^p = 1$, then F^p and G^p are called properly equivalent and if $\det g^p = p - 1$, then F^p and G^p are called improperly equivalent. A form F^p is called ambiguous if it is improperly equivalent to itself. An element $g^p \in \overline{\Gamma}^p$

is called an automorphism of F^p if $g^pF^p = F^p$. If $\det g^p \equiv 1(mod\ p)$, then g^p is called a proper automorphism and if $\det g^p \equiv -1(mod\ p)$, then g^p is called an improper automorphism. Let $Aut(F^p)_p^+$ denote the set of proper automorphisms and let $Aut(F^p)_p^-$ denote the set of improper automorphisms. Let

$$F^p(x,y) \equiv 2x^2 + 3xy + y^2 \quad (mod\ p) \tag{5}$$

be the quadratic form over $\mathbf{F}_p$. Then we can give the following theorem.

Theorem 2.4. *For the quadratic form F^p, we have*

$$\#Aut(F^p)_p^+ = \#Aut(F^p)_p^- = p-1$$

for every primes $p \geq 5$.

Proof. Let $F^p = (2,3,1)$. Then we have the following system of equations

$$\begin{aligned} 2r^2 + 3rs + s^2 &\equiv 2(mod\ p) \\ 4rt + 3ru + 3ts + 2su &\equiv 3(mod\ p) \\ 2t^2 + 3tu + u^2 &\equiv 1(mod\ p). \end{aligned} \tag{6}$$

Then there are $\frac{p-1}{2}$ points r such that (6) has a solution like this $[r_1; s_1; t_1; u_1]$ and $[p-r_1; s_2; t_2; u_2]$ with $ru - st \equiv 1(mod\ p)$. So there are $2\frac{p-1}{2} = p-1$ solutions and hence $\#Aut(F^p)_p^+ = p-1$. Also (6) has a solution $[r_2; s_3; t_3; u_3]$ and $[p-r_2; s_3; t_3; u_3]$ with $ru - st \equiv -1(mod\ p)$. So $\#Aut(F^p)_p^- = p-1$.

Now we will consider the number of representations of integers $n \in \mathbf{F}_p^*$ by universal quadratic form $F = (2,3,1)$. It is known that [7], to each quadratic form F, there corresponds the theta series

$$\wp(\tau; F) = 1 + \sum_{n=1}^{\infty} r(n; F) z^n, \tag{7}$$

where $r(n;F)$ is the number of representations of a positive integer n by the quadratic form F. We redefine (7) to any finite field $\mathbf{F}_p$. Let $F^p = (a,b,c)$ be a quadratic form over $\mathbf{F}_p$. Then (7) becomes

$$\wp^p(\tau; F^p) = 1 + \sum_{n \in \mathbf{F}_p^*} r^p(n; F^p) z^n, \tag{8}$$

where $r^p(n;F)$ is the number of representations of $n \in \mathbf{F}_p^*$ by F^p. Note that the theta series in (8) is determined by $r^p(n;F^p)$. So we have the find out $r^p(n;F^p)$. To get this, we can give the following theorem.

Theorem 2.5. *For the quadratic form F^p in (5), we get*

$$r^p(n;F^p) = \#Aut(F^p)_p^+$$

for every primes $p \geq 5$.

Proof. This is just to solve the quadratic equation

$$F^p(x,y) = 2x^2 + 3xy + y^2 \equiv n(mod\ p)$$

for $n \in \mathbf{F}_p^*$. In fact, it is easily seen that this quadratic congruence has $p-1$ integer solutions. Note that $\#Aut(F^p)_p^+ = p-1$. So $r^p(n;F^p) = \#Aut(F^p)_p^+$.

An elliptic curve E over a finite field F_p is defined by an equation in the Weierstrass form

$$E : y^2 = x^3 + ax^2 + bx, \tag{9}$$

where $a, b \in \mathbf{F}_p$ and $b^2(a^2-4b) \neq 0$ with discriminant $\Delta(E) = 16b^2(a^2-4b)$. If $\Delta(E) = 0$, then E is not an elliptic curve, it is a curve of genus 0 (in fact it is a singular curve). We can view an elliptic curve E as a curve in projective plane P^2, with a homogeneous equation $y^2z = x^3 + ax^2z^2 + bxz^3$, and one point at infinity, namely $(0,1,0)$. This point ∞ is the point where all vertical lines meet. We denote this point by O. The set of rational points (x,y) on E

$$E(\mathbf{F}_p) = \{(x,y) \in \mathbf{F}_p \times \mathbf{F}_p : y^2 = x^3 + ax^2 + bx\} \cup \{O\}$$

is a subgroup of E. The order of $E(\mathbf{F}_p)$, denoted by $\#E(\mathbf{F}_p)$, is defined as the number of the points on E and is given by

$$\#E(\mathbf{F}_p) = p + 1 + \sum_{x \in \mathbf{F}_p} \left(\frac{x^3 + ax^2 + bx}{\mathbf{F}_p} \right),$$

where $(\frac{\cdot}{\mathbf{F}_p})$ denotes the Legendre symbol (for the arithmetic of elliptic curves and rational points on them see [16, 17]).

Now we want to construct a connection between quadratic forms and elliptic curves. For this reason we first give the following definition.

Definition 2.6. *Let $F = (a,b,c)$ be a quadratic form of discriminant Δ. If $b = 1 + ac$, then F is called elliptic form.*

From above definition, we can say that an elliptic form F is a form of the type $F = (a, 1+ac, c)$ of discriminant $\Delta(F) = (1-ac)^2$. Now we can give the following theorem concerning the connection between elliptic forms and elliptic curves.

Theorem 2.7. *Let F be an elliptic form of discriminant $\Delta(F)$. Then there exists an elliptic curve E_F of discriminant $\Delta(E_F) = 16a^2c^2\Delta(F)$.*

Proof. Let $F = (a, b, c)$ be any quadratic form of discriminant $\Delta(F) = b^2 - 4ac$. Then we define the corresponding elliptic curve E_F as

$$E_F : y^2 = ax^3 + bx^2 + cx. \tag{10}$$

If we make the substitution $y' = ay$ and $x' = ax + 1$ in (10), then we get

$$E_F : y'^2 = x'^3 + (b-3)x'^2 + (3 - 2b + ac)x' + (-1 + b - ac). \tag{11}$$

Note that F is elliptic form, that is, $b = 1 + ac$. So (11) becomes

$$E_F : y'^2 = x'^3 + (ac-2)x'^2 + (1-ac)x'. \tag{12}$$

The discriminant of E_F is hence

$$\Delta(E_F) = 16(1-ac)^2[(ac-2)^2 - 4(1-ac)] = 16a^2c^2(1-ac)^2. \tag{13}$$

Since $\Delta(F) = (1-ac)^2$, (13) becomes $\Delta(E_F) = 16a^2c^2\Delta(F)$. This completes the proof.

Now we can return our problem. Note that the form $F = (2, 3, 1)$ is an elliptic form. So the corresponding elliptic curve is hence

$$E_F : y'^2 = x'^3 - x' \tag{14}$$

of discriminant $\Delta(E_F) = 64$ by (13). It is proved in [17] that the order of E_F is $p+1$ if $p \equiv 3(mod\ 4)$; $p+1+2a$ if $p \equiv 1(mod\ 4)$ and 1 is not a 4 th power $mod\ p$ or $p+1-2a$ if $p \equiv 1(mod\ 4)$ and 1 is a fourth power $mod\ p$, where a and b are integers with b is even and $a + b \equiv 1(mod\ 4)$. So we can give the following theorem.

Theorem 2.8. *For the elliptic curve in (14) we have*

$$\#E_F(\mathbf{F}_p) = \begin{cases} p+1 & if\ p \equiv 3(mod\ 4) \\ p+1 \pm 2a & if\ p \equiv 1(mod\ 4), \end{cases}$$

where a and b are integers with b is even and $a + b \equiv 1(mod\ 4)$.

3. Integer Sequence

In this section, we consider the integer sequence associated with the universal form obtained in Section 2. Note that the form $F = (2, 3, 1)$ is universal. Now set

$$Q = F(k, 1) = 2k^2 + 3k + 1 \quad and \quad P = F'(k, 1) = 4k + 3 \tag{15}$$

for an integer $k \neq -1$ (If $k = -1$, then we have the constant sequence $A_n = -1$ for all $n \geq 1$). Then we define the sequence $A = A_n(P, Q)$ as $A_0 = 0$, $A_1 = 1$ and

$$A_n = PA_{n-1} - QA_{n-2} = (4k+3)A_{n-1} - (2k^2+3k+1)A_{n-2} \tag{16}$$

for all $n \geq 2$. The characteristic equation of (16) is $x^2 - (4k+3)x + (2k^2+3k+1) = 0$. The discriminant is $D = (4k+3)^2 - 4(2k^2+3k+1) = 8k^2+12k+5$ and the roots of it are

$$\alpha = \frac{(4k+3)+\sqrt{D}}{2} \quad \textit{and} \quad \beta = \frac{(4k+3)-\sqrt{D}}{2}. \tag{17}$$

Hence by Binet's formula we get

$$A_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{18}$$

for $n \geq 1$. Then we can give the following theorems.

Theorem 3.1. *Let A_n denote the $n-th$ number. Then*

$$\sum_{i=1}^{n} A_i = \frac{A_{n+1} - (2k^2+3k+1)A_n - 1}{-2k^2+k+1}. \tag{19}$$

Proof. Note that $A_n = (4k+3)A_{n-1} - (2k^2+3k+1)A_{n-2}$. So $A_{n+2} = (4k+3)A_{n+1} - (2k^2+3k+1)A_n = A_{n+1} + (4k+2)A_{n+1} - (2k^2+3k+1)A_n$ and hence

$$A_{n+2} - A_{n+1} = (4k+2)A_{n+1} - (2k^2+3k+1)A_n. \tag{20}$$

Applying (20), we deduce that

$$\begin{aligned}
&n = 0 \Rightarrow A_2 - A_1 = (4k+2)A_1 - (2k^2+3k+1)A_0 \\
&n = 1 \Rightarrow A_3 - A_2 = (4k+2)A_2 - (2k^2+3k+1)A_1 \\
&n = 2 \Rightarrow A_4 - A_3 = (4k+2)A_3 - (2k^2+3k+1)A_2 \\
&\cdots \\
&n = n-1 \Rightarrow A_{n+1} - A_n = (4k+2)A_n - (2k^2+3k+1)A_{n-1} \\
&n = n \Rightarrow A_{n+2} - A_{n+1} = (4k+2)A_{n+1} - (2k^2+3k+1)A_n.
\end{aligned} \tag{21}$$

If we sum of both sides of (21), then we obtain

$$\begin{aligned}
A_{n+2} - A_1 \quad = \quad & [(4k+2) - (2k^2+3k+1)](A_1 + A_2 + \cdots + A_n) \\
& + (4k+2)A_{n+1} - (2k^2+3k+1)A_0.
\end{aligned} \tag{22}$$

Note that $A_0 = 0$ and $A_1 = 1$. So (22) becomes

$$A_{n+2} - 1 = (-2k^2+k+1)(A_1 + A_2 + \cdots + A_n) + (4k+2)A_{n+1}$$

and hence

$$A_1 + A_2 + \cdots + A_n = \frac{A_{n+2} - (4k+2)A_{n+1} - 1}{-2k^2 + k + 1}. \tag{23}$$

If we take $A_{n+2} = (4k+3)A_{n+1} - (2k^2+3k+1)A_n$ in (23), then we conclude that

$$A_1 + A_2 + \cdots + A_n = \frac{A_{n+1} - (2k^2+3k+1)A_n - 1}{-2k^2 + k + 1}.$$

Now we want to derive a recurrence relation on A_n numbers. To get this we can give the following theorem.

Theorem 3.2. *Let A_n denote the $n-th$ number. Then*

$$A_{2n} = (12k^2 + 18k + 7)A_{2n-2} - (4k^4 + 12k^3 + 13k^2 + 6k + 1)A_{2n-4}$$

and

$$A_{2n+1} = (12k^2 + 18k + 7)A_{2n-1} - (4k^4 + 12k^3 + 13k^2 + 6k + 1)A_{2n-3}$$

for all $n \geq 2$.

Proof. Recall that $A_n = (4k+3)A_{n-1} - (2k^2+3k+1)A_{n-2}$. So $A_{2n} = (4k+3)A_{2n-1} - (2k^2+3k+1)A_{2n-2}$ and hence

$$\begin{aligned}
A_{2n} &= (4k+3)A_{2n-1} - (2k^2+3k+1)A_{2n-2} \\
&= (4k+3)\Big[(4k+3)A_{2n-2} - (2k^2+3k+1)A_{2n-3}\Big] - (2k^2+3k+1)A_{2n-2} \\
&= \Big[(4k+3)^2 - (2k^2+3k+1)\Big] A_{2n-2} - (4k+3)(2k^2+3k+1)A_{2n-3} \\
&= \Big[(4k+3)^2 - (2k^2+3k+1)\Big] A_{2n-2} \\
&\quad -(4k+3)(2k^2+3k+1)\Big[(4k+3)A_{2n-4} - (2k^2+3k+1)A_{2n-5}\Big] \\
&= \Big[(4k+3)^2 - (2k^2+3k+1)\Big] A_{2n-2} - (4k+3)^2(2k^2+3k+1)A_{2n-4} \\
&\quad +(4k+3)(2k^2+3k+1)^2 A_{2n-5} \\
&= \Big[(4k+3)^2 - (2k^2+3k+1)\Big] A_{2n-2} - (2k^2+3k+1)A_{2n-2} \\
&\quad +(2k^2+3k+1)A_{2n-2} - (4k+3)^2(2k^2+3k+1)A_{2n-4} \\
&\quad +(4k+3)(2k^2+3k+1)^2 A_{2n-5} \\
&= \Big[(4k+3)^2 - 2(2k^2+3k+1)\Big] A_{2n-2} + (2k^2+3k+1)A_{2n-2} \\
&\quad -(4k+3)^2(2k^2+3k+1)A_{2n-4} + (4k+3)(2k^2+3k+1)^2 A_{2n-5} \\
&= \Big[(4k+3)^2 - 2(2k^2+3k+1)\Big] A_{2n-2}
\end{aligned}$$

$$
\begin{aligned}
& +(2k^2+3k+1)\left[(4k+3)A_{2n-3}-(2k^2+3k+1)A_{2n-4}\right] \\
& -(4k+3)^2(2k^2+3k+1)A_{2n-4}+(4k+3)(2k^2+3k+1)^2A_{2n-5} \\
= & \left[(4k+3)^2-2(2k^2+3k+1)\right]A_{2n-2}+(4k+3)(2k^2+3k+1)A_{2n-3} \\
& -(2k^2+3k+1)^2A_{2n-4}-(4k+3)^2(2k^2+3k+1)A_{2n-4} \\
& +(4k+3)(2k^2+3k+1)^2A_{2n-5} \\
= & \left[(4k+3)^2-2(2k^2+3k+1)\right]A_{2n-2} \\
& +(4k+3)(2k^2+3k+1)\left[(4k+3)A_{2n-4}-(2k^2+3k+1)^2A_{2n-5}\right] \\
& -(2k^2+3k+1)^2A_{2n-4}-(4k+3)^2(2k^2+3k+1)A_{2n-4} \\
& +(4k+3)(2k^2+3k+1)^2A_{2n-5} \\
= & \left[(4k+3)^2-2(2k^2+3k+1)\right]A_{2n-2}+(4k+3)^2(2k^2+3k+1)A_{2n-4} \\
& -(4k+3)(2k^2+3k+1)^2A_{2n-5}-(2k^2+3k+1)^2A_{2n-4} \\
& -(4k+3)^2(2k^2+3k+1)A_{2n-4}+(4k+3)(2k^2+3k+1)^2A_{2n-5} \\
= & \left[(4k+3)^2-2(2k^2+3k+1)\right]A_{2n-2}-(2k^2+3k+1)^2A_{2n-4} \\
= & (12k^2+18k+7)A_{2n-2}-(4k^4+12k^3+13k^2+6k+1)A_{2n-4}.
\end{aligned}
$$

The other assertion can be proved similarly.

We can also give the $n-$th number A_n by using the powers of $(4k+3)$ and $(8k^2+12k+5)$. To get this we can give the following theorem.

Theorem 3.3. *Let A_n denote the $n-$th number. Then*

$$
A_n=\frac{1}{2^{n-1}}\begin{cases} \sum_{i=1}^{\frac{n-2}{2}}\begin{pmatrix} n \\ 2i+1\end{pmatrix}(4k+3)^{n-(2i+1)}(8k^2+12k+5)^i & if\ n\ is\ even \\ \\ \sum_{i=1}^{\frac{n-1}{2}}\begin{pmatrix} n \\ 2i+1\end{pmatrix}(4k+3)^{n-(2i+1)}(8k^2+12k+5)^i & if\ n\ is\ odd \end{cases}
$$

for all $n\geq 1$.

Proof. Let n be even. Then by Binet's formula, we get

$$
\begin{aligned}
A_n &= \frac{\alpha^n-\beta^n}{\alpha-\beta} \\
&= \frac{\left(\frac{4k+3+\sqrt{D}}{2}\right)^n-\left(\frac{4k+3-\sqrt{D}}{2}\right)^n}{\sqrt{D}} \\
&= \frac{1}{2^n\sqrt{D}}\left[(4k+3+\sqrt{D})^n-(4k+3-\sqrt{D})^n\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2^{n-1}\sqrt{D}}\left[\begin{array}{c}\binom{n}{1}(4k+3)^{n-1}\sqrt{D}+\binom{n}{3}(4k+3)^{n-3}\sqrt{D}^3+\cdots \\ +\binom{n}{n-1}(4k+3)\sqrt{D}^{n-1}\end{array}\right] \\
&= \frac{1}{2^{n-1}}\left[\begin{array}{c}\binom{n}{1}(4k+3)^{n-1}+\binom{n}{3}(4k+3)^{n-3}D+\cdots \\ +\binom{n}{n-1}(4k+3)D^{\frac{n-2}{2}}\end{array}\right] \\
&= \frac{1}{2^{n-1}}\left[\sum_{i=1}^{\frac{n-2}{2}}\binom{n}{2i+1}(4k+3)^{n-(2i+1)}(8k^2+12k+5)^i\right].
\end{aligned}
$$

The other case can be proved similarly.

We can reformulate the $n-$th number A_n by using the powers of $(4k+3)$ and $(2k^2+3k+1)$. To get this we can give the following theorem without giving its proof since it can be proved as in same way that Theorem 3.3 was proved.

Theorem 3.4. *Let A_n denote the $n-th$ number. Then*

$$
A_n = \begin{cases} \sum_{i=0}^{\frac{n-2}{2}}\binom{n-1-i}{i}(-1)^i(4k+3)^{n-(2i+1)}(2k^2+3k+1)^i & if\ n\ is\ even \\ \\ \sum_{i=0}^{\frac{n-1}{2}}\binom{n-1-i}{i}(-1)^i(4k+3)^{n-(2i+1)}(2k^2+3k+1)^i & if\ n\ is\ odd \end{cases}
$$

Example 3.5. Let $k=5$. Then $A_n = 23A_{n-1} - 66A_{n-2}$ and hence

$$0, 1, 23, 463, 9131, 179455, 35224819, 69226807, 1359578507, 26701336399, \cdots.$$

Let $n=5$. Then

$$A_5 = \sum_{i=0}^{2}\binom{4-i}{i}(-1)^i 23^{5-(2i+1)}66^i = 23^4 - 3\cdot 23^2\cdot 66 + 66^2 = 179455$$

and let $n=8$, then

$$
\begin{aligned}
A_8 &= \sum_{i=0}^{3}\binom{7-i}{i}(-1)^i 23^{8-(2i+1)}66^i \\
&= 23^7 - 6\cdot 23^5\cdot 66 + 10\cdot 23^3\cdot 66^2 - 4\cdot 23\cdot 66^3 \\
&= 1359578507.
\end{aligned}
$$

Now we can give the following theorems related to powers of α and β.

Theorem 3.6. *Let A_n denote the $n-th$ number. Then*

$$A_{n+1}-(2k^2+3k+1)A_{n-1}=\alpha^n+\beta^n \tag{24}$$

for every $n\geq 1$.

Proof. Since $A_{n+1}=(4k+3)A_n-(2k^2+3k+1)A_{n-1}$, we get

$$\begin{aligned}
& A_{n+1}-(2k^2+3k+1)A_{n-1} \\
= & \Big[(4k+3)A_n-(2k^2+3k+1)A_{n-1}\Big]-(2k^2+3k+1)A_{n-1} \\
= & (4k+3)A_n-2(2k^2+3k+1)A_{n-1} \\
= & (4k+3)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)-2(2k^2+3k+1)\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right) \\
= & \frac{4k+3}{\sqrt{D}}(\alpha^n-\beta^n)-\frac{2(2k^2+3k+1)}{\sqrt{D}}\left(\frac{\alpha^n}{\alpha}-\frac{\beta^n}{\beta}\right) \\
= & \frac{4k+3}{\sqrt{D}}(\alpha^n-\beta^n)-\frac{2}{\sqrt{D}}(\alpha^n\beta-\alpha\beta^n) \\
= & \alpha^n\left[\frac{4k+3-2\beta}{\sqrt{D}}\right]+\beta^n\left[\frac{-4-3+2\alpha}{\sqrt{D}}\right] \\
= & \alpha^n+\beta^n.
\end{aligned}$$

Theorem 3.7. *Let A_n denote the $n-th$ number. Then*

$$2A_{n+1}-(4k+3)A_n=\alpha^n+\beta^n \tag{25}$$

for every $n\geq 1$.

Proof. We proved in above theorem that $\alpha^n+\beta^n=A_{n+1}-(2k^2+3k+1)A_{n-1}$. So

$$A_{n-1}=\frac{A_{n+1}-(\alpha^n+\beta^n)}{2k^2+3k+1}$$

and hence

$$\begin{aligned}
A_n+A_{n+1} &= A_n+[(4k+3)A_n-(2k^2+3k+1)A_{n-1}] \\
&= (4k+4)A_n-(2k^2+3k+1)A_{n-1} \\
&= (4k+4)A_n-(2k^2+3k+1)\frac{A_{n+1}-(\alpha^n+\beta^n)}{2k^2+3k+1} \\
&= (4k+4)A_n-A_{n+1}+(\alpha^n+\beta^n).
\end{aligned}$$

Consequently, $2A_{n+1}-(4k+3)A_n=\alpha^n+\beta^n$.

Theorem 3.8. *Let A_n denote the $n-th$ number. Then*

$$(4k+3)A_n - (4k^2+6k+2)A_{n-1} = \alpha^n + \beta^n \tag{26}$$

and

$$\alpha^n + \beta^n = \begin{cases} \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n}{2}} \begin{pmatrix} n \\ 2i \end{pmatrix} (4k+3)^{n-2i}(8k^2+12k+5)^i & if\ n\ is\ even \\ \\ \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-1}{2}} \begin{pmatrix} n \\ 2i \end{pmatrix} (4k+3)^{n-2i}(8k^2+12k+5)^i & if\ n\ is\ odd \end{cases}$$

for $n \geq 1$.

Proof. The first assertion can be proved as in the same way that Theorems 3.6 and 3.7 were proved. The second assertion is just an application to Binomial series.

Now we set the following identities

$$\begin{aligned} M &= \frac{-4k^2-2k+1+\sqrt{D}}{2\sqrt{D}},\ N = -2k^2+k+1,\ L = \frac{4k+5+\sqrt{D}}{2\sqrt{D}} \\ H &= \frac{40k^3+90k^2+73k+21+(14k^2+21k+9)\sqrt{D}}{2\sqrt{D}} \\ K &= \frac{4k^2+2k-1+\sqrt{D}}{(4k^2+6k+2)\sqrt{D}}. \end{aligned}$$

Then we can give the following theorem.

Theorem 3.9. *Let A_n denote the $n-th$ number. Then*

1. *The sum of first non-zero A_n number is $\frac{1}{N}[M\alpha^n - \overline{M}\beta^n - 1]$.*
2. *$A_n + A_{n+1} = L\alpha^n - \overline{L}\beta^n$ for $n \geq 0$.*
3. *$A_{n+1} + A_{n-1} = H\alpha^{n-2} - \overline{H}\beta^{n-2}$ for $n \geq 2$.*
4. *$A_n - A_{n-1} = K\alpha^n - \overline{K}\beta^n$ for $n \geq 1$.*

Proof. 1. We proved in Theorem 3.6 that $\alpha^n + \beta^n = A_{n+1} - (2k^2+3k+1)A_{n-1}$. So

$$\begin{aligned} \alpha^{n+1} + \beta^{n+1} &= A_{n+2} - (2k^2+3k+1)A_n \\ &= [(4k+3)A_{n+1} - (2k^2+3k+1)A_n] - (2k^2+3k+1)A_n \\ &= (4k+3)A_{n+1} - 2(2k^2+3k+1)A_n \\ &= (4k+2)A_{n+1} + A_{n+1} - 2(2k^2+3k+1)A_n \\ &= [A_{n+1} - (2k^2+3k+1)A_n] + (4k+2)A_{n+1} - (2k^2+3k+1)A_n \end{aligned}$$

and hence

$$\begin{aligned}
&A_{n+1}-(2k^2+3k+1)A_n\\
&=\alpha^{n+1}+\beta^{n+1}-(4k+2)A_{n+1}+(2k^2+3k+1)A_n\\
&=\alpha^{n+1}+\beta^{n+1}-(4k+2)\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)+(2k^2+3k+1)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\\
&=\alpha^n\left(\alpha-\frac{(4k+2)\alpha}{\sqrt{D}}+\frac{2k^2+3k+1}{\sqrt{D}}\right)+\beta^n\left(\beta+\frac{(4k+2)\beta}{\sqrt{D}}-\frac{2k^2+3k+1}{\sqrt{D}}\right)\\
&=\alpha^n\left(\frac{-4k^2-2k+1+\sqrt{D}}{2\sqrt{D}}\right)-\beta^n\left(\frac{-4k^2-2k+1-\sqrt{D}}{2\sqrt{D}}\right)\\
&=M\alpha^n-\overline{M}\beta^n.
\end{aligned}$$

Hence applying Theorem 3.1, the result is clear.

2. Note that $A_{n+1}=(4k+3)A_n-(2k^2+3k+1)A_{n-1}$. So

$$\begin{aligned}
A_{n+1}+A_n &= (4k+4)A_n-(2k^2+3k+1)A_{n-1}\\
&= (4k+4)\left(\frac{\alpha^n-\beta^n}{\sqrt{D}}\right)-(2k^2+3k+1)\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{D}}\right)\\
&= (4k+4)\left(\frac{\alpha^n-\beta^n}{\sqrt{D}}\right)-\left(\frac{\beta\alpha^n-\alpha\beta^n}{\sqrt{D}}\right)\\
&= \alpha^n\left(\frac{4k+4-\beta}{\sqrt{D}}\right)-\beta^n\left(\frac{4k+4-\alpha}{\sqrt{D}}\right)\\
&= \alpha^n\left(\frac{4k+5+\sqrt{D}}{2\sqrt{D}}\right)-\beta^n\left(\frac{4k+5-\sqrt{D}}{2\sqrt{D}}\right)\\
&= L\alpha^n-\overline{L}\beta^n.
\end{aligned}$$

3. We proved in Theorem 3.7 that $\alpha^n+\beta^n=2A_{n+1}-(4k+3)A_n$. So we get

$$\begin{aligned}
\alpha^n+\beta^n &= 2A_{n+1}-(4k+3)A_n\\
&= 2A_{n+1}-(4k+3)[(4k+3)A_{n-1}-(2k^2+3k+1)A_{n-2}]\\
&= 2A_{n+1}-(4k+3)^2A_{n-1}+(4k+3)(2k^2+3k+1)A_{n-2}\\
&= 2(A_{n+1}+A_{n-1})-(16k^2+24k+11)A_{n-1}+(8k^3+18k^2+13k+3)A_{n-2}
\end{aligned}$$

and hence

$$\begin{aligned}
&A_{n+1}+A_{n-1}\\
=\ &\frac{\alpha^n+\beta^n+(16k^2+24k+11)A_{n-1}-(8k^3+18k^2+13k+3)A_{n-2}}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^n+\beta^n+\frac{(16k^2+24k+11)}{\sqrt{D}}(\alpha^{n-1}-\beta^{n-1})-\frac{8k^3+18k^2+13k+3}{\sqrt{D}}(\alpha^{n-2}-\beta^{n-2})}{2} \\
&= \frac{\alpha^n}{2}\left[1+\frac{16k^2+24k+11}{\sqrt{D}}\cdot\frac{1}{\alpha}-\frac{8k^3+18k^2+13k+3}{\sqrt{D}}\cdot\frac{1}{\alpha^2}\right] \\
&\quad +\frac{\beta^n}{2}\left[1-\frac{16k^2+24k+11}{\sqrt{D}}\cdot\frac{1}{\beta}+\frac{8k^3+18k^2+13k+3}{\sqrt{D}}\cdot\frac{1}{\beta^2}\right] \\
&= \alpha^{n-2}\left[\frac{40k^3+90k^2+73k+21+(14k^2+21k+9)\sqrt{D}}{2\sqrt{D}}\right] \\
&\quad -\beta^{n-2}\left[\frac{40k^3+90k^2+73k+21-(14k^2+21k+9)\sqrt{D}}{2\sqrt{D}}\right] \\
&= H\alpha^{n-2}-\overline{H}\beta^{n-2}.
\end{aligned}
$$

4. We proved in Theorem 3.6 that $\alpha^n+\beta^n = A_{n+1}-(2k^2+3k+1)A_{n-1}$. Hence

$$A_{n+1} = \alpha^n+\beta^n+(2k^2+3k+1)A_{n-1}. \tag{27}$$

Further $A_{n+1}-(2k^2+3k+1)A_n = M\alpha^n-\overline{M}\beta^n$. Hence

$$A_{n+1} = M\alpha^n-\overline{M}\beta^n+(2k^2+3k+1)A_n. \tag{28}$$

Applying (27) and (28), we obtain $\alpha^n+\beta^n+(2k^2+3k+1)A_{n-1} = M\alpha^n-\overline{M}\beta^n+(2k^2+3k+1)A_n$ and hence

$$
\begin{aligned}
A_n-A_{n-1} &= \frac{\alpha^n+\beta^n-M\alpha^n+\overline{M}\beta^n}{2k^2+3k+1} \\
&= \frac{\alpha^n(1-M)+\beta^n(1+\overline{M})}{2k^2+3k+1} \\
&= \frac{\alpha^n\left(1-\frac{-4k^2-2k+1+\sqrt{D}}{2\sqrt{D}}\right)+\beta^n\left(1+\frac{-4k^2-2k+1-\sqrt{D}}{2\sqrt{D}}\right)}{2k^2+3k+1} \\
&= \alpha^n\left(\frac{4k^2+2k-1+\sqrt{D}}{(4k^2+6k+2)\sqrt{D}}\right)-\beta^n\left(\frac{4k^2+2k+1-\sqrt{D}}{(4k^2+6k+2)\sqrt{D}}\right) \\
&= K\alpha^n-\overline{K}\beta^n.
\end{aligned}
$$

Now we can formulate the sum of even and odd numbers A_{2n} and A_{2n-1}, respectively by using the powers of α and β.

Theorem 3.10. *Let A_n denote the $n-th$ number. Then*

$$\sum_{i=1}^{n} A_{2i} = \begin{cases} H\sum_{i=1}^{\frac{n}{2}} \alpha^{4i-3} - \overline{H}\sum_{i=1}^{\frac{n}{2}} \beta^{4i-3} & if\ n\ is\ even \\ \frac{\alpha^{2n}}{\sqrt{D}} + H\sum_{i=1}^{\frac{n-1}{2}} \alpha^{4i-3} - \frac{\beta^{2n}}{\sqrt{D}} - \overline{H}\sum_{i=1}^{\frac{n-1}{2}} \beta^{4i-3} & if\ n\ is\ odd \end{cases}$$

and

$$\sum_{i=1}^{n} A_{2i-1} = \begin{cases} H\sum_{i=1}^{\frac{n}{2}} \alpha^{4i-4} - \overline{H}\sum_{i=1}^{\frac{n}{2}} \beta^{4i-4} & if\ n\ is\ even \\ \frac{\alpha^{2n-1}}{\sqrt{D}} + H\sum_{i=1}^{\frac{n-1}{2}} \alpha^{4i-4} - \frac{\beta^{2n-1}}{\sqrt{D}} - \overline{H}\sum_{i=1}^{\frac{n-1}{2}} \beta^{4i-4} & if\ n\ is\ odd \end{cases}$$

Proof. We proved in (3) of Theorem 3.9 that $A_{n+1} + A_{n-1} = H\alpha^{n-2} - \overline{H}\beta^{n-2}$ for $n \geq 2$. Now let n be even. Then

$$\begin{aligned} \sum_{i=1}^{n} A_{2i} &= (A_2 + A_4) + (A_6 + A_8) + \cdots + (A_{2n-2} + A_{2n}) \\ &= (H\alpha - \overline{H}\beta) + (H\alpha^5 - \overline{H}\beta^5) + \cdots + (H\alpha^{2n-3} - \overline{H}\beta^{2n-3}) \\ &= H(\alpha + \alpha^5 + \cdots + \alpha^{2n-3}) - \overline{H}(\beta + \beta^5 + \cdots + \beta^{2n-3}) \\ &= H\sum_{i=1}^{\frac{n}{2}} \alpha^{4i-3} - \overline{H}\sum_{i=1}^{\frac{n}{2}} \beta^{4i-3} \end{aligned}$$

and let n be odd, then

$$\begin{aligned} \sum_{i=1}^{n} A_{2i} &= (A_2 + A_4) + (A_6 + A_8) + \cdots + (A_{2n-4} + A_{2n-2}) + A_{2n} \\ &= (H\alpha - \overline{H}\beta) + (H\alpha^5 - \overline{H}\beta^5) + \cdots + (H\alpha^{2n-5} - \overline{H}\beta^{2n-5}) + \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \\ &= \frac{\alpha^{2n}}{\sqrt{D}} + H(\alpha + \alpha^5 + \cdots + \alpha^{2n-5}) - \frac{\beta^{2n}}{\sqrt{D}} - \overline{H}(\beta + \beta^5 + \cdots + \beta^{2n-5}) \\ &= \frac{\alpha^{2n}}{\sqrt{D}} + H\sum_{i=1}^{\frac{n-1}{2}} \alpha^{4i-3} - \frac{\beta^{2n}}{\sqrt{D}} - \overline{H}\sum_{i=1}^{\frac{n-1}{2}} \beta^{4i-3}. \end{aligned}$$

The other assertion can be proved similarly.

Applying the formal power series we get the following theorem.

Theorem 3.11. *Let A_n denote the $n-th$ number. Then*

$$\sum_{n=0}^{\infty} A_n x^n = \frac{x}{1 - (4k+3)x + (2k^2 + 3k + 1)x^2}.$$

For the A_n numbers, we set

$$M(A_n) = \begin{pmatrix} 4k+3 & -2k^2-3k-1 \\ 1 & 0 \end{pmatrix} \quad and \quad N(A_n) = \begin{pmatrix} 4k+3 & 1 \\ 1 & 0 \end{pmatrix} \tag{29}$$

Then we can give the following theorem.

Theorem 3.12. *Let A_n denote the $n-th$ number. Then*

1.
$$\begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = M(A_n)^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{30}$$
for all $n \geq 2$.

2.
$$\begin{pmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{pmatrix} = M(A_n)^{n-1} N(A_n) \tag{31}$$
for all $n \geq 1$.

Proof. 1. We prove the theorem by induction on n. Let $n = 2$. Then

$$\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} = \begin{pmatrix} 4k+3 & -2k^2-3k-1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4k+3 \\ 1 \end{pmatrix}.$$

So (30) is true for $n = 2$. Let us assume that this relation is satisfied for $n-1$, that is,

$$\begin{pmatrix} A_{n-1} \\ A_{n-2} \end{pmatrix} = M(A_n)^{n-2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then it is easily seen that

$$\begin{aligned} \begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} &= M(A_n)^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M(A_n) \cdot M(A_n)^{n-2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M(A_n) \begin{pmatrix} A_{n-1} \\ A_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} (4k+3)A_{n-1} - (2k^2+3k+1)A_{n-2} \\ A_{n-1} \end{pmatrix}. \end{aligned}$$

Hence (30) is true for n since $A_n = (4k+3)A_{n-1} - (2k^2+3k+1)A_{n-2}$.

2. We prove it by induction on n. Let $n = 1$. Then

$$\begin{pmatrix} A_2 & A_1 \\ A_1 & A_0 \end{pmatrix} = N = \begin{pmatrix} A_2 & A_1 \\ A_1 & A_0 \end{pmatrix}.$$

So (31) is true for $n = 1$. Let us assume that this relation is satisfied for $n-1$, that is,

$$\begin{pmatrix} A_n & A_{n-1} \\ A_{n-1} & A_{n-2} \end{pmatrix} = M(A_n)^{n-2}N(A_n).$$

Then it is easily seen that

$$\begin{aligned} \begin{pmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{pmatrix} &= M(A_n)M(A_n)^{n-2}N(A_n) = M(A_n)\begin{pmatrix} A_n & A_{n-1} \\ A_{n-1} & A_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} (4k+3)A_n - (2k^2+3k+1)A_{n-1} & (4k+3)A_{n-1} - (2k^2+3k+1)A_{n-2} \\ A_n & A_{n-1} \end{pmatrix}. \end{aligned}$$

This completes the proof.

From above theorem we can give the following result.

Theorem 3.13. *Let A_n denote the $n-th$ number. Then*

1. $A_{n+1}A_{n-1} - A_n^2 = -(2k^2+3k+1)^{n-1}$.
2. $A_{n+1}^2 - (4k+3)A_{n+1}A_n + (2k^2+3k+1)A_n^2 = (2k^2+3k+1)^n$.

Proof. 1. Note that $\det(N(A_n)) = -1$ and $\det(M(A_n)) = 2k^2+3k+1$. So taking the determinant of both sides of (31) yields $A_{n+1}A_{n-1} - A_n^2 = -(2k^2+3k+1)^{n-1}$.

2. Recall that $A_n = (4k+3)A_{n-1} - (2k^2+3k+1)A_{n-2}$. So $A_{n+1} = (4k+3)A_n - (2k^2+3k+1)A_{n-1}$ and hence

$$\begin{aligned} & A_{n+1}^2 - (4k+3)A_{n+1}A_n + (2k^2+3k+1)A_n^2 \\ = \; & [(4k+3)A_n - (2k^2+3k+1)A_{n-1}]^2 \\ & -(4k+3)[(4k+3)A_n - (2k^2+3k+1)A_{n-1}]A_n + (2k^2+3k+1)A_n^2 \\ = \; & (4k+3)^2A_n^2 - 2(4k+3)(2k^2+3k+1)A_nA_{n-1} + (2k^2+3k+1)^2A_{n-1}^2 \\ & -(4k+3)^2A_n^2 + (4k+3)(2k^2+3k+1)A_{n-1}A_n + (2k^2+3k+1)A_n^2 \\ = \; & -(4k+3)(2k^2+3k+1)A_nA_{n-1} + (2k^2+3k+1)^2A_{n-1}^2 + (2k^2+3k+1)A_n^2 \\ = \; & -(2k^2+3k+1)A_{n-1}[(4k+3)A_n - (2k^2+3k+1)A_{n-1}] + (2k^2+3k+1)A_n^2 \\ = \; & -(2k^2+3k+1)A_{n-1}A_{n+1} + (2k^2+3k+1)A_n^2 \\ = \; & -(2k^2+3k+1)[A_{n+1}A_{n-1} - A_n^2] \\ = \; & -(2k^2+3k+1)[-(2k^2+3k+1)^{n-1}] \\ = \; & (2k^2+3k+1)^n. \end{aligned}$$

3.1. Simple Continued Fraction Expansion of A_n Numbers In this sub section, we want to consider the continued fraction expansion of A_n numbers. Recall that a continued fraction is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cdots \cdots a_{n-3} + \cfrac{b_{n-3}}{a_{n-2} + \frac{b_{n-2}}{a_{n-1}}}}}. \tag{32}$$

In general, the a_n' s and b_n' s of (32) may be real or complex numbers. However, if each b_n is equal to 1 and each a_n is an integer such that $a_n > 0$ for $n > 1$, then the continued fraction is called simple continued fraction. So a simple continued fraction of order n is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cdots + \frac{1}{a_n}}} \tag{33}$$

which can be abbreviated as $[a_0; a_1, a_2, \cdots, a_n]$. Now we first give the following result.

Theorem 3.14. *Let A_n denote the $n-th$ number.*

1. *If $k = 1$, then $A_n = 6^{n-1} + 6^{n-2} + \cdots + 6^1 + 1$ for every $n \geq 1$.*
2. *If $k = 1$, then $A_{n+1} - 6A_n = 1$ for every $n \geq 1$.*
3. *If $k = 0$, then $A_{n+1} - A_{n-1} = \alpha^n + \beta^n$ for every $n \geq 1$.*

Proof. 1. Let $k = 1$. Then $A_n = 7A_{n-1} - 6A_{n-2}$ and also $\alpha = 6$ and $\beta = 1$. So by Binet's formula we deduce that

$$A_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{6^n - 1}{6 - 1} = 6^{n-1} + 6^{n-2} + \cdots + 6^1 + 1.$$

2. Applying Binet's formula we get

$$A_{n+1} - 6A_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - 6\frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^n\left(\frac{\alpha - 6}{5}\right) + \beta^n\left(\frac{-\beta + 6}{5}\right) = \beta^n = 1$$

since $\alpha = 6$ and $\beta = 1$.

3. Let $k=0$. Then $A_n = 3A_{n-1} - A_{n-2}$. Also $\alpha = \frac{3+\sqrt{5}}{3}$ and $\beta = \frac{3-\sqrt{5}}{2}$. So

$$\begin{aligned} A_{n+1} - A_{n-1} &= \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) \\ &= \alpha^n \left(\frac{\alpha - \frac{1}{\alpha}}{\sqrt{5}}\right) + \beta^n \left(\frac{-\beta + \frac{1}{\beta}}{\sqrt{5}}\right) \\ &= \alpha^n \left(\frac{\alpha^2 - 1}{\alpha\sqrt{5}}\right) + \beta^n \left(\frac{1 - \beta^2}{\beta\sqrt{5}}\right) \\ &= \alpha^n + \beta^n. \end{aligned}$$

Now we can return our problem.

Theorem 3.15. *Let A_n denote the $n-th$ number.*

1. *If $k=1$, then*

$$\begin{aligned} \frac{A_{n+1}}{A_n} &= [6; 6^{n-1} + 6^{n-2} + \cdots + 6^1 + 1] \\ \frac{A_{2n+1}}{A_{2n-1}} &= [36; 6^{2n-3} + 6^{2n-5} + \cdots + 6, 6+1] \end{aligned}$$

for every $n \geq 1$ and

$$\frac{A_{2n}}{A_{2n-2}} = [36; 6^{2n-4} + 6^{2n-6} + \cdots + 6^2 + 1]$$

for every $n \geq 3$.

2. *If $k=0$, then*

$$\begin{aligned} \frac{A_{n+1}}{A_n} &= [2; (\overline{1,1})_{2n-3}, 2] \\ \frac{A_{2n+1}}{A_{2n-1}} &= [6; (\overline{1,5})_{n-2}, 1, 6+1], \end{aligned}$$

where $(\overline{1,1})_{2n-3}$ means that there are $2n-3$ successive terms $1,1$ and $(\overline{1,5})_{n-2}$ means that there are $n-2$ successive terms $1,5$ for every $n \geq 2$ and

$$\frac{A_{2n}}{A_{2n-2}} = [6; (\overline{1,5})_{n-3}, 1, 6],$$

where $(\overline{1,5})_{n-3}$ means that there are $n-3$ successive terms $1,5$ for every $n \geq 3$.

Proof. 1. Let $k=1$. Then $A_n = 6^{n-1}+6^{n-2}+\cdots+6^1+1$ by (1) of Theorem 3.14. A straightforward calculation shows that

$$\begin{aligned}
[6; 6^{n-1}+6^{n-2}+\cdots+6^1+1] &= 6+\frac{1}{6^{n-1}+6^{n-2}+\cdots+6+1}\\
&= \frac{6^n+6^{n-1}+\cdots+6+1}{6^{n-1}+6^{n-2}+\cdots+6+1}\\
&= \frac{A_{n+1}}{A_n}.
\end{aligned}$$

Similarly we find that

$$\begin{aligned}
[36; 6^{2n-3}+6^{2n-5}+\cdots+6, 6+1] &= 36+\frac{1}{6^{2n-3}+6^{2n-5}+\cdots+6+\frac{1}{6+1}}\\
&= 36+\frac{6+1}{(6+1)(6^{2n-3}+6^{2n-5}+\cdots+6)+1}\\
&= \frac{6^{2n}+6^{2n-1}+\cdots+6^2+6+1}{6^{2n-2}+6^{2n-3}+\cdots+6+1}\\
&= \frac{A_{2n+1}}{A_{2n-1}}
\end{aligned}$$

and

$$\begin{aligned}
[36; 6^{2n-4}+6^{2n-6}+\cdots+6^2+1] &= 36+\frac{1}{6^{2n-4}+6^{2n-6}+\cdots+6^2+1}\\
&= \frac{6^{2n-2}+6^{2n-4}+\cdots+6^4+6^2}{6^{2n-4}+6^{2n-6}+\cdots+6^2+1}\\
&= \frac{6^{2n-1}+6^{2n-2}+\cdots+6^2+6+1}{6^{2n-3}+6^{2n-4}+\cdots+6^2+6+1}\\
&= \frac{A_{2n}}{A_{2n-2}}.
\end{aligned}$$

2. It can be proved similarly.

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