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SOME FIXED POINT THEOREMS FOR WEAK CONTRACTION CONDITIONS OF INTEGRAL TYPE

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Abstract. In this paper, we shall establish two fixed point theorems by following the concept of [9, 21] and using weak contractions of the integral type.

Our results are generalizations of the classical Banach's fixed point theorem [1, 2, 3, 4, 5, 7, 25] as well as extensions of some other results of Berinde [4, 5, 6, 7], Berinde and Berinde [8], Branciari [9], Chatterjea [10], Kannan [16] and Zamfirescu [24].

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1. Introduction

Let (X, d) be a complete metric space and $f : X \to X$ a selfmap of X. Suppose that $F_f = \{x \in X \mid f(x) = x\}$ is the set of fixed points of f. The classical Banach's fixed point theorem is established in Banach [3] by using the following contractive definition: there exists $c \in [0, 1)$ (fixed) such that $\forall\ x,\ y \in X$, we have

$$d(f(x), f(y)) \leq c\ d(x, y). \tag{1}$$

In a recent paper of Branciari [9], a generalization of Banach [3] is established. In that paper, Branciari [9] employed the following contractive integral inequality condition: there exists $c \in [0, 1)$ such that $\forall\ x,\ y \in X$, we have

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt, \tag{2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$.
Rhoades [21] used the conditions

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt,\ \forall\ x,\ y \in X, \tag{3}$$

where $m(x, y) = \max\left\{\mathrm{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x,f(y))+d(y,f(x))}{2}}\right\}$,
and

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq k \int_0^{M(x,y)} \varphi(t)dt, \; \forall \; x, \; y \in X, \tag{4}$$

with $M(x, y) = \max\{\mathrm{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))}\}$,
where $k \in [0, 1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ in both cases is as defined in (2). Condition (4) is the integral form of Ciric's condition in Ciric [12].

Literature abounds with several generalizations of the classical Banach's fixed point theorem since 1922. For some of these generalizations of the classical Banach's fixed point theorem and various contractive definitions that have been employed, we refer the readers to [1, 2, 4, 5, 6, 8, 11, 12, 20, 24] and other references listed in the reference section of this paper.

In this paper, we shall establish a fixed point result similar to those of Branciari [9] and Rhoades [21] by employing a weak contraction of the integral type.

Our result is a generalization of the classical Banach's fixed point theorem [1, 3, 5, 25] as well as an extension of some results of Berinde [6], Berinde and Berinde [8], Branciari [9], Chatterjea [10], Kannan [16] and Zamfirescu [24].
The following definition is taken from Berinde [6]:
Definition 1.1 *A single-valued mapping $f : X \to X$ is called a weak contraction or $(\delta, \; L)-$weak contraction if and only if there exist two constants, $\delta \in [0, 1)$ and $L \geq 0$, such that*

$$d(f(x), f(y)) \leq \delta d(x, y) + Ld(y, f(x)), \; \forall \; x, \; y \in X. \tag{5}$$

For the extension of the Banach's fixed point theorem in the sense of multi-valued mapping, the reader is referred to Berinde and Berinde [8]. We shall employ the following definitions in the sequel to obtain our results :
Definition 1.2 *A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if it satisfies the following conditions:*
(i) ψ is monotone increasing;
(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0, \; \forall \; t \geq 0.$
Remark 1.3 Every comparison function satisfies $\psi(0) = 0$.
Definition 1.4 *We shall say that a single-valued mapping $f : X \to X$ is a $(L, \; \psi)-$weak contraction of integral type if and only if there exist a constant $L \geq 0$ and a continuous comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall \; x, \; y \in X$,*

$$\begin{aligned}\textstyle\int_0^{d(f(x),f(y))} \varphi(t)d\nu(t) \quad &\leq L \left(\textstyle\int_0^{d(x,f(x))} \varphi(t)d\nu(t)\right)^r \left(\textstyle\int_0^{d(y,f(x))} \varphi(t)d\nu(t)\right) \\ &\quad + \psi\left(\textstyle\int_0^{d(x,y)} \varphi(t)d\nu(t)\right),\end{aligned} \tag{6}$$

where $r \geq 0$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative such that for each* $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)d\nu(t) > 0$ and $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ *is a monotone increasing function.*

Definition 1.5 *We shall say that a single-valued mapping* $f : X \to X$ *is a* $(\Phi,\ \psi)-$*weak contraction of integral type if and only if there exists a continuous monotone increasing function* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\Phi(0) = 0$ *and a continuous comparison function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\forall\ x,\ y \in X$,

$$\begin{aligned} \int_0^{d(f(x),f(y))} \varphi(t)d\nu(t) &\leq \Phi\left(\int_0^{d(x,f(x))} \varphi(t)d\nu(t)\right)\left(\int_0^{d(y,f(x))} \varphi(t)d\nu(t)\right) \\ &\quad + \psi\left(\int_0^{d(x,y)} \varphi(t)d\nu(t)\right), \end{aligned} \tag{7}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative such that for each* $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)d\nu(t) > 0$ and $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ *is also monotone increasing.*

Remark 1.6 The contractive conditions (6) and (7) do not require any additional conditions for the uniqueness of the fixed point of f. This is an improvement on the result of Berinde [6].

Remark 1.7 The contractive condition (6) reduces to (5) if $r = 0,\ \nu(t) = t,\ \varphi(t) = 1$,
$\forall\ t \in \mathbb{R}^+$ and $\psi(\mathrm{u}) = \delta\mathrm{u}\ \forall\ \mathrm{u} \in \mathbb{R}^+$. In a similar manner, condition (7) also reduces to (5), (6) and some other interesting and well-known contractive conditions in the literature. Also, see Olatinwo [18] for more fixed point theorems.

2.The second section

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3.The main results

Theorem 2.1 *Let* (X, d) *be a complete metric space and* $f : X \to X$ *a* $(\Phi,\ \psi)-$*weak contraction of integral type. Suppose that* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous comparison function and* $\nu,\ \Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *are monotone increasing functions such that* Φ *is continuous and* $\Phi(0) = 0$. *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative, and such that for each* $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)d\nu(t) > 0$, where $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ *is also an increasing function. Then,* f *has a unique fixed point* $x^* \in X$ *such that for each* $x \in X$, $\lim\limits_{n\to\infty} f^n x = x^*$.

Proof. Let $x_0 \in X$ and let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = f(x_{n-1}) = f^n x_0,\ n =$

$1, 2, \cdots$, be the Picard iteration associated to f. From (7), we have that

$$\begin{aligned} \int_0^{d(x_n,x_{n+1})} \varphi(t)d\nu(t) &= \int_0^{d(f(x_{n-1}),f(x_n))} \varphi(t)d\nu(t) \leq \psi\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)d\nu(t)\right) \\ &\quad + \Phi\left(\int_0^{d(x_{n-1},f(x_{n-1}))} \varphi(t)d\nu(t)\right)\left(\int_0^{d(x_n,f(x_{n-1}))} \varphi(t)d\nu(t)\right) \\ &= \psi\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)d\nu(t)\right) \leq \cdots \leq \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)d\nu(t)\right). \quad (8) \end{aligned}$$

Taking the limit in (8) as $n \to \infty$ yields

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0, \qquad (\star)$$

since $\lim_{n\to\infty} \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)d\nu(t)\right) = 0$ and $\int_0^{\epsilon} \varphi(\mathrm{t})\mathrm{d}\nu(\mathrm{t}) > 0$ for each $\epsilon > 0$. Therefore, it follows from $(\star)$ that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \qquad (9)$$

We now establish that $\{x_n\}$ is a Cauchy sequence. Suppose it is not so. Then, there exists an $\epsilon > 0$ and subsequences $\left\{x_{m(p)}\right\}$ and $\left\{x_{n(p)}\right\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \epsilon, \quad d(x_{m(p)}, x_{n(p)-1}) < \epsilon. \qquad (10)$$

Again, by using (7), then we have that

$$\begin{aligned} \int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)d\nu(t) &= \int_0^{d(f(x_{m(p)-1}),f(x_{n(p)-1}))} \varphi(t)d\nu(t) \\ &\leq \psi\left(\int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)d\nu(t)\right) \\ &\quad + \Phi\left(\int_0^{d(x_{m(p)-1},f(x_{m(p)-1}))} \varphi(t)d\nu(t)\right) \\ &\left(\int_0^{d(x_{n(p)-1},f(x_{m(p)-1}))} \varphi(t)d\nu(t)\right) \\ &= \psi\left(\int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)d\nu(t)\right) \\ &\quad + \Phi\left(\int_0^{d(x_{m(p)-1},x_{m(p)})} \varphi(t)d\nu(t)\right) \\ &\left(\int_0^{d(x_{n(p)-1},x_{m(p)})} \varphi(t)d\nu(t)\right). \quad (11) \end{aligned}$$

By using (9), (10) and the triangle inequality, we get

$$\begin{aligned} d(x_{m(p)-1}, x_{n(p)-1}) &\leq d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) \\ &< d(x_{m(p)-1}, x_{m(p)}) + \epsilon \to \epsilon \text{ as p} \to \infty. \end{aligned} \qquad (12)$$

By applying (9) again and the conditions on Φ, we get

$$\lim_{p\to\infty} \Phi\left(\int_0^{d(x_{m(p)-1},x_{m(p)})} \varphi(t)d\nu(t)\right) = 0. \tag{13}$$

Using (10), (12) and (13) in (11), then we get

$$\int_0^{\epsilon} \varphi(t)d\nu(t) \leq \int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)d\nu(t) \leq \psi\left(\int_0^{\epsilon} \varphi(t)d\nu(t)\right) < \int_0^{\epsilon} \varphi(t)d\nu(t), \tag{14}$$

which is a contradction. Therefore, we must have that $\int_0^{\epsilon} \varphi(t)dt = 0$, that is, $\epsilon = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence and hence convergent. Since (X, d) is a complete metric space, $\{x_n\}$ converges to some $z \in X$, that is, $\lim\limits_{n\to\infty} x_n = z$.
Again, from (7), we have that

$$\begin{aligned}
\int_0^{d(x_{n+1},f(z))} \varphi(t)d\nu(t) &= \int_0^{d(f(x_n),f(z))} \varphi(t)d\nu(t) \leq \psi\left(\int_0^{d(x_n,z)} \varphi(t)d\nu(t)\right) \\
&\quad + \Phi\left(\int_0^{d(x_n,f(x_n))} \varphi(t)d\nu(t)\right)\left(\int_0^{d(z,f(x_n))} \varphi(t)d\nu(t)\right) \\
&= \psi\left(\int_0^{d(x_n,z)} \varphi(t)d\nu(t)\right) \\
&\quad + \Phi\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)d\nu(t)\right)\left(\int_0^{d(z,x_{n+1})} \varphi(t)d\nu(t)\right).
\end{aligned} \tag{15}$$

By taking the limits in (15) as $n \to \infty$, then we get

$$\int_0^{d(z,f(z))} \varphi(t)d\nu(t) \leq \psi(0) = 0, \tag{16}$$

so that from (16), we have again, a contradiction. Therefore, by the condition on φ, we have that $\int_0^{d(z,f(z))} \varphi(t)d\nu(t) = 0$, from which it follows that $d(z, f(z)) = 0$, or z = f(z).

We now prove that f has a unique fixed point: Suppose this is not true. Then, there exist $w_1,\ w_2 \in F_f,\ w_1 \neq w_2,\ d(w_1, w_2) > 0$. Therefore, we obtain by using (7) again that

$$\begin{aligned}
\int_0^{d(w_1,w_2)} \varphi(t)d\nu(t) &= \int_0^{d(f(w_1),f(w_2))} \varphi(t)d\nu(t) \\
&\leq \psi\left(\int_0^{d(w_1,w_2)} \varphi(t)d\nu(t)\right) \\
&\quad + \Phi\left(\int_0^{d(w_1,f(w_1))} \varphi(t)d\nu(t)\right)\left(\int_0^{d(w_2,f(w_1))} \varphi(t)d\nu(t)\right) \\
&= \psi\left(\int_0^{d(w_1,w_2)} \varphi(t)d\nu(t)\right) < \int_0^{d(w_1,w_2)} \varphi(t)d\nu(t),
\end{aligned}$$

leading to a contradiction again. Therefore, by the condition on φ again, we get $\int_0^{d(w_1,w_2)} \varphi(t)d\nu(t) = 0$, from which it follows that $d(w_1, w_2) = 0$, or $\mathrm{w}_1 = \mathrm{w}_2$. Hence, f has a unique fixed point.

Theorem 2.2 *Let (X, d) be a complete metric space and $f : X \to X$ a $(L, \psi)-$weak contraction of integral type. Suppose that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous comparison function and $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)d\nu(t) > 0$. Then, f has a unique fixed point $x^* \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = x^*$.*

Proof. The proof of this theorem follows similar argument as in Theorem 2.1.

Remark 2.3. Both Theorem 2.1 and Theorem 2.2 are generalizations and extensions of the celebrated Banach's fixed point [1, 2, 3, 5, 25] as well as extensions of the results of Branciari [9], Chatterjea [10], Kannan [16] and Zamfirescu [24]. Theorem 2.1 and Theorem 2.2 are also extensions of some results of Berinde [4, 5, 6, 7] as well as Theorem 2 of Berinde and Berinde [8].

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