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A NEW INTEGRAL OPERATOR FOR MEROMORPHIC FUNCTIONS

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Abstract. In this paper, a new integral operator

$$H(f_1, f_2, ...f_n)(z) = \frac{1}{z^2}\int\limits_0^z (uf_1(u))^{\gamma_1} ...,(uf_n(u))^{\gamma_n} du\ , \quad z \in U^*$$

for $f_i(z) \in \Sigma$ is defined. In addition, starlike condition will be derived. Moreover, a new subclass of meromorphic function satisfying the condition $-\Re\left(\frac{zf''(z)}{f(z)} + 1\right) < \beta, \quad z \in U^*,\ \beta > 1$, is introduced and sufficient conditions for this new subclass are studied.

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Introduction

Let Σ denotes the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic and univalent in the punctured open unit disk

$$U^* = \{z \in :\ 0 < |z| < 1\} = U - \{0\},$$

where U is the open unit disk $U = \{z \in :\ |z| < 1\}$.
A function $f \in \Sigma$ is said to be meromorphic univalent starlike of order α if

$$-\Re\frac{zf'(z)}{f(z)} > \alpha \quad (z \in U^*;\ 0 \leq \alpha < 1),$$

and we denote this class by $\Sigma^*(\alpha)$.

A function $f \in \Sigma$ is said to be meromorphic univalent convex of order α if

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U^*;\ 0 \leq \alpha < 1),$$

and we denote this class by $\Sigma_k(\alpha)$.

Let A denote the class of functions f normalized by $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disc U.

Let $\mathrm{N}(\beta)$ be the subclass of A, consisting of functions $f(z)$, which satisfy the inequality:

$$\Re\left(\tfrac{zf''(z)}{f(z)}+1\right) < \beta, \quad z \in U,\ \beta > 1.$$

This class studied by Uralegaddi etal in [1], and Owa and Srivastava in[2].

Analogous to $\mathrm{N}(\beta)$ we introduce the class $\Sigma_{\mathrm{N}}(\beta)$ as the following:

A function $f \in \Sigma$ is in the class $\Sigma_{\mathrm{N}}(\beta)$ if it satisfies the inequality

$$-\Re\left(\tfrac{zf''(z)}{f(z)}+1\right) < \beta, \quad z \in U^*,\ \beta > 1.$$

The study of integral operators has been rapidly investigated by many authors in the field of univalent functions. Recently, various integral operators have been introduced for certain class of analytic univalent functions in the unit disk. In this study, we follow the similar approach by introducing an integral operator for the class of meromorphic functions Σ and study its sufficient conditions.

Definition 1. *Let $n \in N, i \in \{1,2,3,...,n\}, \gamma_i > 0$. We define the integral operator $H(f_1, f_2, ...f_n)(z) : \Sigma^n \to \Sigma$ by*

$$H(f_1, f_2, ...f_n)(z) = \frac{1}{z^2}\int\limits_0^z (uf_1(u))^{\gamma_1}...,(uf_n(u))^{\gamma_n} du\ , \quad z \in U^*, \tag{1}$$

where $f_i(z) \in \Sigma$.

It is easy to see that this integral operator is analytic operator.

Remark 1. For $n = 1$, $\gamma_1 = \gamma,\ \gamma_2 = \gamma_3 = ... = \gamma_n = 0$, we obtain the integral operator

$$G[f](z)=\frac{1}{z^2}\int_0^z (uf(u))^{\gamma}du. \tag{2}$$

Remark 2. For $n=1$, $\gamma_1=1,\ \gamma_2=\gamma_3=...=\gamma_n=0$, we obtain the integral operator

$$F(z)=\frac{1}{z^2}\int_0^z uf(u)du. \tag{3}$$

Main Result

In order to prove our main results (Theorem 1 and Theorem 2 below) we shall need the following lemma due to S. S. Miller and P. T. Mocanu [3]:

Lemma 1. *Let the function* $\Psi: C^2 \to C$ *satisfy*

$$\Re\Psi(ix,y)\leq 0$$

for all real x *and for all real* y, $y\leq -(1+x^2)/2$. *If* $p(z)=1+p_1z+...$ *is analytic in the unit disc* $U=\{z: z\in C\ \ |z|<1\}$ *and*

$$\Re\Phi(p(x),zp'(x))>0,\ \ z\in U,$$

then

$$\Re p(z)>0\ \ for\ z\in U.$$

First, we prove the following starlike result of the operator $H[f](z)$.

Theorem 1. *Let* $f_i\in\Sigma,\ i\in\{1,2,3,...,n\},\ \gamma_i>0.$

If

$$-\Re\frac{zf_i'(z)}{f_i(z)}>\sum_{i=1}^{n}\gamma_i,$$

then $H[f](z) \in \Sigma^*$ *(the class of meromorphically starlike functions), where* $H[f](z)$ *is the integral operator define as in (1).*

Proof. On successive differentiation of $H[f](z)$, we get

$$z^2\mathrm{H}[f]'(z) + 2z\mathrm{H}[f](z) = (z\, f_1(z))^{\gamma_1}...(z\, f_n(z))^{\gamma_n},$$

$$z^2\mathrm{H}[f]''(z) + 4z\mathrm{H}[f]'(z) + 2\mathrm{H}[f](z)) = \sum_{i=1}^{n} \gamma_i (zf_i(z))^{\gamma_i - 1} \left(zf_i'(z) + f_i(z) \right) \prod_{j=1, j\neq i}^{n} (zf_j(z))^{\gamma_j}.$$

Then

$$\frac{z^2\mathrm{H}[f]''(z)+4z\mathrm{H}[f]'(z)+2\mathrm{H}[f](z)}{z^2\mathrm{H}[f]'(z)+2z\mathrm{H}[f](z)} = \sum_{i=1}^{n} \gamma_i \left[\frac{f_i'(z)}{f_i(z)} + \frac{1}{z} \right].$$

By multiplying the above expression with z we obtain

$$\frac{z^2\mathrm{H}[f]''(z)+4z\mathrm{H}[f]'(z)+2\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)+2\mathrm{H}[f](z)} = \sum_{i=1}^{n} \gamma_i \left[\frac{zf_i'(z)}{f_i(z)} + 1 \right],$$

that is equivalent to

$$\frac{\frac{z\mathrm{H}''[f](z)}{\mathrm{H}[f]'(z)} + 2\frac{\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)} + 4}{1 + 2\frac{\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)}} = \sum_{i=1}^{n} \gamma_i \left[\frac{zf_i'(z)}{f_i(z)} + 1 \right]. \tag{4}$$

Thus

$$\frac{-(\left(\frac{z\mathrm{H}[f]''(z)}{\mathrm{H}[f]'(z)} + 1\right)) - 2\frac{\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)} - 3}{1 + 2\frac{\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)}} = -\sum_{i=1}^{n} \gamma_i \left[\frac{zf_i'(z)}{f_i(z)} + 1 \right]. \tag{5}$$

We define the regular function p in U by

$$p(z) = -\frac{zH'[f](z)}{H[f]}, \qquad z \in U^* \tag{6}$$

and $p(0) = 1$. Differentiating $p(z)$ logarithmically, we obtain

$$p(z) - \frac{zp'(z)}{p(z)} = -\left(1 + \frac{zH[f]''(z)}{H[f]'(z)}\right). \tag{7}$$

¿From (5),(6) and (7) we obtain

$$p(z) - 1 - \frac{zp'(z)}{p(z)-2} = -\sum_{i=1}^{n} \gamma_i \left[\frac{zf_i'(z)}{f_i(z)} + 1\right]. \tag{8}$$

¿From (4) and (8) we get

$$\Re\left[p(z) - 1 - \frac{zp'(z)}{p(z)-2}\right] = \Re\left\{-\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} + 1\right)\right\} > 0.$$

We define the function Ψ by

$$\Psi(u,v) = u - 1 - \frac{v}{u-2}.$$

In order to use Lemma 1 we must verify that $\Psi(ix, y) \leq 0$ whenever x and y are real numbers such that $y \leq -(1+x^2)/2$.

We have

$$\Re\Psi(ix,y) = \Re\left(ix - 1 - \frac{y}{ix-2}\right) = -1 + \frac{2y}{4+x^2} \leq -1 - \frac{1+x^2}{4+x^2} < 0.$$

Then

$$\Re\Psi(p(z), zp'(z)) > 0, \;\; z \in U.$$

By Lemma 1, we conclude that$\Re p(z) > 0, \;\; z \in U$, and so

$$-\Re\frac{zH'[f](z)}{H[f]} > 0. \tag{9}$$

Therefore $H[f](z) \in \Sigma^*$.

Corollary 1. *Let $f \in \Sigma$. If*

$$-\Re\frac{zf'(z)}{f(z)} > \gamma.$$

Then $G[f](z) \in \Sigma^*$(the class of meromorphically starlike functions), where $G[f](z)$ is the integral operator define as in (2).

Proof. From Theorem 1, by considering $n = 1, \quad \gamma_1 = \gamma, \; \gamma_2 = \gamma_3 = ... = \gamma_n = 0$.

Corollary 2. *Let $f \in \Sigma$. If*

$$-\Re\frac{zf'(z)}{f(z)} > 1$$

Then $F[f](z) \in \Sigma^*$ *(the class of meromorphically starlike functions), where* $F[f](z)$ *is the integral operator define as in (3).*

Proof. From Corollary 1, let $\gamma = 1$.

Our next result is the following:

Theorem 2. *Let* $f_i \in \Sigma,\ i \in \{1, 2, 3, ..., n\}$.

and let

$$1 < \sum_{i=1}^{n} \gamma_i < 2.$$

If

$$-\Re\frac{zf_i'(z)}{f_i(z)} > \sum_{i=1}^{n} \gamma_i.$$

Then

$$\mathrm{H}[f](z) \in \Sigma_{\mathrm{N}}(\beta),\ \beta > 1,$$

where $H[f](z)$ *is the integral operator define as in (1).*

Proof. From (5) we know that

$$\frac{-\left(\frac{z\mathrm{H}[f]''(z)}{\mathrm{H}[f]'(z)}+1\right)-2\frac{\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)}-3}{1+2\frac{\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)}} = -\sum_{i=1}^{n} \gamma_i \left[\frac{zf_i'(z)}{f_i(z)} + 1\right].$$

Let

$$h(z) = -\frac{\mathrm{H}[f](z)}{z\mathrm{H}[f]'(z)}.$$

It is easy to see from (9) and Theorem 1 that $\Re\, h(z) = \Re\, 1/p(z) > 0$.

Then we have

$$\frac{-(\left(\frac{z\mathrm{H}[f]''(z)}{\mathrm{H}[f]'(z)}+1\right))+2h(z)-3}{1-2h(z)} = -\sum_{i=1}^{n}\gamma_i\left[\frac{zf_i'(z)}{f_i(z)}+1\right].$$

That is equivalent to

$$-\left(\frac{z\mathrm{H}[f]''(z)}{\mathrm{H}[f]'(z)}+1\right) = \sum_{i=1}^{n}\gamma_i\left(-\frac{zf_i'(z)}{f_i(z)}\right)[1-2h(z)]+2h(z)[\sum_{i=1}^{n}\gamma_i-1]+3-\sum_{i=1}^{n}\gamma_i. \tag{10}$$

Taking real parts in (10), we get

$$-\Re\left(\frac{z\mathrm{H}[f]''(z)}{\mathrm{H}[f]'(z)}+1\right) = \Re\sum_{i=1}^{n}\gamma_i\left(-\frac{zf_i'(z)}{f_i(z)}\right)[1-2h(z)]+2[\sum_{i=1}^{n}\gamma_i-1]\Re h(z)+3-\sum_{i=1}^{n}\gamma_i.$$

Since $\Re(z) < |z|$, we can write

$$-\Re\left(\frac{z\mathrm{H}[f]''(z)}{\mathrm{H}[f]'(z)}+1\right) \leq \sum_{i=1}^{n}\gamma_i\left|-\frac{zf_i'(z)}{f_i(z)}[1-h(z)]\right|+2[\sum_{i=1}^{n}\gamma_i-1]\Re h(z)+3-\sum_{i=1}^{n}\gamma_i.$$

Let

$$\beta = \sum_{i=1}^{n}\gamma_i\left|-\frac{zf_i'(z)}{f_i(z)}[1-2h(z)]\right|+2[\sum_{i=1}^{n}\gamma_i-1]\Re h(z)+3-\sum_{i=1}^{n}\gamma_i$$

and since

$$\sum_{i=1}^{n}\gamma_i\left|-\frac{zf_i'(z)}{f_i(z)}[1-2h(z)]\right| > 0$$

we can conclude that

$$\beta > 2[\sum_{i=1}^{n}\gamma_i-1]\Re h(z)+3-\sum_{i=1}^{n}\gamma_i > 3-\sum_{i=1}^{n}\gamma_i > 1,$$

and so $-\Re\left(\frac{z\mathrm{H}[f]''(z)}{\mathrm{H}[f]'(z)}+1\right) < \beta, \quad \beta > 1$. Therefore $\mathrm{H}[f](z) \in \Sigma_{\mathrm{N}}(\beta)$, $\beta > 1$.

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