Acta Universitatis Apulensis
ISSN: 1582-5329

No. 24/2010
pp. 95-107

BOUNDEDNESS FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR ON HARDY AND HERZ-HARDY SPACES

Jiasheng Zeng

Abstract. In this paper, the $(H^p_{\vec{b}}, L^p)$ and $(H\dot{K}^{\alpha,p}_{q,\vec{b}}, \dot{K}^{\alpha,p}_q)$ type boundedness for the multilinear commutator associated with the Littlewood-Paley operator are obtained.

2000 *Mathematics Subject Classification*: 42B20, 42B25.

1. Introduction and definition

Let $0 < q < \infty$ and $L^q_{loc}(R^n) = \{f^q$ is locally integrable on $R^n\}$. Suppose $f \in L^1_{loc}(R^n)$, $B = B(x_0, r) = \{x \in R^n : |x - x_0| < r\}$ denotes a ball of R^n centered at x_0 and having radius r, write $f_B = |B|^{-1}\int_B f(x)dx$ and $f^{\#}(x) = \sup_{x\in B}|B|^{-1}\int_B |f(x) - f_B|dx < \infty$. f is said to belong to $BMO(R^n)$, if $f^{\#} \in L^\infty(R^n)$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$.

Let T be the Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see[2]) proved that the commutator $[b, T]$ is bounded on $L^r(R^n)$ for any $1 < r < \infty$. However, it was observed that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ when $0 < p \leq 1$. But if $H^p(R^n)$ is replaced by a suitable atomic space $H^p_{\vec{b}}(R^n)$(see [1][7][12]), then $[b, T]$ maps continuously $H^p_{\vec{b}}(R^n)$ into $L^p(R^n)$, and a similar results holds for Herz-type spaces. In addition we have easily known that $H^p_{\vec{b}}(R^n) \subset H^p(R^n)$. The main purpose of this paper is to consider the continuity of the multilinear commutators related to the Littlewood-Paley operators and $BMO(R^n)$ functions on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions(see [1][3-10][12][13]).

Definition 1. *Let $\varepsilon > 0$, $n > \delta > 0$ and ψ be a fixed function which satisfies the following properties:*

(1) $\int_{R^n} \psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,
(3) $|\psi(x+y)-\psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon-\delta)}$ *when* $2|y|<|x|$;
The Littlewood-Paley multilinear commutator is defined by

$$S_\delta^{\vec{b}}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x)-b_j(z))\right] \psi_t(y-z)f(z)dz,$$

$\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ *and* $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ *for* $t > 0$. *Set* $F_t(f)(y) = \int_{R^n} \psi_t(y-x)f(x)dx$. *We also define that*

$$S_\delta(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator with $\delta = 0$*(see [15]).*

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

Definition 2. *Let* b_i $(i = 1, \cdots, m)$ *be a locally integrable functions and* $0 < p \le 1$*. A bounded measurable function* a *on* R^n *is called a* $(p, \vec{b})$ *atom, if*
(1) *supp* $a \subset B = B(x_0, r)$
(2) $||a||_{L^\infty} \le |B(x_0,r)|^{-1/p}$
(3) $\int_B a(y)dy = \int_B a(y) \prod_{l\in\sigma} b_l(y)dy = 0$ *for any* $\sigma \in C_j^m$, $1 \le j \le m$.

A temperate distribution(see [14][15]) f is said to belong to $H_{\vec{b}}^p(R^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x).$$

where $a_j's$ are $(p, \vec{b})$ atoms, $\lambda_j \in C$ and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Moreover, $||f||_{H_{\vec{b}}^p} \approx (\sum_{j=1}^\infty |\lambda_j|^p)^{1/p}$.

Definition 3. *Let* $0 < p, q < \infty$, $\alpha \in R$*. For* $k \in Z$*, set* $B_k = \{x \in R^n : |x| \le 2^k\}$ *and* $C_k = B_k \setminus B_{k-1}$*, and* $\chi_k = \chi_{C_k}$ *for* $k \in Z$*, where* χ_{C_k} *is the characteristic function of set* C_k*. Denote the characteristic function of* B_0 *by* χ_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n)=\left\{f\in L_{loc}^q(R^n\setminus\{0\}):||f||_{\dot{K}_q^{\alpha,p}}<\infty\right\}.$$

where

$$||f||_{\dot{K}_q^{\alpha,p}}=\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n)=\left\{f\in L_{loc}^q(R^n):||f||_{K_q^{\alpha,p}}<\infty\right\}.$$

where

$$||f||_{K_q^{\alpha,p}}=\left[\sum_{k=1}^{\infty}2^{k\alpha p}||f\chi_k||_{L^q}^p+||f\chi_0||_{L^q}^p\right]^{1/p}.$$

Definition 4. *Let* $\alpha\in R^n,\ 1<q<\infty,\ \alpha\geq n(1-1/q),\ \ b_i\in BMO(R^n),\ 1\leq i\leq m$. *A function* $a(x)$ *is called a central* $(\alpha,q,\vec{b})$ *-atom (or a central* $(\alpha,q,\vec{b})$*-atom of restrict type), if*

(1) $supp\ a\subset B=B(x_0,r)$ *(or for some* $r\geq 1$*),*

(2) $||a||_{L^q}\leq|B(x_0,r)|^{-\alpha/n}$,

(3) $\int_B a(x)x^\beta dx=\int_B a(x)x^\beta\prod_{i\in\sigma}b_i(x)dx=0$ *for any* $\sigma\in C_j^m$ *,* $1\leq j\leq m$, $0\leq|\beta|\leq\alpha$, *where* $\beta=(\beta_1,...,\beta_n)$ *is the multi-indices with* $\beta_i\in N$ *for* $1\leq i\leq n$ *and* $|\beta|=\sum_{i=1}^n\beta_i$.

A temperate distribution f is said to belong to $H\dot{K}_{q,\vec{b}}^{\alpha,p}(R^n)$(or $HK_{q,\vec{b}}^{\alpha,p}(R^n)$), if it can be written as $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$ (or $f=\sum_{j=0}^{\infty}\lambda_j a_j$), in the $S'(R^n)$ sense, where a_j is a central $(\alpha,q,\vec{b})$-atom(or a central $(\alpha,q,\vec{b})$-atom of restrict type) supported on $B(0,2^j)$ and $\sum_{-\infty}^{\infty}|\lambda_j|^p<\infty$(or $\sum_{j=0}^{\infty}|\lambda_j|<\infty$). Moreover,

$$||f||_{H\dot{K}_{q,\vec{b}}^{\alpha,p}}(\text{ or }||f||_{HK_{q,\vec{b}}^{\alpha,p}})=\inf(\sum_j|\lambda_j|^p)^{1/p},$$

where the infimum are taken over all the decompositions of f as above.

2. Theorems and Proofs

To prove the theorems, we need the following lemmas.

Lemma 1.(see [12]) *Let* $1<p<q<n/\alpha$, $1/q=1/p-\alpha/n$. *Then* S_δ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Lemma 2.(see [12]) *Let* $1<p<q<n/\alpha$, $1/q=1/p-\alpha/n$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Theorem 1. *Let* $\varepsilon > 0$, $n/(n+\varepsilon-\delta) < p \leq 1$, $1/q = 1/p - \delta/n$, $b_i \in BMO$, $1 \leq i \leq m$, $\vec{b} = (b_1, \cdots, b_m)$. *Then the multilinear commutator* $S_\delta^{\vec{b}}$ *is bounded from* $H^p_{\vec{b}}(R^n)$ *to* $L^q(R^n)$.

Proof. It suffices to show that there exist a constant $C > 0$, such that for every $(p, \vec{b})$ atom a,

$$||S_\delta^{\vec{b}}(a)||_{L^q} \leq C.$$

Let a be a $(p, \vec{b})$ atom supported on a ball $B = B(x_0, d)$. When $m = 1$ see [7], and now we prove $m > 1$. Write

$$\int_{R^n} |S_\delta^{\vec{b}}(a)(x)|^q dx = \int_{|x-x_0|\leq 2d} |S_\delta^{\vec{b}}(a)(x)|^q dx + \int_{|x-x_0|>2d} |S_\delta^{\vec{b}}(a)(x)|^q dx = I + II.$$

For I, taking $r, s > 1$ with $q < s < n/\delta$ and $1/r = 1/s - \delta/n$, by Hölder's inequality and the (L^s, L^r)-boundedness of $S_\delta^{\vec{b}}$, we have

$$\begin{aligned} I &\leq \left(\int_{|x-x_0|\leq 2d} |S_\delta^{\vec{b}}(a)(x)|^r dx \right)^{q/r} \cdot |B(x_0, 2d)|^{1-q/r} \\ &\leq C||S_\delta^{\vec{b}}(a)(x)||^q_{L^r} \cdot |B(x_0, d)|^{1-q/r} \\ &\leq C||a||^q_{L^s} |B|^{1-q/r} \\ &\leq C. \end{aligned}$$

For II, denoting $\lambda = (\lambda_1, \cdots, \lambda_m)$ with $\lambda_i = (b_i)_B$, $1 \leq i \leq m$, where $(b_i)_B =$

$|B(x_0,r)|^{-1}\int_{B(x_0,r)} b_i(x)dx$, by the vanishing moment of a, we get

$$\begin{aligned}
II &= \sum_{k=1}^{\infty}\int_{2^{k+1}r\geq|x-x_0|>2^kd}|S_\delta^{\vec{b}}(a)(x)|^q dx\\
&\leq C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}d)|^{1-q}\left(\int_{2^{k+1}d\geq|x-x_0|>2^kd}|S_\delta^{\vec{b}}(a)(x)|dx\right)^q\\
&\leq C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}d)|^{1-q}\\
&\times\left[\int_{2^{k+1}d\geq|x-x_0|>2^kd}\left(\int\int_{\Gamma(x)}|\int_B\prod_{j=1}^m(b_j(x)-b_j(z))\psi_t(y-z)a(z)dz|^2\frac{dydt}{t^{n+1}}\right)^{1/2}dx\right]^q\\
&\leq C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}d)|^{1-q}[\int_{2^{k+1}d\geq|x-x_0|>2^kd}\\
&\times\left(\int\int_{\Gamma(x)}\left(\int_B|\psi_t(y-z)-\psi_t(y-x_0)|\prod_{j=1}^m|(b_j(x)-b_j(z))||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}dx]^q;
\end{aligned}$$

noting that $z\in B,\ x\in B(x_0,2^{k+1}d)\setminus B(x_0,2^kd)$, then

$$\begin{aligned}
&S_\delta^{\vec{b}}(a)(x)\\
=&\left[\int\int_{\Gamma(x)}\left(\int_B|\psi_t(y-z)-\psi_t(y-x_0)|\prod_{j=1}^m|b_j(x)-b_j(z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
\leq& C\left[\int\int_{\Gamma(x)}\left(\int_B t^{-n+\delta}|a(z)|\prod_{j=1}^m|b_j(x)-b_j(z)|\frac{(|x_0-z|/t)^\varepsilon}{(1+|x_0-y|/t)^{n+1+\varepsilon-\delta}}dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
\leq& C\left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|x_0-y|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2}\int_B\prod_{j=1}^m|b_j(x)-b_j(z)||x_0-z|^\varepsilon|a(z)|dz\\
\leq& C\left(\int\int_{\Gamma(x)}\frac{t^{1-n}2^{2(n+1+\varepsilon-\delta)}}{(2t+2|x_0-y|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2}\int_B\prod_{j=1}^m|b_j(x)-b_j(z)||x_0-z|^\varepsilon|a(z)|dz;
\end{aligned}$$

Notice that $2t+|x_0-y|>2t+|x_0-x|-|x-y|>t+|x_0-x|$ when $|x-y|<t$,

and it is easy to calculate that

$$\int_0^\infty \frac{tdt}{(t+|x-x_0|)^{2(n+1+\varepsilon-\delta)}} = C|x-x_0|^{-2(n+\varepsilon-\delta)};$$

then, we deduce

$$\begin{aligned}
& S_\delta^{\vec{b}}(a)(x) \\
\leq\ & C\left(\int\int_{\Gamma(x)} \frac{t^{1-n}}{(2t+|x_0-y|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2} \int_B \prod_{j=1}^m |b_j(x)-b_j(z)||x_0-z|^\varepsilon |a(z)|dz \\
\leq\ & C\left(\int\int_{\Gamma(x)} \frac{t^{1-n}}{(t+|x-x_0|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2} \int_B \prod_{j=1}^m |b_j(x)-b_j(z)||x_0-z|^\varepsilon |a(z)|dz \\
\leq\ & C\left(\int_0^\infty \frac{tdt}{(t+|x-x_0|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2} \int_B \prod_{j=1}^m |b_j(x)-b_j(z)||x_0-z|^\varepsilon |a(z)|dz \\
\leq\ & C|B|^{\varepsilon/n-1/p}|x-x_0|^{-(n+\varepsilon-\delta)} \int_B \prod_{j=1}^m |b_j(x)-b_j(z)|dz.
\end{aligned}$$

So

$$\begin{aligned}
II &\leq C|B|^{(\varepsilon/n-1/p)q}\sum_{k=1}^{\infty}|B(x_0,2^{k+1}r)|^{1-q}\\
&\quad\times\left[\int_{2^{k+1}d\geq|x-x_0|>2^kd}|x-x_0|^{-(n+\varepsilon-\delta)}\left(\int_B\prod_{j=1}^m|b_j(x)-b_j(z)|dz\right)dx\right]^q\\
&\leq C|B|^{(\varepsilon/n-1/p)q}\sum_{k=1}^{\infty}|B(x_0,2^{k+1}r)|^{1-q}\\
&\quad\times\left[\sum_{j=0}^m\sum_{\sigma\in C_j^m}\int_{2^{k+1}d\geq|x-x_0|>2^kd}|x-x_0|^{-(n+\varepsilon-\delta)}|(\vec{b}(x)-\lambda)_\sigma|dx\int_B|(\vec{b}(z)-\lambda)_{\sigma^c}|dz\right]^q\\
&\leq C|B|^{(\varepsilon/n-1/p)q}\sum_{j=0}^m\sum_{\sigma\in C_j^m}\left(\int_B|(\vec{b}(z)-\lambda)_{\sigma^c}|dz\right)^q\\
&\quad\times\sum_{k=1}^{\infty}|B(x_0,2^{k+1}d)|^{1-q}\left[\int_{2^{k+1}d\geq|x-x_0|>2^kd}|x-x_0|^{-(n+\varepsilon-\delta)}|(\vec{b}(x)-\lambda)_\sigma|dx\right]^q\\
&\leq C\sum_{j=0}^m\sum_{\sigma\in C_j^m}||\vec{b}_{\sigma^c}||_{BMO}^q\cdot||\vec{b}_\sigma||_{BMO}^q\sum_{k=1}^{\infty}|B(x_0,2^{k+1}d)|^{1-(n+\varepsilon-\delta)q/n}k^q|B|^{(1+\varepsilon/n-1/p)q}\\
&\leq C||\vec{b}||_{BMO}^q\sum_{k=1}^{\infty}k^q\cdot 2^{k(n-q-qn+q\delta)}\\
&\leq C.
\end{aligned}$$

This finishes the proof of Theorem 1.

Theorem 2. *Let* $\varepsilon>0$, $0<p<\infty$, $1<q_1,q_2<\infty$, $1/q_1-1/q_2=\delta/n$, $n(1-1/q_1)\leq\alpha<n(1-1/q_1)+\varepsilon$ *and* $b_i\in BMO(R^n), 1\leq i\leq m$, $\vec{b}=(b_1,\cdots,b_m)$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $H\dot{K}_{q_1,D^mA}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}(R^n)$.

Proof. Let $f\in H\dot{K}_{q_1,\vec{b}}^{\alpha,p}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_ja_j(x)$ be the atomic decomposition for f as in Definition 4, we write

$$\begin{aligned}
||S_\delta^{\vec{b}}(f)(x)||_{\dot{K}_{q_2}^{\alpha,p}} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||S_\delta^{\vec{b}}(f)\chi_k||_{L^{q_2}}^p\right)^{1/p} \\
&\leq C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=-\infty}^{\infty}|\lambda_j|||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}})^p\right]^{1/p} \\
&\leq C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=-\infty}^{k-3}|\lambda_j|||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}})^p\right]^{1/p} \\
&\quad +C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=k-2}^{\infty}|\lambda_j|||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}})^p\right]^{1/p} \\
&= I+II.
\end{aligned}$$

For II, by the (L^{q_1}, L^{q_2})-boundedness of $S_\delta^{\vec{b}}$ and the Hölder's inequality, we have

$$\begin{aligned}
II &\leq C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||S_\delta^{\vec{b}}(a_j)\chi_j||_{L^{q_2}}\right)^p\right]^{1/p} \\
&\leq C\left[\sum_{-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p\right]^{1/p} \\
&\leq C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\cdot 2^{-j\alpha}\right)^p\right]^{1/p} \\
&\leq C\begin{cases} \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right]^{1/p}, & 0<p\leq 1 \\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}2^{(k-j)p/2}\right]^{1/p}, & 1<p<\infty \end{cases} \\
&\leq C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \\
&\leq C||f||_{H\dot{K}_{q_1,\vec{b}}^{\alpha,p}}.
\end{aligned}$$

For I, when m=1, let $b_j^1 = |B_j|^{-1}\int_{B_j} b_1(x)dx$. We have

$$\begin{aligned}
S_\delta^{b_1}(a_j)(x) &= \left[\int\int_{\Gamma(x)} |\int_{B_j}(b_1(x)-b_1(z))\psi_t(y-z)a_j(z)dz|^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\le \left[\int\int_{\Gamma(x)}\left(\int_{B_j}|\psi_t(y-z)-\psi_t(y)||b_1(y)-b_1(z)||a_j(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\le C\left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|y|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2}\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(x)-b_1(z)|dz\\
&\le C\left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|x|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2}\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(x)-b_1(z)|dz\\
&\le C\left(\int_0^\infty\frac{tdt}{(t+|x|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2}\left(\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(x)-b_1(z)|dz\right)\\
&\le C|x|^{-(n+\varepsilon-\delta)}\int_{B_j}|z|^\varepsilon|a_j(z)|b_1(x)-b_1(z)|dz\\
&\le C|x|^{-(n+\varepsilon-\delta)}\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(x)-b_j^1|dz\\
&\quad +C|x|^{-(n+\varepsilon-\delta)}\int_{B_j}|z|^\varepsilon|a_j(z)||b_1(z)-b_j^1|dz\\
&\le C|x|^{-(n+\varepsilon-\delta)}\left(|b_1(x)-b_j^1|2^{j(\varepsilon+n(1-1/q_1)-\alpha)}+2^{j(\varepsilon+n(1-1/q_1)-\alpha)}||b_1||_{BMO}\right).
\end{aligned}$$

So

$$\begin{aligned}
&||S_\delta^{b_1}(a_j)\chi_k||_{L^{q_2}}\\
\le\ & C2^{j(\varepsilon+n(1-1/q_1)-\alpha)}[\left(\int_{B_k}|b_1(x)-b_j^1||x|^{-q_2(n+\varepsilon-\delta)}dx\right)^{1/q_2}\\
&+\left(\int_{B_k}|x|^{-q_2(n+\varepsilon-\delta)}dx\right)^{1/q_2}||b_1||_{BMO}]\\
\le\ & C2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\left[2^{-k(n+\varepsilon-\delta)}\cdot|B_k|^{1/q_2}||b_1||_{BMO}+2^{-k(n+\varepsilon-\delta)}\cdot|B_k|^{1/q_2}||b_1||_{BMO}\right]\\
\le\ & C||b_1||_{BMO}2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]};
\end{aligned}$$

thus

$$\begin{aligned}
I &= C\left[\sum_{j=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||S_\delta^{b_1}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p} \\
&\le C||b_1||_{BMO}\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]}\right)^p\right]^{1/p} \\
&\le C||b_1||_{BMO} \\
&\times \begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]p}\right]^{1/p}, & 0<p\le 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{p[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]/2}\right)\right. & \\ \quad \left.\times\left(\sum_{j=-\infty}^{k-3} 2^{p'[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty \end{cases} \\
&\le C||b_1||_{BMO}\begin{cases} \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p}\right]^{1/p}, & 0<p\le 1 \\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p/2}\right]^{1/p}, & 1<p<\infty \end{cases} \\
&\le C||b_1||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \\
&\le C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.
\end{aligned}$$

When $m > 1$, Let $b_j^i = |B_j|^{-1}\int_{B_j} b_i(x)dx$, $1 \le i \le m, \vec{b'} = (b_j^1, \cdots, b_j^m)$. We have

$$\begin{aligned}
S_\delta^{\vec{b}}(a_j)(x) &= \left[\int\int_{\Gamma(x)} |\int_{B_j}\prod_{i=1}^m (b_i(x)-b_i(z))\psi_t(y-z)a_j(z)dz|^2 \frac{dydt}{t^{n+1}}\right]^{1/2} \\
&\le \left[\int\int_{\Gamma(x)} \left(\int_{B_j}\prod_{i=1}^m |b_i(x)-b_i(z)||\psi_t(y-z)-\psi_t(y)||a_j(z)|dz\right)^2 \frac{dydt}{t^{n+1}}\right]^{1/2} \\
&\le C|x|^{-(n+\varepsilon-\delta)}\int_{B_j}|z|^\varepsilon|a_j(z)|\prod_{i=1}^m |b_i(x)-b_i(z)|dz \\
&\le C|x|^{-(n+\varepsilon-\delta)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b})_\sigma|\int_{B_j}|z|^\varepsilon|a_j(z)||(\vec{b}(y)-\vec{b})_{\sigma^c}|dz \\
&\le C|x|^{-(n+\varepsilon-\delta)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b})_\sigma|2^{j\varepsilon}\cdot 2^{-j\alpha}\cdot 2^{jn(1-1/q_1)}||\vec{b}_{\sigma^c}||_{BMO} \\
&\le C|x|^{-(n+\varepsilon-\delta)}\cdot 2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b})_\sigma|||\vec{b}_{\sigma^c}||_{BMO};
\end{aligned}$$

So

$$\begin{aligned}
&||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^q} \\
\le\ & C2^{j(\varepsilon+n(1-1/q_1)-\alpha)}||\vec{b}_{\sigma^c}||_{BMO}\left[\int_{B_k}\left(|x|^{-(n+\varepsilon)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b})_\sigma|\right)^q dx\right]^{1/q} \\
\le\ & C||\vec{b}||_{BMO}2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\cdot 2^{-k(n+\varepsilon-\delta)+kn/q_2};
\end{aligned}$$

then

$$
\begin{aligned}
I &= C\left[\sum_{j=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p} \\
&\le C||\vec{b}||_{BMO}\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]}\right)^p\right]^{1/p} \\
&\le C||\vec{b}||_{BMO} \\
&\times \begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]p}\right]^{1/p}, & 0<p\le 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{p[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]/2}\right)\right. & \\ \quad \left.\times\left(\sum_{j=-\infty}^{k-3}2^{p'[j(\varepsilon+n(1-1/q_1)-\alpha)-k(n+\varepsilon-\delta)+kn/q_2]}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty \end{cases} \\
&\le C||\vec{b}||_{BMO}\begin{cases} \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p}\right]^{1/p}, & 0<p\le 1 \\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p/2}\right]^{1/p}, & 1<p<\infty \end{cases} \\
&\le C||\vec{b}||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \\
&\le C||f||_{H\dot{K}_{q_1,\vec{b}}^{\alpha,p}}.
\end{aligned}
$$

Remark. Theorem 2 also hold for nonhomogeneous Herz-type spaces, we omit the details.

References

[1] J. Alvarez, *Continuity properties for linear commutators of Calderón-Zygmund operators*, Collect. Math., 49(1998), 17-31.

[2] R. Coifman, R. Rocherg and G. Weiss, *Factorization theorem for Hardy space in several variables*, Ann. of Math., 103(1976), 611-635.

[3] J. Garcia-Cuerva and M. L. Herrero, *A Theory of Hardy spaces associated to Herz Spaces*, Proc. London Math. Soc., 69(1994), 605-628.

[4] L. Z. Liu, *Weighted weak type* (H^1, L^1) *estimates for commutators of Littlewood-Paley operators*, Indian J. of Math., 45(1)(2003), 71-78.

[5] L. Z. Liu, *Weighted Block-Hardy spaces estimates for commutators of Littlewood-Paley operators*, Southeast Asian Bull. of Math., 27(2004), 833-838.

[6] L. Z. Liu, *Weighted weak type estimates for commutators of Littlewood-Paley operator*, Japanese J. of Math., 29(1)(2003), 1-13.

[7] L. Z. Liu, S.Z.Lu and J. S. Xu, *Boundedness for commutators of Littlewood-Paley operators*, Adv. in Math.(China), 32(2003), 473-480.

[8] S. Z. Lu and D. C. Yang, *The local versions of $H^p(R^n)$ spaces at the origin*, Studia. Math., 116(1995), 147-158.

[9] S. Z. Lu and D. C. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. in China(ser.A), 38(1995), 147-158.

[10] S. Z. Lu and D. C. Yang, *The weighted Herz type Hardy spaces and its applications*, Sci. in China(ser.A), 38(1995), 662-673.

[11] S. Z. Lu and D. C. Yang, *The continuity of commutators on Herz-type space*, Michigan Math. J., 44(1997), 255-281.

[12] C. Pérez, *Endpoint estimate for commutators of singular integral operators*, J. Func. Anal., 128(1995), 163-185.

[13] C. Pérez and R. Trujillo-Gonzalez, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc., 65(2002), 672-692.

[14] E. M. Stein, *Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ., 1993.

[15] A. Torchinsky, *The real variable methods in harmonic analysis*, Pure and Applied Math., 123, Academic Press, New York, 1986.

Jiasheng Zeng
Department of Mathematics
Hunan Business College
Changsha, 410205, P. R. of China
E-mail: *zengjiashenga@163.com*