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SOME PROPERTIES OF NEW INTEGRAL OPERATOR

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Abstract. Certain integral operator $J_p^m(\lambda,\ell)(\lambda \geq 0; \ell \geq 0; p \in N; m \in N_0 = N \cup \{0\})$, where $N = \{1, 2, ...\}$ is introduced for functions of the form $f(z) = z^p + \sum\limits_{k=p+1}^{\infty} a_k z^k$ which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. The opject of the present paper is to give an applications of the above operator to the differential inequalities.

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1.Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, ...\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. In [3] Catas extended the multiplier transformation and defined the operator $I_p^m(\lambda,\ell)f(z)$ on $A(p)$ by the following infinite series

$$I_p^m(\lambda,\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m a_n z^n$$

$$(\lambda \geq 0; \ell \geq 0; p \in N, m \in N_0; z \in U). \tag{1.2}$$

We note that:

$$I_p^0(1,0)f(z) = f(z) \quad \text{and} \quad I_p^1(1,0)f(z) = \frac{zf'(z)}{p} .$$

By specializing the parameters m, λ, ℓ and p, we obtain the following operators studied by various authors :

(i) $I_p^m(1,\ell) = I_p(m,\ell)f(z)$ (see Kumar et al. [9] and Srivastava et al. [16]);
(ii) $I_p^m(1,0)f(z) = D_p^m f(z)$ (see [2], [8] and [11]);
(iii) $I_1^m(1,\ell)f(z) = I_\ell^m f(z)$ (see Cho and Srivastava [4] and Cho and Kim [5]);
(iv) $I_1^m(1,0) = D^m f(z)$ $(m \in N_0)$ (see Salagean [14]);
(v) $I_1^m(\lambda,0) = D_\lambda^m$ (see Al-Aboudi [1]);
(vi) $I_1^m(1,1) = I^m f(z)$ (see Uralegaddi and Somanatha [17]);
(vii) $I_p^m(\lambda,0) = D_{\lambda,p}^m f(z)$, where $D_{\lambda,p}^m f(z)$ is defined by

$$D_{\lambda,p}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\lambda(k-p)}{p}\right]^m a_k z^k.$$

Furthermore we define the integral operator $J_p^m(\lambda,\ell)f(z)$ as follows:

$$J_p^0(\lambda,\ell)f(z) = f(z),$$

$$J_p^1(\lambda,\ell)f(z) = \left(\frac{p+\ell}{\lambda}\right) z^{p-(\frac{p+\ell}{\lambda})} \int_0^z t^{(\frac{p+\ell}{\lambda})-p-1} f(t)dt \quad (f(z) \in A(p); z \in U),$$

$$J_p^2(\lambda,\ell)f(z) = \left(\frac{p+\ell}{\lambda}\right) z^{p-(\frac{p+\ell}{\lambda})} \int_0^z t^{(\frac{p+\ell}{\lambda})-p-1} J_p^1(\lambda,\ell)f(t)dt \quad (f(z) \in A(p); z \in U),$$

and, in general,

$$J_p^m(\lambda,\ell)f(z) = \left(\frac{p+\ell}{\lambda}\right) z^{p-(\frac{p+\ell}{\lambda})} \int_0^z t^{(\frac{p+\ell}{\lambda})-p-1} J_p^{m-1}(\lambda,\ell)f(t)dt$$

$$= \underbrace{J_p^1(\lambda,\ell)\left(\frac{z^p}{1-z}\right) * J_p^1(\lambda,\ell)\left(\frac{z^p}{1-z}\right) * \ldots . J_p^1(\lambda,\ell)\left(\frac{z^p}{1-z}\right)}_{m-times} * f(z)$$

$$(f(z) \in A(p); m \in N; z \in U). \tag{1.3}$$

We note that if $f(z) \in A(p)$, then from (1.1) and (1.3), we have

$$J_p^m(\lambda,\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell}{p+\ell+\lambda(k-p)}\right)^m a_k z^k \quad (m \in N_0 = N \cup \{0\}). \tag{1.4}$$

From (1.4), it is easy to verify that

$$\lambda z(J_p^{m+1}(\lambda,\ell)f(z))' = (\ell+p)J_p^m(\lambda,\ell)f(z) - [\ell+p(1-\lambda)]\, J_p^{m+1}(\lambda,\ell)f(z) \ (\lambda > 0). \tag{1.5}$$

We note that:
(i)$J_1^m(\lambda,0)f(z) = I_\lambda^{-m}f(z)$ (see Patel [12])

$$= \left\{ f(z) \in A(1) : I_\lambda^{-m}f(z) = z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^{-m}a_k z^k, m \in N_0 \right\};$$

(ii) $J_1^\alpha(1,1)f(z) = I^\alpha f(z)$ (see Jung et al. [7]);

$$= \left\{ f(z) \in A(1) : I^\alpha f(z) = z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^\alpha a_k z^k; \alpha > 0; z \in U \right\};$$

(iii) $J_p^\alpha(1,1)f(z) = I_p^\alpha f(z)$ (see Shams et al. [15]);

$$= \left\{ f(z) \in A(p) : I_p^\alpha f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+1}{k+1}\right)^\alpha a_k z^k; \alpha > 0; z \in U \right\};$$

(iv) $J_p^m(1,1)f(z) = D^m f(z)$ (see Patel and Sahoo [13]);

$$= \left\{ f(z) \in A(p) : D^m f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+1}{k+1}\right)^m a_k z^k; m \text{ is any integer; } z \in U \right\}.$$

(v) $J_1^m(1,1)f(z) = I^m f(z)$ (see Flett [6]);

$$= \left\{ f(z) \in A(1) : I^m f(z) = z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^m a_k z^k; m \in N_0; z \in U \right\};$$

(iv) $J_1^m(1,0)f(z) = I^m f(z)$ (see Salagean [14])

$$= \left\{ f(z) \in A(1) : I^m f(z) = z + \sum_{k=2}^{\infty} k^{-m} a_k z^k; m \in N_0; z \in U \right\}.$$

Also we note that:
(i)$J_p^m(1,0)f(z) = J_p^m f(z)$

$$= \left\{ f(z) \in A(p) : J_p^m f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{p}{k}\right)^m a_k z^k; m \in N_0; z \in U \right\};$$

(ii) $J_p^m(1,\ell)f(z) = J_p^m(\ell)f(z)$

$$= \left\{ f(z) \in A(p) : J_p^m(\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell}{k+\ell}\right)^m a_k z^k; m \in N_0; \ell \geq 0; z \in U \right\};$$

(iii) $J_p^m(\lambda,0)f(z) = J_{\lambda,p}^m f(z)$

$$= \left\{ f(z) \in A(p) : J_{\lambda,p}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)}\right)^m a_k z^k; m \in N_0; \lambda \geq 0; z \in U \right\}.$$

By using the operator $J_p^m(\lambda,\ell)$, we define the following classes of functions.

Definition 1. *Let Φ be the set of complex-valued functions $\varphi(r,s,t)$;*

$$\varphi(r,s,t) : C^3 \to C \ (C \text{ is complex plane})$$

such that

(i) *$\varphi(r,s,t)$ is continuous in a domain $D \subset C^3$;*

(ii) *$(0,0,0) \in D$ and $|\varphi(0,0,0)| < 1$;*

(iii) $\left| \left(\varphi e^{i\theta}, \frac{1}{\left(\frac{p+\ell}{\lambda}\right)} \left(\zeta + \left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right) e^{i\theta}, \frac{1}{\left(\frac{p+\ell}{\lambda}\right)^2} \left[\left\{ \left(1 + 2\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right) \zeta + \left[\frac{p(1-\lambda)+\ell}{\lambda}\right]^2 \right\} e^{i\theta} + M \right] \right) \right| > 1,$

whenever $\left(e^{i\theta}, \frac{1}{\left(\frac{p+\ell}{\lambda}\right)} \left(\zeta + \left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right) e^{i\theta}, \frac{1}{\left(\frac{p+\ell}{\lambda}\right)^2} \left[\left\{ \left(1 + 2\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right) \zeta + \left[\frac{p(1-\lambda)+\ell}{\lambda}\right]^2 \right\} e^{i\theta} + M \right] \right) \in D, \lambda > 0, \ell \geq 0,$ *with* $\operatorname{Re}\left\{e^{-i\theta}M\right\} \geq \zeta(\zeta-1)$, *for all $\theta \in R$, and for all $\zeta \geq p \geq 1$.*

Definition 2. *Let H be a set of complex-valued functions $h(r,s,t)$;*

$$h(r,s,t) : C^3 \to C$$

(i) *$h(r,s,t)$ is continuous in a domain $D \subset C^3$;*

(ii) *$(1,1,1) \in D$ and $|h(1,1,1)| < M$ $(M > 1)$;*

(iii) $\left| h \left(M e^{i\theta}, \frac{\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta}}{\left(\frac{p+\ell}{\lambda}\right)}, \frac{1}{\left(\frac{p+\ell}{\lambda}\right)} \left[\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta} + \right. \right. \right.$

$$\left.\left.\left.\frac{\zeta-\zeta^2+\left(\frac{p+\ell}{\lambda}\right)\zeta Me^{i\theta}+L}{\zeta+\left(\frac{p+\ell}{\lambda}\right)Me^{i\theta}}\right]\right)\right| \geq M,$$

whenever

$$\left(Me^{i\theta}, \frac{\zeta+\left(\frac{p+\ell}{\lambda}\right)Me^{i\theta}}{\left(\frac{p+\ell}{\lambda}\right)}, \frac{1}{\left(\frac{p+\ell}{\lambda}\right)}\left[\zeta+\left(\frac{p+\ell}{\lambda}\right)Me^{i\theta}+\right.\right.$$

$$\left.\left.+\frac{\zeta-\zeta^2+\left(\frac{p+\ell}{\lambda}\right)\zeta Me^{i\theta}+L}{\zeta+\left(\frac{p+\ell}{\lambda}\right)Me^{i\theta}}\right]\right) \in D,$$

where $\operatorname{Re}\{L\} \geq \zeta(\zeta-1)$ *for all* $\theta \in R$ *and for all* $\zeta \geq \dfrac{M-1}{M+1}$.

2.Main Results

We recall the following lemmas due to Miller and Mocenu [10].

Lemma 1 [10]. *Let* $w(z) = b_p z^p + b_{p+1}z^{p+1} + ...(p \in N)$ *be regular in the unit disc* U *with* $w(0) \neq 0$ $(z \in U)$.

If $z_0 = r_0 e^{i\theta_0}(0 < r_0 < 1)$ *and* $|w(z_0)| = \max\limits_{|z|\leq z_0} |w(z)|$, *then*

$$z_0 w^{'}(z_0) = \zeta w(z_0) \tag{2.1}$$

and

$$\operatorname{Re}\left\{1+\frac{z_0 w^{''}(z_0)}{w^{'}(z_0)}\right\} \geq \zeta \tag{2.2}$$

and

where ζ *is real and* $\zeta \geq p \geq 1$.

Lemma 2 [10]. *Let* $w(z) = a + w_\nu z^\nu + ...$ *be regular in* U *with* $w(z) \neq a$ *and* $\nu \geq 1$ *If* $z_0 = r_0 e^{i\theta_0}(0 < r_0 < 1)$ *and* $|w(z_0)| = \max\limits_{|z|\leq z_0} |w(z)|$, *then*

$$z_0 w^{'}(z_0) = \zeta w(z_0)$$

and

$$\operatorname{Re}\left\{1+\frac{z_0 w^{''}(z_0)}{w^{'}(z_0)}\right\} \geq \zeta,$$

where ζ is a real number and

$$\zeta \geq \nu \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq \nu \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}. \tag{2.3}$$

Theorem 1. *Let $\varphi(r,s,t) \in \Phi$ and let $f(z)$ belonging to the class $A(p)$ satisfy*

$$(i)\ (J_p^m(\lambda,\ell)f(z)\ ,\ J_p^{m-1}(\lambda,\ell)f(z)\ ,\ J_p^{m-2}(\lambda,\ell)f(z)) \in D \subset C^3$$

and

$$(ii)\ \left|\varphi(J_p^m(\lambda,\ell)f(z)\ ,\ I_p^{m-1}(\lambda,\ell)f(z)\ ,\ I_p^{m-2}(\lambda,\ell)f(z))\right| < 1$$

for $m > 2, p \in N, \lambda > 0, \ell \geq 0$ and $z \in U$. Then we have

$$\left|J_p^m(\lambda,\ell)f(z)\right| < 1 \quad (z \in U). \tag{2.4}$$

Proof. We define the analytic function $w(z)$ by

$$J_p^m(\lambda,\ell)f(z) = w(z) \quad (m > 2; p \in N; \lambda > 0; \ell \geq 0) \tag{2.5}$$

for $f(z)$ belonging to the class $A(p)$. Then, it folllows that $w(z) \in A(p)$ and $w(z) \neq 0$ $(z \in U)$. With the aid of the idenlitty (1.5), we have

$$J_p^{m-1}(\lambda,\ell)f(z) = \frac{1}{\left(\frac{p+\ell}{\lambda}\right)} \left\{ \left[\frac{p(1-\lambda)+\ell}{\lambda}\right] w(z) + zw^{'}(z) \right\} \tag{2.6}$$

and

$$J_p^{m-2}(\lambda,\ell)f(z) = \frac{1}{\left(\frac{p+\ell}{\lambda}\right)^2} \left\{ \left[\frac{p(1-\lambda)+\ell}{\lambda}\right]^2 w(z) + \left(1 + 2\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right) zw^{'}(z) + z^2 w^{''}(z) \right\}. \tag{2.7}$$

Suppose that $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1; \theta_0 \in R)$ and

$$w(z_0) = \max_{|z| \leq |z_0|} |w(z)| = 1. \tag{2.8}$$

Then, letting $w(z_0) = e^{i\theta_0}$ and using (2.1) of Lemma 1, we obtain

$$J_p^m(\lambda,\ell)f(z_0) = w(z_0) = e^{i\theta_o}, \tag{2.9}$$

$$J_p^{m-1}(\lambda,\ell)f(z_0)=\frac{1}{(\frac{p+\ell}{\lambda})}\left\{\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]w(z_0)+z_0w^{'}(z_0)\right\}$$
$$=\frac{1}{(\frac{p+\ell}{\lambda})}\left(\zeta+\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right)e^{i\theta_0}$$

and

$$J_p^{m-2}(\lambda,\ell)f(z_0)=\frac{1}{(\frac{p+\ell}{\lambda})^2}\left[\left\{\left(1+2\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right)\zeta+\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]^2\right\}e^{i\theta_0}+z_0^2w^{''}(z_0)\right]$$

$$=\frac{1}{(\frac{p+\ell}{\lambda})^2}\left[\left\{\left(1+2\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right)\zeta+\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]^2\right\}e^{i\theta_0}+M\right], \qquad (2.11)$$

where $M=z_0^2w^{''}(z_0)$ and $\zeta\geq p\geq 1$.

Further, an application of (2.2) in Lemma 1, gives

$$\operatorname{Re}\left\{\frac{z_0w^{''}(z_0)}{w^{'}(z_0)}\right\}=\operatorname{Re}\left\{\frac{z_0^2w^{''}(z_0)}{\zeta e^{i\theta_0}}\right\}\geq\zeta-1, \qquad (2.12)$$

or

$$\operatorname{Re}\left\{e^{-i\theta_0}M\right\}\geq\zeta(\zeta-1) \quad (\theta_0\in R;\zeta\geq 1). \qquad (2.13)$$

Since $\varphi(r,s,t)\in\Phi$, we also have

$$\left|\varphi(J_p^m(\lambda,\ell)f(z_0),J^{m-1}(\lambda,\ell)f(z_0),J^{m-2}(\lambda,\ell)f(z_0))\right|$$
$$=\left|\varphi\left(e^{i\theta_0},\frac{1}{(\frac{p+\ell}{\lambda})}\left(\zeta+\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right)e^{i\theta_0},\frac{1}{(\frac{p+\ell}{\lambda})^2}\left[\left\{\left(1+2\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]\right)\zeta+\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]^2\right\}e^{i\theta_0}+M\right]\right)\right|>1 \qquad (2.14)$$

which contradicts the condition (ii) of Theorem 1 Therefore, we conclude that

$$|w(z)|=\left|J_p^m(\lambda,\ell)f(z)\right|<1 \quad (z\in U;\quad m>2).$$

Corollary 1. *Let* $\varphi_1(r,s,t)=s$ *and let* $f(z)\in A(p)$ *satisfy the conditions in Theorem 1 for* $m>2;p\in N;\lambda>0;\ell\geq 0$ *and* $z\in U$. *Then*

$$\left|(J_p^{m+i}(\lambda,\ell)f(z)\right|<1 \ (i=0,1,...;m>2;p\in N;\lambda>0;\ell\geq 0)$$

Proof. Note that $\varphi_1(r,s,t) = s \quad in\ \Phi$, with the aid of Theorem 1, we have

$$\left|J_p^{m-1}(\lambda,\ell)f(z)\right| < 1 \Rightarrow \left|J_p^{m}(\lambda,\ell)f(z)\right| < 1 \quad (m > 2; p \in N; \lambda > 0; \ell \geq 0)$$

$$\Rightarrow \left|J_p^{m+i}(\lambda,\ell)f(z)\right| < 1 \quad (i = 0,1,2,...; m > 2; p \in N; \lambda > 0; \ell \geq 0; z \in U).$$

Theorem 2. *Let* $h(r,s,t) \in H$, *and let* $f(z)$ *belonging to* $A(p)$ *satisfying*

$$(i) \quad \left(\frac{J_p^{m-1}(\lambda,\ell)f(z)}{J_p^{m}(\lambda,\ell)f(z)}, \frac{J_p^{m-2}(\lambda,\ell)f(z)}{J_p^{m-1}(\lambda,\ell)f(z)}, \frac{J_p^{m-3}(\lambda,\ell)f(z)}{J_p^{m-2}(\lambda,\ell)f(z)}\right) \in D \subset C^3$$

and

$$(ii) \quad \left|h\left(\frac{J_p^{m-1}(\lambda,\ell)f(z)}{J_p^{m}(\lambda,\ell)f(z)}, \frac{J_p^{m-2}(\lambda,\ell)f(z)}{J_p^{m-1}(\lambda,\ell)f(z)}, \frac{J_p^{m-3}(\lambda,\ell)f(z)}{J_p^{m-2}(\lambda,\ell)f(z)}\right)\right| < M$$

for some λ, ℓ, m, p, M $(\lambda > 0; \ell \geq 0; m > 3; p \in N; M > 1)$ *and for all* $z \in U$. *Then we have*

$$\left|\frac{J_p^{m-1}(\lambda,\ell)f(z)}{J_p^{m}(\lambda,\ell)f(z)}\right| < M \quad (z \in U) \tag{2.16}$$

Proof. We define the function $w(z)$ by

$$\left|\frac{J_p^{m-1}(\lambda,\ell)f(z)}{J_p^{m}(\lambda,\ell)f(z)}\right| = w(z) \quad (\lambda > 0, \ell \geq 0; m > 3; p \in N) \tag{2.17}$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z)$ is either analytic or meromorphic in U, $w(0) = 1$, and $w(z) \neq 1$. With the aid of the identity (1.5), we have

$$\left|\frac{J_p^{m-2}(\lambda,\ell)f(z)}{J_p^{m-1}(\lambda,\ell)f(z)}\right| = \frac{1}{\left(\frac{p+\ell}{\lambda}\right)}\left[\left(\frac{p+\ell}{\lambda}\right)w(z) + \frac{zw'(z)}{w(z)}\right] \tag{2.18}$$

and

$$\left|\frac{J_p^{m-3}(\lambda,\ell)f(z)}{J_p^{m-2}(\lambda,\ell)f(z)}\right| = \frac{1}{\left(\frac{p+\ell}{\lambda}\right)}\left[\left(\frac{p+\ell}{\lambda}\right)w(z) + \frac{zw'(z)}{w(z)}\right] +$$

$$\left.\frac{\left(\frac{p+\ell}{\lambda}\right)zw'(z) + \frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - (\frac{zw'(z)}{w(z)})^2}{\left(\frac{p+\ell}{\lambda}\right)w(z) + \frac{zw'(z)}{w(z)}}\right]. \tag{2.19}$$

Suppose that $z_0 = r_0 e^{i\theta_0} (0 < r_0 < 1; \theta_0 \in R)$ and $|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| = M$. Letting $w(z_0) = M e^{i\theta_0}$ and applying Lemma 2 with $a = \nu = 1$, we see that

$$\frac{J_p^{m-2}(\lambda, \ell) f(z)}{J_p^{m-1}(\lambda, \ell) f(z)} = \frac{1}{\left(\frac{p+\ell}{\lambda}\right)} \left[\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta_0}\right] \tag{2.20}$$

and

$$\frac{J_p^{m-3}(\lambda, \ell) f(z_0)}{J_p^{m-2}(\lambda, \ell) f(z_0)} = \frac{1}{\left(\frac{p+\ell}{\lambda}\right)} \left\{\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta_0} + \frac{\zeta - \zeta^2 + \left(\frac{p+\ell}{\lambda}\right) \zeta M e^{i\theta_0} + L}{\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta_0}}\right\},$$

where $L = \frac{z_0^2 w''(z_0)}{w(z_0)}$ and $\zeta \geq \frac{M-1}{M+1}$.

Further, an application of (ii) in Lemma 2 gives

$$\operatorname{Re}\{L\} \geq \zeta(\zeta - 1).$$

Since $h(r, s, t) \in H$, we have

$$\left| h\left(\frac{J_p^{m-1}(\lambda, \ell) f(z_0)}{J_p^{m}(\lambda, \ell) f(z_0)}, \frac{J_p^{m-2}(\lambda, \ell) f(z_0)}{J_p^{m-1}(\lambda, \ell) f(z_0)}, \frac{J_p^{m-3}(\lambda, \ell) f(z_0)}{J_p^{m-2}(\lambda, \ell) f(z_0)}\right)\right|$$

$$= \left| h\left(M e^{i\theta_0}, \frac{\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta_0}}{\left(\frac{p+\ell}{\lambda}\right)}, \frac{1}{\left(\frac{p+\ell}{\lambda}\right)} \left[\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta_0} + \frac{\zeta - \zeta^2 + \left(\frac{p+\ell}{\lambda}\right) \zeta M e^{i\theta_0} + L}{\zeta + \left(\frac{p+\ell}{\lambda}\right) M e^{i\theta_0}}\right\}\right)\right| \geq M,$$

which contradicts condition (ii) of Theorem 2. Therefore, we conclude that

$$|w(z)| = \left|\frac{J_p^{m-1}(\lambda, \ell) f(z)}{J_p^{m}(\lambda, \ell) f(z)}\right| < M \tag{2.23}$$

for all $\lambda > 0, \ell \geq 0, m > 3, p \in N$ and $z \in U$. This completes the proof of Theorem 2.

Remarks. (i) Putting $\lambda = 1$ in the above results, we obtain the corresponding results for the operatr $J_p^m(\ell) f(z)$;

(ii) Putting $\ell = 0$ in the above results, we obtain the corresponding results for the operatr $J_{\lambda,p}^m f(z)$;

(iii) Putting $\lambda = \ell = 1$ in the above results, we obtain the corresponding results for the operatr $D^m f(z)$.

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