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DIRICHLET BOUNDARY VALUE PROBLEMS OF NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS WITH JACOBI OPERATORS

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Abstract. In this paper, the solutions to Dirichlet boundary value problems of nonlinear functional difference equations with Jacobi operators are investigated. By using critical point theory, the existence and multiplicity results are obtained.

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1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. Δ is the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$. k is a positive integer and * is the transpose sign for a vector.

Consider the second order functional difference equation

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}), \tag{1}$$

with boundary value conditions

$$u_0 = A, \ u_{k+1} = B, \tag{2}$$

where the operator L is the Jacobi operator

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n,$$

a_n and b_n are real valued for each $n \in \mathbf{Z}$, $f \in C(\mathbf{R}^4, \mathbf{R})$, A and B are constants.

We may think of Eq. (1) as being a discrete analogue of the second order functional differential equation

$$Su(t) + f(t, u(t+1), u(t), u(t-1)) = 0, \ t \in \mathbf{R} \tag{3}$$

which includes the following equation

$$c^2u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)),\ t \in \mathbf{R} \tag{4}$$

where S is the Sturm-Liouville differential expression and $f \in C(\mathbf{R}^4, \mathbf{R})$. Eq. (4) has been studied extensively by many scholars. For example, Smets and Willem have obtained the existence of solitary waves of lattice differential equations, see [21] and the references cited therein.

Jacobi operators appear in a variety of applications [22]. They can be viewed as the discrete analogue of Sturm-Liouville operators and their investigation has many similarities with Sturm-Liouville theory. Whereas numerous books about Sturm-Liouville operators have been written, only few on Jacobi operators exist. In particular, there are currently fewer researches available which cover some basic topics (like positive solutions, periodic operators, boundary value problems, etc.) typically found in textbooks on Sturm-Liouville operators [12].

It is well known that difference equations occur widely in numerous setting and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields, see for examples [1,6,9,10,14,19]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity and results on oscillation and other topics, see [1-4,6-11,14-16,18-20,23-25]. However, to our best knowledge, no similar results are obtained in the literature for the boundary value problem (BVP) (1) with (2). Since f in Eq. (1) depends on u_{n+1} and u_{n-1}, the traditional ways of establishing the functional in [2,23-25] are inapplicable to our case.

Our aim in this paper is to use the critical point theory to give some sufficient conditions for the existence and multiplicity of the BVP (1) with (2). The main idea in this paper is to transfer the existence of the BVP (1) with (2) into the existence of the critical points [13] of some functional.

Our main results are as follows.

Let

$$p_{\max} = \max\{a_n : n \in \mathbf{Z}(0,k)\},\ p_{\min} = \min\{a_n : n \in \mathbf{Z}(0,k)\},$$

$$q_{\max} = \max\{b_n + a_{n-1} + a_n : n \in \mathbf{Z}(1,k)\},\ q_{\min} = \min\{b_n + a_{n-1} + a_n : n \in \mathbf{Z}(1,k)\}.$$

Theorem 1. *Assume that the following hypotheses are satisfied:*

(F_1) *there exists a constant* $M_0 > 0$ *and a functional* $F(n,\cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(0,\cdot) = 0$ *such that for any* $n \in \boldsymbol{Z}(1,k)$,

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),$$

$$\left|\frac{\partial F(n, v_1, v_2)}{\partial v_1}\right| \leq M_0, \ \left|\frac{\partial F(n, v_1, v_2)}{\partial v_2}\right| \leq M_0; \tag{5}$$

(F_2) *for any* $n \in \boldsymbol{Z}(0, k)$, $a_n > 0$; *for any* $n \in \boldsymbol{Z}(1, k)$, $b_n + a_{n-1} + a_n > 0$;

(F_3) $4p_{\max} < q_{\min}$.

Then the BVP (1) with (2) possesses at least one solution.

Remark 1. (5) implies that there exists a constant $M_1 > 0$ such that

$$|F(n, v_1, v_2)| \leq M_1 + M_0(|v_1| + |v_2|), \ \forall n \in \mathbf{Z}(1, k). \tag{6}$$

Corollary 1. *Suppose that* (F_1) *and* (F_3) *are satisfied. And if*

(F_4) *for any* $n \in \boldsymbol{Z}(0, k)$, $a_n < 0$; *for any* $n \in \boldsymbol{Z}(1, k)$, $b_n + a_{n-1} + a_n < 0$.

Then the BVP (1) with (2) possesses at least one solution.

Corollary 2. *Assume that* (F_1)*is satisfied. And if*

(F_5) *for any* $n \in \boldsymbol{Z}(0, k)$, $a_n < 0$; *for any* $n \in \boldsymbol{Z}(1, k)$, $b_n + a_{n-1} + a_n > 0$.

Then the BVP (1) with (2) possesses at least one solution.

Theorem 2. *Suppose that the following hypotheses are satisfied:*

(F_6) *there exists a functional* $F(n, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(0, \cdot) = 0$ *such that*

$$\lim_{r \to 0} \frac{F(n, v_1, v_2)}{r^2} = 0, \ r = \sqrt{v_1^2 + v_2^2}, \ \forall n \in \boldsymbol{Z}(1, k);$$

(F_7) *there exists a constant* $\beta > 2$ *such that for any* $n \in \boldsymbol{Z}(1, k)$,

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),$$

$$\frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \leq \beta F(n, v_1, v_2) < 0, \ \forall (v_1, v_2) \neq 0; \tag{7}$$

(F_8) *for any* $n \in \boldsymbol{Z}(0, k)$, $a_n > 0$; *for any* $n \in \boldsymbol{Z}(1, k)$, $b_n + a_{n-1} + a_n \leq 0$;

(F_9) $A = B = 0$.

Then the BVP (1) with (2) possesses at least two nontrivial solutions.

Remark 2. (7) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$F(n, v_1, v_2) \leq -a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\beta} + a_2, \ \forall n \in \mathbf{Z}(1, k). \tag{8}$$

The rest of the paper is organized as follows. In Sect. 2 we shall establish the variational framework for the BVP (1) with (2) in order to apply the critical point method and give some useful lemmas. In Sect. 3 we shall complete the proof of the main results and give an example to illustrate the result.

2. Variational Structure and Some Lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (1) with (2) and give some basic notations and useful lemmas.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v\rangle = \sum_{j=1}^{k} u_j v_j, \ \forall u, v \in \mathbf{R}^k, \tag{9}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in \mathbf{R}^k. \tag{10}$$

On the other hand, we define the norm $\|\cdot\|_r$ on $\mathbf{R}^k$ as follows:

$$\|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r\right)^{\frac{1}{r}}, \tag{11}$$

for all $u \in \mathbf{R}^k$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $c_1, \ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_r \leq c_2\|u\|_2, \ \forall u \in \mathbf{R}^k. \tag{12}$$

Clearly, $\|u\| = \|u\|_2$. For the BVP (1) with (2), consider the functional J on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{2}\sum_{n=0}^{k} a_n(\Delta u_n)^2 - \frac{1}{2}\sum_{n=1}^{k}(b_n + a_{n-1} + a_n)u_n^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n), \tag{13}$$

$\forall u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k, \ u_0 = A, \ u_{k+1} = B.$

Clearly, $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $u = \{u_n\}_{n=0}^{k+1} = (u_0, u_1, \ldots, u_{k+1})^*$, by using $u_0 = A, \ u_{k+1} = B$, we can compute the partial derivative as

$$\begin{aligned}\frac{\partial J}{\partial u_n} &= -a_n\Delta u_n + a_{n-1}\Delta u_{n-1} - (b_n + a_{n-1} + a_n)u_n + f(n, u_{n+1}, u_n, u_{n-1}) \\ &= -Lu_n + f(n, u_{n+1}, u_n, u_{n-1}), \ n \in \mathbf{Z}(1, k).\end{aligned}$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}),\ \forall n \in \mathbf{Z}(1,k).$$

We reduce the existence of the BVP (1) with (2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1) with (2).

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to be satisfying the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(k)}\right\} \subset E$ for which $\left\{J\left(u^{(k)}\right)\right\}$ is bounded and $J'\left(u^{(k)}\right) \to 0 (k \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 1 *(Mountain Pass Lemma [17]). Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfies the P.S. condition. If $J(0) = 0$ and*
(J_1) there exist constants $\rho,\ a > 0$ such that $J|_{\partial B_\rho} \geq a$, and
(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.
Then J possesses a critical value $c \geq a$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)), \tag{14}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{15}$$

Lemma 2. *Suppose that $(F_6) - (F_9)$ is satisfied. Then the functional J satisfies the P.S. condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k,\ l \in \mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded. Then there exists a positive constant M_2 such that

$$-M_2 \leq J\left(u^{(l)}\right) \leq M_2,\ \forall l \in \mathbf{N}.$$

By (1.8), we have

$$-M_2 \leq J\left(u^{(l)}\right) = \frac{1}{2}\sum_{n=0}^{k} a_n \left(\Delta u_n^{(l)}\right)^2 - \frac{1}{2}\sum_{n=1}^{k}(b_n + a_{n-1} + a_n)\left(u_n^{(l)}\right)^2 + \sum_{n=1}^{k} F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right)$$

$$\leq p_{\max}\sum_{n=0}^{k}\left[\left(u_{n+1}^{(l)}\right)^2 + \left(u_n^{(l)}\right)^2\right] - \frac{q_{\min}}{2}\left\|u^{(l)}\right\|^2 - a_1\sum_{n=1}^{k}\left[\sqrt{\left(u_{n+1}^{(l)}\right)^2 + \left(u_n^{(l)}\right)^2}\right]^\beta + a_2 k$$

$$\leq 2p_{\max}\left\|u^{(l)}\right\|^2 - \frac{q_{\min}}{2}\left\|u^{(l)}\right\|^2 - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + a_2 k.$$

That is,

$$a_1 c_1^{\beta} \left\| u^{(l)} \right\|^{\beta} - \left(2p_{\max} - \frac{q_{\min}}{2} \right) \left\| u^{(l)} \right\|^{2} \le M_2 + a_2 k.$$

Since $\beta > 2$, there exists a constant $M_3 > 0$ such that

$$\left\| u^{(l)} \right\| \le M_3, \ \forall l \in \mathbf{N}.$$

Therefore, $\left\{ u^{(l)} \right\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\left\{ u^{(l)} \right\}$ possesses a convergence subsequence in $\mathbf{R}^k$. And thus the P.S. condition is verified.

3. Proof of the Main Results

In this section, we shall complete the proof of Theorems 1 and 2.

3.1. Proof of Theorem 1

Proof. By (6), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{n=0}^{k} a_n (\Delta u_n)^2 - \frac{1}{2}\sum_{n=1}^{k} (b_n + a_{n-1} + a_n) u_n^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\ &\le p_{\max} \sum_{n=0}^{k} \left(u_{n+1}^2 + u_n^2 \right) - \frac{q_{\min}}{2} \|u\|^2 + M_0 \sum_{n=1}^{k} \left(|u_{n+1}| + |u_n| \right) + M_1 k \\ &\le 2p_{\max} \sum_{n=1}^{k} u_n^2 + p_{\max} \left(A^2 + B^2 \right) - \frac{q_{\min}}{2} \|u\|^2 + M_0 \left(2\sum_{n=1}^{k} |u_n| + |B| \right) + M_1 k \\ &\le \left(2p_{\max} - \frac{q_{\min}}{2} \right) \|u\|^2 + 2M_0 \sqrt{k} \|u\| + M_0 |B| + M_1 k + p_{\max} \left(A^2 + B^2 \right) \\ &\to -\infty, (\|u\| \to +\infty). \end{aligned}$$

By continuity of J on $\mathbf{R}^k$ and above argument, there exists $u_0 \in \mathbf{R}^k$ such that

$$J(u_0) = \max \left\{ J(u) | u \in \mathbf{R}^k \right\}.$$

Clearly, u_0 is a critical point of the functional J. The proof of Theorem 1 is complete.

3.2. Proof of Theorem 2

Proof. By (F_6), for any $\epsilon = \frac{1}{8} p_{\min} \lambda_1$($\lambda_1$ can be referred to (16)), there exists $\rho > 0$, such that

$$|F(n, v_1, v_2)| \le \frac{1}{8} p_{\min} \lambda_1 \left(v_1^2 + v_2^2 \right), \forall n \in \mathbf{Z}(1, k),$$

for $\sqrt{v_1^2+v_2^2} \leq \sqrt{2}\rho$.

For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$ and $\|u\| \leq \rho$, we have $|u_n| \leq \rho,\ n \in \mathbf{Z}(1,k)$.

For any $n \in \mathbf{Z}(1,k)$,

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{n=0}^{k} a_n(\Delta u_n)^2 - \frac{1}{2}\sum_{n=1}^{k}(b_n + a_{n-1} + a_n)u_n^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\ &\geq \frac{1}{2}p_{\min}\sum_{n=0}^{k}(\Delta u_n)^2 - \frac{1}{8}p_{\min}\lambda_1\sum_{n=1}^{k}\left(u_{n+1}^2 + u_n^2\right) \\ &\geq \frac{1}{2}p_{\min}(u^* D u) - \frac{1}{4}p_{\min}\lambda_1\|u\|^2, \end{aligned}$$

where $u^* = (u_1, u_2, \cdots, u_k),\ u \in \mathbf{R}^k$,

$$D = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{k\times k.}$$

Clearly, D is positive definite. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the other eigenvalues of D. Applying matrix theory, we know $\lambda_j > 0,\ j = 1, 2, \cdots, k$. Without loss of generality, we may assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k, \tag{16}$$

then

$$\begin{aligned} J(u) &\geq \frac{1}{2}p_{\min}\lambda_1\|u\|^2 - \frac{1}{4}p_{\min}\lambda_1\|u\|^2 \\ &= \frac{1}{4}p_{\min}\lambda_1\|u\|^2. \end{aligned}$$

Take $a\frac{1}{4}p_{\min}\lambda_1\|\rho\|^2 > 0$. Therefore,

$$J(u) \geq a > 0,\ \forall u \in \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \geq a$. That is to say, J satisfies the condition (J_1) of the Mountain Pass Lemma.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify other conditions of the Mountain Pass

Lemma. By Lemma 2, J satisfies the P.S. condition. So it suffices to verify the condition (J_2).

From the proof of the P.S. condition, we know

$$J(u) \leq 2p_{\max}\|u\|^2 - \frac{q_{\min}}{2}\|u\|^2 - a_1c_1^{\beta}\|u\|^{\beta} + a_2k.$$

Since $\beta > 2$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u}) < 0$.

By the Mountain Pass Lemma, J possesses a critical value $c \geq a > 0$, where

$$c = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s)),$$

and

$$\Gamma = \{h \in C([0,1], \mathbf{R}^k) \mid h(0) = 0,\ h(1) = \bar{u}\}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. Similar to the proof of the P.S. condition, we know that there exists $\hat{u} \in \mathbf{R}^k$ such that

$$J(\hat{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s)).$$

Clearly, $\hat{u} \neq 0$. If $\tilde{u} \neq \hat{u}$, then the conclusion of Theorem 2 holds. Otherwise, $\tilde{u} = \hat{u}$. Then $c = J(\tilde{u}) = c_{\max} = \max\limits_{s\in[0,1]} J(h(s))$. That is,

$$\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s\in[0,1]} J(h(s)),\ \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$ such that

$$J\left(h\left(s_0\right)\right) = c_{\max}.$$

Choose $h_1,\ h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_1(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1,\ s_2 \in (0,1)$ such that

$$J\left(h_1\left(s_1\right)\right) = J\left(h_2\left(s_2\right)\right) = c_{\max}.$$

Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by

$$u^1 = h_1\left(s_1\right),\ u^2 = h_2\left(s_2\right).$$

The above argument implies that the BVP (1) with (2) possesses at least two non-trivial solutions. The proof of Theorem 2 is finished.

Remark 3. As an application of Theorem 2, finally, we give an example to illustrate our result.

For $n \in \mathbf{Z}(1,k)$, assume that

$$2u_{n+1} + 2u_{n-1} - 8u_n = -\beta u_n \left[\varphi(n) \left(u_{n+1}^2 + u_n^2 \right)^{\frac{\beta}{2}-1} + \varphi(n-1) \left(u_n^2 + u_{n-1}^2 \right)^{\frac{\beta}{2}-1} \right], \tag{17}$$

with boundary value conditions

$$u_0 = 0, \;\; u_{k+1} = 0, \tag{18}$$

where $\beta > 2$, φ is continuously differentiable and $\varphi(n) > 0$, $n \in \mathbf{Z}(1,k)$ with $\varphi(0) = 0$.

We have

$$a_n = a_{n-1} \equiv 2, \;\; b_n \equiv -8,$$

$$f(n, v_1, v_2, v_3) = -\beta v_2 \left[\varphi(n) \left(v_1^2 + v_2^2 \right)^{\frac{\beta}{2}-1} + \varphi(n-1) \left(v_2^2 + v_3^2 \right)^{\frac{\beta}{2}-1} \right]$$

and

$$F(n, v_1, v_2) = -\varphi(n) \left(v_1^2 + v_2^2 \right)^{\frac{\beta}{2}}.$$

Then

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = -\beta v_2 \left[\varphi(n) \left(v_1^2 + v_2^2 \right)^{\frac{\beta}{2}-1} + \varphi(n-1) \left(v_2^2 + v_3^2 \right)^{\frac{\beta}{2}-1} \right].$$

It is easy to verify all the assumptions of Theorem 2 are satisfied and then the BVP (17) with (18) possesses at least two nontrivial solutions.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, New York, 1992.

[2] R.P. Agarwal, K. Perera and D. O'Regan, *Multiple positive solutions of singular discrete p-Laplacian problems via variational methods*, Adv. Difference Equ. **2005**(2005), 93-99.

[3] D. R. Anderson, R.I. Avery and J. Henderson, *Existence of solutions for a one dimensional p-Laplacian on time-scales*, J. Difference Equ. Appl. **10**(2006), 889-896.

[4] S. Chen, *Disconjugacy, disfocality, and oscillation of second order difference equations*, J. Differential Equations **107**(1994), 383-394.

[5] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985.

[6] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 1996.

[7] P. Eloe, *A boundary value problem for a system of difference equations*, Nonlinear Anal. **7**(1983), 813-820.

[8] J. Henderson and H. B. Thompson, *Existence of multiple solutions for second-order discrete boundary value problems*, Comput. Math. Appl. **43**(2002), 1239-1248.

[9] W.G. Kelly and A.C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, 1991.

[10] V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Application*, Kluwer Academic Publishers, Boston, 1993.

[11] H. Matsunaga, T. Hara and S. Sakata, *Global attractivity for a nonlinear difference equation with variable delay*, Computers Math. Appl. **41**(2001), 543-551.

[12] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[13] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.

[14] R.E. Mickens, *Difference Equations: Theory and Application*, Van Nostrand Reinhold, New York, 1990.

[15] A. Pankov and N. Zakharchenko, *On some discrete variational problems*, Acta Appl. Math. **65**(2001), 295-303.

[16] A. Peterson, *Boundary value problems for an nth order linear difference equations*, SIAM J. Math. Anal. **15**(1984), 124-132.

[17] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Amer. Math. Soc., Providence, RI, New York, 1986.

[18] Y. Rodrigues, *On nonlinear discrete boundary value problems*, J. Math. Anal. Appl. **114**(1986), 398-408.

[19] A.N. Sharkovsky, Y.L. Maistrenko and E.Y. Romanenko, *Difference Equations and Their Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[20] H.-P. Shi, W.-P. Ling, Y.-H. Long and H.-Q. Zhang, *Periodic and subharmonic solutions for second order nonlinear functional difference equations*, Commun. Math. Anal. **5**(2) (2008), 50-59.

[21] D. Smets and M. Willem, *Solitary waves with prescribed speed on infinite lattices*, J. Funct. Anal. **149**(1997), 266-275.

[22] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Amer. Math. Soc., Providence, RI, New York, 2000.

[23] J.-S. Yu and Z.-M. Guo, *Boundary value problems of discrete generalized Emden-Fowler equation*, Sci. China **49**(A)(2006), 1303-1314.

[24] J.-S. Yu and Z.-M. Guo, *On boundary value problems for a discrete generalized Emden-Fowler equation*, J. Differential Equations, **231**(2006), 18-31.

[25] J.-S. Yu, *The minimal period problem for the classical forced pendulum equation*, J. Differential Equations, **247**(2009), 672-684.

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