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SOME RESULTS REGARDING THE BERNSTEIN POLYNOMIALS

Ovidiu T. Pop, Dan Bărbosu and Petru I. Braica

Abstract. The images of the test functions $e_p : [0,1] \to [0,1]$, $e_p(x) = x^p$, $p \in \mathbb{N}$ by the Bernstein's operators $B_m : C([0,1]) \to C([0,1])$, $(B_m e_p)(x)$, $m \in \mathbb{N}$ are established and some applications are given.

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1. Preliminaries

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $m \in \mathbb{N}_0$ and $k \in \mathbb{Z}\backslash\{0, 1, \dots, m\}$, then

$$\binom{m}{k} = 0,\ \binom{0}{0} = 1,\ A_m^k = 0,\ A_0^0 = 1.$$

For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ be the Bernstein operator, defined for any function $f \in C([0,1]$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental Bernstein's polynomials defined for any $x \in [0,1]$ and any $k \in \{0, 1, \dots, m\}$ by

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}. \tag{1.2}$$

2. Main results

Theorem 2.1. *If $k, m \in \mathbb{N}_0$ and $p \in \mathbb{N}$, the following identities*

$$k^p \binom{m}{k} = \sum_{i=0}^{p-1} a_p^{(i)} A_m^{p-i} \binom{m-p+i}{k-p+i}, \tag{2.1}$$

$$a_{p+1}^{(i)} = (p-i+1)a_p^{i-1} + a_p^{(i)}, \tag{2.2}$$

$$a_p^{(i)} > 0 \tag{2.3}$$

hold, where $1 \leq 1 \leq p-1$ *and*

$$a_p^{(0)} = a_p^{(p-1)} = 1. \tag{2.4}$$

Proof. Using the mathematical induction with respect p, we suppose that (2.1) holds. Next, we get successively:

$$\begin{aligned}
k^{p+1}\binom{m}{k} &= k \cdot k^p \binom{m}{p} = k \sum_{i=0}^{p-1} a_p^{(i)} A_m^{p-i} \binom{m-p+i}{k-p+i} \\
&= \sum_{i=0}^{p-1} \{(k-p+i)+(p-i)\} a_p^{(i)} A_m^{p-i} \binom{m-p+i}{k-p+i} \\
&= (m-p) a_p^{(0)} A_m^p \binom{m-p-1}{k-p-1} + \{p a_p^{(0)} A_m^p + (m-p+1) a_p^{(1)} A_m^{p-1} \binom{m-p}{k-p} \\
&\quad + \left\{(p-1) a_p^{(1)} A_m^{p-1} + (m-p+2) a_p^{(2)} A_m^{p-2}\right\} \binom{m-p+1}{k-p+1} + \ldots \\
&\quad + \left\{2 a_p^{(p-2)} A_m^2 + (m-1) a_p^{(p-1)} A_m^1\right\} \binom{m-2}{k-2} + a_p^{(p-1)} A_m^1 \binom{m-1}{k-1} \\
&= A_m^{p+1} \binom{m-p-1}{k-p-1} + \left\{p a_p^{(0)} a_p^{(1)}\right\} A_m^p \binom{m-p}{k-p} \\
&\quad + \left\{(p-1) a_p^{(1)} + a_p^{(2)}\right\} A_m^{p-1} \binom{m-p+1}{k-p+1} + \ldots \\
&\quad + \left\{2 a_p^{(p-2)} + a_p^{(p-1)}\right\} A_m^2 \binom{m-2}{k-2} + a_p^{(p-1)} A_m^1 \binom{m-1}{k-1},
\end{aligned}$$

i.e. the conclusions follow.

Theorem 2.2. *For any $m, p \in \mathbb{N}$ and any $x \in [0,1]$, the following*

$$(B_m e_p)(x) = \frac{1}{n^p} \sum_{i=0}^{p-1} a_p^{(i)} A_m^{p-i} x^{p-i} \tag{2.5}$$

holds.

Proof. Taking the identity (2.1) into account, we get

$$\begin{aligned}(B_m e_p)(x) &= \sum_{k=0}^{m} x^k (1-x)^{m-k} \binom{m}{k} \left(\frac{k}{m}\right)^p \\ &= \sum_{k=0}^{m} x^k (1-x)^{m-k} \frac{1}{m^p} \left\{ k^p \binom{m}{k} \right\} \\ &= \sum_{k=0}^{m} x^k (1-x)^{m-k} \frac{1}{m^p} A_m^{p-i} \binom{m-p+i}{k-p+i} \\ &= \frac{1}{m^p} \sum_{i=0}^{p-1} a_p^{(i)} A_m^{p-i} x^{p-i} \sum_{k=p-i}^{m} x^{k-p+i} (1-x)^{m-k} \binom{m-p+i}{k-p+i}\end{aligned}$$

and because

$$\sum_{k=p-i}^{m} x^{k-p+i} (1-x)^{m-k} \binom{m-p+i}{k-p+i} = 1,$$

the identity (2.5) yields.

From Theorem 2.2 it is immediately the following

Corollary 2.1. *If $n \in \mathbb{N}$, $p \in \mathbb{N}_0$ and $d_{m,p}$ denotes the degree of the polynomial $(B_m e_p)(x)$, then*

$$d_{m,p} = \begin{cases} p, & \text{if } p \le m; \\ m, & \text{if } p > m. \end{cases} \tag{2.6}$$

Considering $m \in \mathbb{N}$ and $x \in [0,1]$ we shall present some applications of the Theorem 2.2.

Application 2.1. For $p = 1$, we get

$$(B_m e_1)(x) = \frac{1}{m} a_1^{(0)} A_m^1 x = x.$$

Application 2.2. For $p = 2$, it follows

$$(B_m e_2)(x) = \frac{1}{m^1} \sum_{i=0}^{1} a_2^{(i)} A_m^{2-i} x^{2-i} = \frac{1}{m^2} \left\{ a_2^{(0)} A_m^2 x^2 + a_2^{(1)} A_m^1 x \right\}$$

and taking (2.3) into account one obtains

$$(B_m e_2)(x) = x^2 + \frac{x(1-x)}{m}.$$

Application 2.3. For $p = 3$, it follows

$$(B_m e_3)(x) = \frac{1}{m^3} \sum_{i=0}^{2} a_3^{(i)} A_m^{3-i} x^{3-i}.$$

But $a_3^{(0)} = a_3^{(2)} = 1$, $a_3^{(1)} = 2a_2^{(0)} + a_2^{(1)} = 3$ and we get

$$(B_m e_3)(x) = \frac{(m-1)(m-2)}{m^2} x^3 + \frac{3(m-1)}{m^2} x^2 + \frac{1}{m^2} x.$$

Corollary 2.2.*If $p \in \mathbb{N}$ and $i \in \{1, 2, \dots, p-1\}$, then*

$$a_p^{(i)} = \sum_{k_i=1}^{p-i} k_i \sum_{k_{i-1}=1}^{k_i} k_{i-1} \cdots \sum_{k_2=1}^{k_3} k_2 \sum_{k_1=1}^{k_2} k_1. \tag{2.7}$$

Proof. We use the mathematical induction with respect i. For $i = 1$, from (2.2) follows $a_{p+1}^{(1)} = pa_p^{(0)} + a^{(1)}$ and because $a_p^{(0)} = 1$, we get $a_{p+1}^{(1)} = p + a_p^{(1)}$. In the above equality, we put successively $p \in \{2, 3, \dots, p-1\}$ and summing the obtained identities one arrives to

$$a_p^{(1)} = (p-1) + (p-2) + \cdots + 2 + a_2^{(1)} = 1 + 2 + \cdots + (p-1) = \sum_{k_1=1}^{p-1} k_1.$$

Suppose that $a_l^{(s)} = \sum\limits_{k_s=1}^{l-s} k_s \sum\limits_{k_{s-1}}^{k_s} k_{s-1} \cdots \sum\limits_{k_1=1}^{k_2} k_1$ for $s \in \{1, 2, \dots, l-2\}$ and $l \in \{1, 2, \dots, p\}$.

Using the identity (2.2), we get

$$a_p^{(s+1)} = (p-s-1)a_{p-1}^{(s)} + (p-s-2)a_{p-2}^{(s)} + (p-s-3)a_{p-3}^{(s)} + \cdots + 2a_{s+2}^{(s)} + a_{s+2}^{(s+1)}.$$

But, from (2.4) it follows $a_{s+2}^{(s+1)} = 1$ and next, using the induction hypothesis, one arrives to

$$a_p^{(s+1)} = \sum_{k_{s+1}=1}^{p-s-1} k_{s+1} a_{s+k_{s+1}}^{(s)} = \sum_{k_{s+1}=1}^{p-s-1} k_{s+1} \sum_{k_s=1}^{k_{s+1}} k_s \cdots \sum_{k_1=1}^{k_2} k_1,$$

which ends the proof.

References

[1] Agratini, O., *Aproximare prin operatori liniari*, Presa Universitară Clujeană, Cluj-Napoca, 2000 (Romanian)

[2] Bărbosu, D., *Introducere în analiza numerică şi teoria aproximării*, Ed. Univ. de Nord Baia Mare, 2009 (Romanian)

[3] Lorentz, G. G., *Bernstein polynomials*, University of Toronto Press, Toronto, 1953

[4] Pop, O. T., *About Bernstein polynomial*, Octogon Mathematical Magazine, **12** (2004), No. 1, 117-122

[5] Stancu, D. D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Analiză numerică şi teoria aproxmării*, **I**, Presa Universitară Clujeană, Cluj-Napoca, 2001 (Romanian)

Ovidiu T. Pop
National College "Mihai Eminescu"
No. 5, Mihai Eminescu Street
440014 Satu Mare, Romania
e-mail: ovidiutiberiu@yahoo.com

Dan Bărbosu
North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
e-mail: barbosudan@yahoo.com

Petru I. Braica
Secondary School "Grigore Moisil"
No. 1 Mileniului Street
440037 Satu Mare, Romania
e-mail: petrubr@yahoo.com