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A BETTER NUMERICAL SOLUTION STABILITY BY USING THE FUNCTION DEVELOPMENTS IN CAZACU'S PROPER SERIES

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Abstract. Starting from the centred finite difference definitions for an analytic function, we obtain a better approximation of its value in a neighbouring point with respect to that, in which one gives the function and its partial differential values.

Using the partial differential expressions, extracted from these proper series and calculating the algebraic relation associated to the stream line function, corresponding to the Navier-Stokes' partial differential equations of viscous fluid flow hydrodynamics, one establishes the error propagation relations by numerical calculus in the domain, tracing the grid relaxation diagrams, which show the error propagation manner in the net on different calculus directions and the limit values of the product between the global Reynolds number, the local relative velocity and the grid relative step.

The use of these proper series gives a better numerical solution stability, because the 4th order partial differentials of the stream line function, which intervene in the stable linear part of the stream line partial differential equation, are greater of about 2 time than these obtained by using of Taylor's series developments and of 32.667 time greater than these obtained by the finite difference method.

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1. Introduction

My former professor and Ph.D. surveyor, Dr.doc. engineer and mathematician Dumitru Dumitrescu, member of Romanian Academy, introduced in 1955 [1] in Romania the numerical integration of partial differential equations for solving of some hydrodynamic problems, by so called relaxation method, using the finite differentials method to express the function discrete values in a grid.

Because I was interested to solve many technical problems, described by complicated non-linear partial differential equation systems, I introduced in my doctoral thesis asserted in 1957 [2] the **function developments in series** of the renowned English mathematician **Brook Taylor** (1685-1731) for the numerical solving of hyperbolic equations, for which the grid relaxation have no-sense and I obtained the first important results between 1964 [3][4][6] and 1970 [5], when I studied the viscous fluid steady flows, given by the non-linear equation system of Navier and Stokes and I developed the numerical integration techniques, concerning a better approximation expressions of the partial differentials higher order, by Brook Taylor's series developments, the boundary conditions in viscous fluid hydrodynamics and the stability of numerical solutions.

In 1979 I introduced the **stability of linearity** idea [7], showing that the 4th order partial differentials deduced by Brook Taylor series, which intervened in the stable linear part of the stream line function equation, are sixteenth time greater than those obtained by finite difference method $\Psi^{iv}|_{DTS} = 16 \times \Psi^{iv}|_{FD}$ [8] and in 2005 I introduced my proper series developments [9].

2. The developments in proper series of a function

Generally, it is very known the fact that the approximation of a continuous, uniform and bounded analytical function in the neighbourhood of a regular point of a domain, by development in infinite series, consist in addition of the partial derivatives in the regular considered point, multiplied by the convenient exponent of the step and some coefficients, whose size diminish in a sufficient measure with the partial differential order, to assure the process of convergence of the approximated function value.

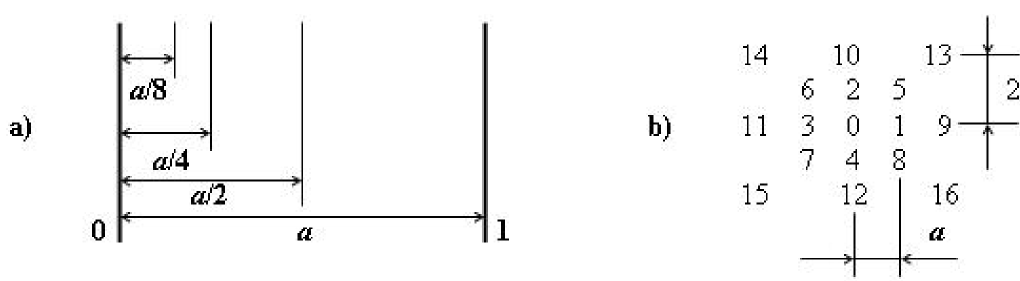

Figure 1: a) grid step division and b) knots numbering in two-dimensional grid

Considering the idea of centred finite differences, I writhed the development in series as

$$f_1 = f_0 + a f'_x(0) + \frac{a^2}{2} f''_{x^2}(0) + \frac{a^3}{8} f'''_{x^3}(0) + \frac{a^4}{64} f^{iv}_{x^4}(0) + ... = f_0 + \sum_{n=0}^{\infty} \left(\prod_{i=0}^{n} \frac{a}{2^i} \right) f^{(n+1)}_{x^{n+1}}(0), \tag{1}$$

justified by the expressions of the centred finite differences:

$$\frac{f_1 - f_0}{a} = f'_x(a/2) = f'_x(0) + \frac{a}{2} f''_{x^2}(0) + \frac{a^2}{8} f'''_{x^3}(0) + \frac{a^3}{64} f^{iv}_{x^4}(0) + ..., \tag{2}$$

$$\frac{\frac{f_1 - f_0}{a} - f'_x(0)}{a/2} = f''_{x^2}(a/4) = f''_{x^2}(0) + \frac{a}{4} f'''_{x^3}(0) + \frac{a^2}{32} f^{iv}_{x^4}(0) + ..., \tag{3}$$

$$\frac{\frac{\frac{f_1 - f_0}{a} - f'_x(0)}{a/2} - f''_{x^2}(0)}{a/4} = f'''_{x^3}(a/8) = f'''_{x^3}(0) + \frac{a}{8} f^{iv}_{x^4}(0) + ..., \tag{4}$$

$$\frac{\frac{\frac{\frac{f_1 - f_0}{a} - f'_x(0)}{a/2} - f''_{x^2}(0)}{a/4} - f'''_{x^3}(0)}{a/8} = f^{iv}_{x^4}(a/16) = f^{iv}_{x^4}(0) + ..., \tag{5}$$

Using the formula (1) we shall fused written the following function developments in finite series for a two-dimensional grid with different steps $a \neq b$ [7], for the neighbouring points of an ordinary point 0, corresponding to the notations from fig.1,b: for the points $1 \div 4$, disposed at the first steps in the small cross position:

$$f_{1,3} = f_0 \pm a f'_x + \frac{a^2}{2} f''_{x^2} \pm \frac{a^3}{8} f'''_{x^3} + \frac{a^4}{64} f^{iv}_{x^4}(0) \pm ..., \tag{6}$$

$$f_{2,4} = f_0 \pm b f'_y + \frac{b^2}{2} f''_{y^2} \pm \frac{b^3}{8} f'''_{y^3} + \frac{b^4}{64} f^{iv}_{y^4}(0) \pm ..., \tag{7}$$

for the points numbered with $5 \div 8$, placed in the small square position, we have:

$$\begin{aligned} f_{5,7} = f_0 \pm \left(a f'_x + b f'_y \right) + \frac{1}{2} \left(a^2 f''_{x^2} + 2ab f''_{xy} + b^2 f''_{y^2} \right) \\ \pm \frac{1}{8} \left(a^3 f'''_{x^3} + 3a^2 b f'''_{x^2 y} + 3ab^2 f'''_{xy^2} + b^3 f'''_{y^3} \right) \\ + \frac{1}{64} \left(a^4 f^{iv}_{x^4} + 4a^3 b f^{iv}_{x^3 y} + 6a^2 b^2 f^{iv}_{x^2 y^2} + 4ab^3 f^{iv}_{xy^3} + b^4 f^{iv}_{y^4} \right) \pm ..., \end{aligned} \tag{8}$$

$$\begin{aligned} f_{6,8} = f_0 \pm \left(-a f'_x + b f'_y \right) + \frac{1}{2} \left(a^2 f''_{x^2} - 2ab f''_{xy} + b^2 f''_{y^2} \right) \\ \pm \frac{1}{8} \left(-a^3 f'''_{x^3} + 3a^2 b f'''_{x^2 y} - 3ab^2 f'''_{xy^2} + b^3 f'''_{y^3} \right) \end{aligned}$$

$$+\frac{1}{64}\left(a^4 f^{iv}_{x^4} - 4a^3 b f^{iv}_{x^3 y} + 6a^2 b^2 f^{iv}_{x^2 y^2} - 4ab^3 f^{iv}_{xy^3} + b^4 f^{iv}_{y^4}\right) \pm ..., \qquad (9)$$

for the points numbered with $9 \div 12$, placed at the double step in the big cross position:

$$f_{9,11} = f_0 \pm 2af'_x + 2a^2 f''_{x^2} \pm a^3 f'''_{x^3} + \frac{a^4}{4} f^{iv}_{x^4}(0) \pm ..., \qquad (10)$$

$$f_{10,12} = f_0 \pm 2bf'_y + 2b^2 f''_{y^2} \pm b^3 f'''_{x^3} + \frac{b^4}{4} f^{iv}_{y^4}(0) \pm ..., \qquad (11)$$

and for the points numbered with $13 \div 16$, placed at the double steps in the big square position:

$$\begin{aligned} f_{13,15} &= f_0 \pm 2\left(af'_x + bf'_y\right) + 2\left(a^2 f''_{x^2} + 2abf''_{xy} + b^2 f''_{y^2}\right) \\ &\pm \left(a^3 f'''_{x^3} + 3a^2 b f'''_{x^2 y} + 3ab^2 f'''_{xy^2} + b^3 f'''_{y^3}\right) \\ &+ \frac{1}{4}\left(a^4 f^{iv}_{x^4} + 4a^3 b f^{iv}_{x^3 y} + 6a^2 b^2 f^{iv}_{x^2 y^2} + 4ab^3 f^{iv}_{xy^3} + b^4 f^{iv}_{y^4}\right) \pm ..., \qquad (12) \end{aligned}$$

$$\begin{aligned} f_{14,16} &= f_0 \pm 2\left(-af'_x + bf'_y\right) + 2\left(a^2 f''_{x^2} - 2abf''_{xy} + b^2 f''_{y^2}\right) \\ &\pm \left(-a^3 f'''_{x^3} + 3a^2 b f'''_{x^2 y} - 3ab^2 f'''_{xy^2} + b^3 f'''_{y^3}\right) \\ &+ \frac{1}{4}\left(a^4 f^{iv}_{x^4} - 4a^3 b f^{iv}_{x^3 y} + 6a^2 b^2 f^{iv}_{x^2 y^2} - 4ab^3 f^{iv}_{xy^3} + b^4 f^{iv}_{y^4}\right) \pm ..., \qquad (13) \end{aligned}$$

3. The partial differential expressions

By simple algebraic calculus we can deduce the new partial differential expressions, fused written, which are the same with these obtained from Taylor's series developments till the 2nd order, and the same with these deduced [3] from the finite difference method:

$$f'_{x,y} = \frac{f_{1,2} - f_{3,4}}{2(a,b)}, \quad f''_{(x,y)^2} = \frac{f_{1,2} - 2f_0 + f_{3,4}}{(a,b)^2}, \quad f''_{xy} = \frac{f_5 + f_7 - f_6 - f_8}{4ab} \qquad (14)$$

and also for the 1st and 2nd order partial differentials with respect to the finite Taylor's series developments to the 4th order partial differentials:

$$-v_0, u_0 = f'_{x,y} = \frac{1}{a,b}\left[\frac{2}{3}(f_{1,2} - f_{3,4}) - \frac{1}{12}(f_{9,10} - f_{11,12})\right], \qquad (15)$$

$$f''_{(x,y)^2} = \frac{1}{(a,b)^2}\left[\frac{4}{3}(f_{1,2} + f_{3,4}) - \frac{5}{2}f_0 - \frac{1}{12}(f_{9,10} + f_{11,12})\right], \qquad (16)$$

instead what for the partial differential of superior orders we have the expressions:

$$f'''_{(x,y)^3} = \frac{1}{(a,b)^3}\left[\frac{2}{3}(f_{9,10} - f_{11,12}) - \frac{4}{3}(f_{1,2} + f_{3,4})\right], \qquad (17)$$

$$f'''_{x^2y} = \frac{1}{a^2b}\left[\frac{2}{3}(f_5 + f_6 - f_7 - f_8) - \frac{4}{3}(f_2 - f_4)\right],$$

$$f'''_{xy^2} = \frac{1}{ab^2}\left[\frac{2}{3}(f_5 - f_6 - f_7 + f_8) - \frac{4}{3}(f_1 - f_3)\right], \qquad (18)$$

the partial differentials of 3rd order being 4/3 time greater as these obtained from Taylor's series developments and 16/3 time greater as these obtained from finite difference method, instead that the differentials of 4th order being 8/3 time greater as these obtained from Taylor's series developments and 42.66... time greater as these obtained by finite difference method

$$f^{iv}_{(xy)^4} = \frac{1}{(a,b)^4}\left[16f_0 - \frac{32}{3}(f_{1,2} + f_{3,4}) + \frac{8}{3}(f_{9,10} + f_{11,12})\right],$$

$$f^{iv}_{x^2y^2} = \frac{1}{a^2b^2}\left[\frac{32}{3}f_0 - \frac{16}{3}\sum_{i=1}^{4} f_i + \frac{8}{3}\sum_{i=5}^{8} f_i\right]. \qquad (19)$$

4. Algebraic relation, associated to partial differential equation, in the case of $a = b = \chi$

Introducing the partial differential expressions (15) ÷ (18) in the stream line equation, deduced from the Navier and Stokes' equation system of a viscous fluid flow [3][4] in the case of square grid

$$\Psi^{iv}_{x^4} + 2\Psi^{iv}_{x^2y^2} + \Psi^{iv}_{y^4} = \mathrm{Re}\left[\Psi'_y\left(\Psi'''_{x^3} + \Psi'''_{xy^2}\right) - \Psi'_x\left(\Psi'''_{x^2y} + \Psi'''_{y^3}\right)\right], \qquad (20)$$

one obtain the associate algebraic relation as general numerical solution for the stream line current

$$\Psi_0 = \frac{2}{5}\sum_{i=1}^{4}\Psi_i - \frac{1}{10}\sum_{i=5}^{8}\Psi_i - \frac{1}{20}\sum_{i=9}^{12}\Psi_i +$$

$$\mathrm{Re}\left\{\begin{array}{l} [8(\Psi_2 - \Psi_4) + \Psi_{12} - \Psi_{10}]\,[\Psi_5 + \Psi_8 + \Psi_8 - \Psi_6 - \Psi_7 - \Psi_{11} + 4(\Psi_3 - \Psi_1)] + \\ + [8(\Psi_3 - \Psi_1) + \Psi_9 - \Psi_{11}]\,[\Psi_5 + \Psi_6 + \Psi_{10} - \Psi_7 - \Psi_8 - \Psi_{12} + 4(\Psi_4 - \Psi_2)] \end{array}\right\} \qquad (21)$$

4.1. The solution stability and the relaxation diagram

For the case of a parallel current with $0x$ axle, corresponding to a stable relaxed grid, the error propagation relations in the current direction and perpendicularly to it [5] are for local Reynolds number denoted by $R = \mathrm{Re}\chi u$:

$$\delta\Psi_{n+1}^{\pm x} = \left(\frac{2}{5} \pm \frac{\mathrm{Re}\chi u}{20}\right)\delta\Psi_n - \left(\frac{1}{20} \pm \frac{R}{80}\right)\delta\Psi_{n-1} \text{ and } \delta\Psi_{n+1}^{\pm x} = \frac{2}{5}\delta\Psi_n - \frac{1}{20}\delta\Psi_{n-1}, \tag{22}$$

(22) with that we can trace the error propagation diagram as in figure 2

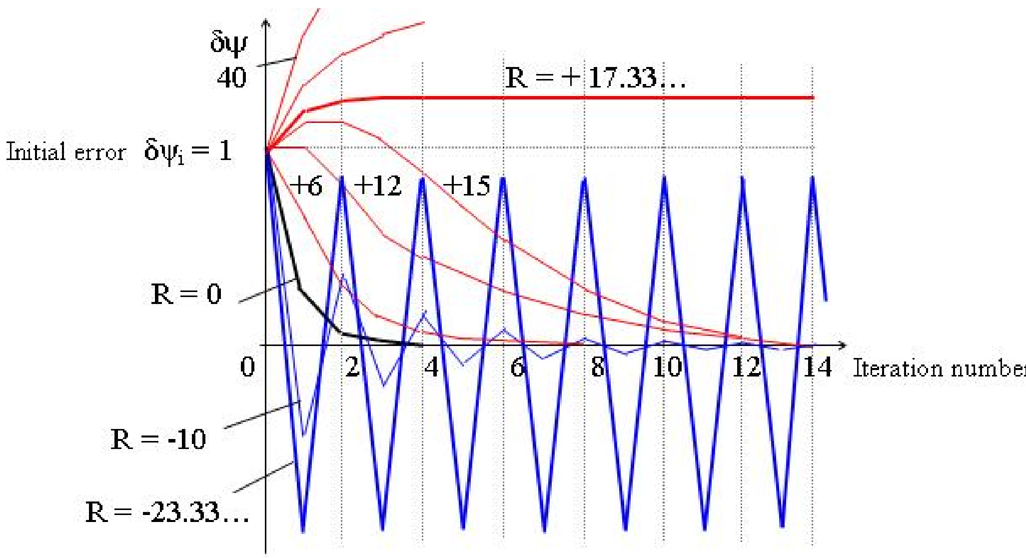

Figure 2: Error relaxation diagram on the $\pm x$ current direction and perpendicularly on it $\pm y$.

where, we represented the extreme curves, corresponding to the limit values of so called Reynolds local number $R = \mathrm{Re}\chi\ (u \text{ or } v)$, which assure the numerical solution stability by diminishing of the errors.

In the case, when we lead the calculus on the diagonal directions, the stability is better

$$\delta\Psi_{n+1} = \left(\frac{2}{5} \pm \frac{\mathrm{Re}\chi u}{20}\right)\delta\Psi_n - \left(\frac{1}{20} \pm \frac{R}{80}\right)\delta\Psi_{n-1}$$

and

$$\delta\Psi_{n+1} = \left(\frac{2}{5} \pm \frac{\mathrm{Re}\chi u}{20}\right)\delta\Psi_n - \left(\frac{1}{20} \pm \frac{R}{80}\right)\delta\Psi_{n-1}.$$

5. Algebraic relation, associated to partial differential equation, in the case $a \neq b$

In the case of a rectangular grid with different steps $a \neq b$, the associated algebraic relation (21) becomes a more complicated expression (23), as well as its error propagation relations on the different calculus direction in the domain, but offers a better numerical solution stability for any special kinds of grid passing (fig. 3).

$$\begin{aligned}
\Psi_0 \;=\;& \frac{1}{16\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)+\frac{64}{3}} + \left\{ \frac{\mathrm{Re}Sh}{\tau}\left[b^2(\Psi_1+\Psi_3-\Psi_1^- -\Psi_3^- -2\Psi_0+2\Psi_0^-)\right.\right.\\
+\;& \left. a^2(\Psi_2+\Psi_4-\Psi_2^- -\Psi_4^- -2\Psi_0+2\Psi_0^-)\right]\\
+\;& \frac{32}{3}\left[\frac{b^2}{a^2}(\Psi_1+\Psi_3)+\frac{a^2}{b^2}(\Psi_2+\Psi_4)+\sum_{i=1}^{4}\Psi_i\right]\\
-\;& \frac{8}{3}\left[\frac{b^2}{a^2}(\Psi_9+\Psi_{11})+\frac{a^2}{b^2}(\Psi_{10}+\Psi_{12})+\sum_{i=5}^{8}\Psi_i\right]\\
+\;& \mathrm{Re}u_0 a\left\{\frac{4}{3}(\Psi_3-\Psi_1)+\frac{2}{3}(\Psi_5+\Psi_8-\Psi_6-\Psi_7)\right.\\
&\left.+\frac{b^2}{a^2}\left[\frac{4}{3}(\Psi_3-\Psi_1)+\frac{2}{3}(\Psi_9-\Psi_{11})\right]\right\}\\
+\;& \mathrm{Re}v_0 a\left\{\frac{4}{3}(\Psi_4-\Psi_2)+\frac{2}{3}(\Psi_5+\Psi_6-\Psi_7-\Psi_8)\right.\\
&\left.\left.+\frac{a^2}{b^2}\left[\frac{4}{3}(\Psi_4-\Psi_2)+\frac{2}{3}(\Psi_{10}-\Psi_{12})\right]\right\}\right\}.
\end{aligned} \tag{23}$$

5.1. The solution stability and the relaxation diagram

Considering the three possible directions to calculus leading and that the error come from the back of the 0 point, the relations of error propagation on different directions being in this case:

-on the fluid flow direction

$$\delta\Psi_{n+1}^{\pm} = \frac{1+\alpha^2}{6\alpha^4+8\alpha^2+6}(4\pm\mathrm{Re}ua)\delta\Psi_n - \frac{1}{6\alpha^4+8\alpha^2+6}\left(1\pm\frac{\mathrm{Re}ua}{2}\right)\delta\Psi_{n-1}, \tag{24}$$

-perpendicularly on the current

$$\delta\Psi_{n+1} = \frac{4\alpha^2(1+\alpha^2)}{6\alpha^4+8\alpha^2+6}\delta\Psi_n - \frac{\alpha^4}{6\alpha^4+8\alpha^2+6}\Psi_{n-1}, \tag{25}$$

-and on the diagonal directions

$$\delta\Psi_{n+1}^{\times} = \frac{-\alpha^2}{6\alpha^4 + 8\alpha^2 + 6}\left(1 \pm \frac{\mathrm{Re}ua}{2}\right)\delta\Psi_n, \tag{26}$$

what put into the evidence the advantageous (favorable) manner to run (traverse) the rectangular grid in different directions, the local Reynolds number having the expressions

$$Re = \mathrm{Re}ua^+ = \frac{12\alpha^4 + 8\alpha^2 + 6}{\alpha(1 + 2\alpha^2)}, \quad Re = \mathrm{Re}ua^- = \frac{12\alpha^4 + 24\alpha^2 + 22}{\alpha(3 + 2\alpha^2)}, \tag{27}$$

respectively on the current direction, and their variation with the mesh ratio as in diagram fig. 3, that in the condition of the sub unitary coefficients and for the step admitted values $b = 0,1$ respectively $a = 0,5$ for a interesting slope $tg\alpha = b/a = 1/5$, conducted us to the following diagram:

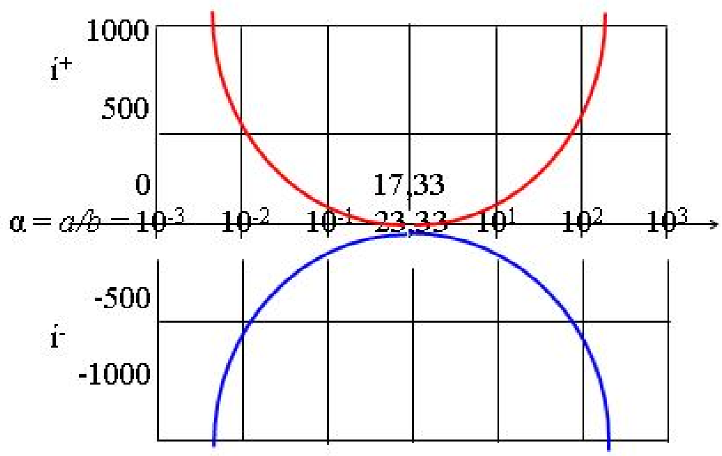

Figure 3: Variation with network step ratio $\alpha = a/b$ of the local Reynolds number, which assure the numerical solution stability at the net run in the current direction

6. Conclusions

The main results of this research are the followings:

- for any non-linear with partial differential equation systems we can assure the numerical solution stability, determining for all directions of iterative calculus the **error propagation relations**, deduced from the algebraic relations associated to the equations with partial differentials and respecting the condition that the error propagation coefficients be sub-unitary,

- the numerical solution **stability is included in even the partial differential equations**, depending of: the shape of equations, the manner of the grid running by calculus, the fineness of the grid, the local characteristic parameter (for instance in viscous fluid hydrodynamics, of the local Reynolds number , equal to the product between the global Reynolds number Re, the grid relative pitch χ and the local dimensionless velocity u, calculated after the formula $Re \cdot \chi \cdot u = \frac{U_m B}{v} \cdot \frac{h}{B} \cdot \frac{U}{U_m} = \frac{U \cdot h}{v}$).

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