Acta Universitatis Apulensis
ISSN: 1582-5329

No. 21/2010
pp.21-34

ON A PROPERTY OF GROUPS WITH COVERINGS

Ahmad Erfanian and Francesco G. Russo

Abstract. A subgroup H of a group G is conjugately dense in G if for each element g in G the intersection of H with the conjugates of g in G is nonempty. Conjugately dense subgroups deal with interesting open problems, related to parabolic groups. In the present paper we study them with respect to suitable coverings.

2000 *Mathematics Subject Classification*: 20E45, 20B35, 20F99.
Keywords and phrases: Conjugately dense subgroups; coverings of subgroups.

1. Introduction

If H is a subgroup of a group G, H is said to be *conjugately dense* in G if for each element g of G the intersection of H with the conjugates of g in G is nonempty. These subgroups arouse interest in various situations. For instance, [34, Problem 8.8b] is an open question on the existence of a noncyclic finitely presented group with a conjugately dense cyclic subgroup. It is conjectured that each conjugately dense irreducible subgroup of $GL(n, K)$, the general linear group of degree n over an arbitrary field K, where n is a fixed positive integer, coincides always with the whole group $GL(n, K)$, except when K is quadratically closed, $n = 2$ and the characteristic of K is 2. There is not too much literature on the topic: mainly [23, 24, 39, 40, 41].

The validity of [34, Problem 8.8b] in the general case would imply that each conjugately dense subgroups of $GL(n, K)$ is parabolic, but [41, Theorem 2] shows that this is not true. From the argument of [41, Theorem 2], a positive answer to [34, Problem 8.8b] depends on a hand by the choice of K, by the characteristic of K and by n, on an other hand by the properties of stability of the conjugately dense subgroups of $GL(n, K)$ with respect to certain usual operations between subgroups such as intersections, products and conjugation. A big problem in working with conjugately dense subgroups is due to the fact that they do not form a class of groups closed with respect to subgroups and subdirect products. This difficulty has been often mentioned in [23, 24, 39, 40], noting that the intersection of two conjugately dense subgroups of a group G can not be a conjugately dense subgroup of G. Here we will study conjugately dense subgroups in some specific contexts.

The main results of the present paper have been proved in successive steps. Section 2 recalls the definitions of classes of generalized FC-groups. Section 3 gives some notions in theory of groups with coverings. Then we can formulae the main results of the present paper in Section 4. Section 5 is devoted to general properties of the conjugately dense subgroups. Finally, the proofs of the main results of this paper are contained in Section 6.

Most of our notation is standard and referred to [8] and [30]. The properties of conjugately dense subgroups are referred to [6, 23, 24, 39, 40, 41]. The properties of generalized FC-groups are referred to [9, 10, 19, 20, 28, 29, 30]. The properties of groups with coverings are referred to [12, 13, 15, 16, 31, 32, 33].

2. Some classes of Generalized FC-groups

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Let $\mathfrak{X}$ be a class of groups. An element x of a group G is said to be an $\mathfrak{X}C$-*element* if $G/C_G(\langle x\rangle^G)$ satisfies $\mathfrak{X}$. A group whose elements are all $\mathfrak{X}C$-elements is said to be an $\mathfrak{X}C$-group. Sometimes $G/C_G(\langle x\rangle^G)$ is denoted by $Aut_G(\langle x\rangle^G)$ to recall that $G/C_G(\langle x\rangle^G)$ is a group of automorphisms of $\langle x\rangle^G$ (see [30, Chapter 3]).

If $\mathfrak{X}$ is the class of finite groups, we find the notion of FC-element and FC-group (see [30, Chapter 4]). If $\mathfrak{X}$ is the class of Chernikov groups, we find the notion of CC-element and CC-group (see [28] and [29]). If $\mathfrak{X}$ is the class of polycyclic-by-finite groups, we find the notion of PC-element and PC-group (see [9]). If $\mathfrak{X}$ is the class of (soluble minimax)-by-finite groups, we find the notion of MC-element and MC-group (see [19, 20, 21]).

It could be opportune to recalling that a group H is said to be (soluble minimax)-by-finite if it contains a normal subgroup K such that K has a finite characteristic series $1 = K_0 \lhd K_1 \lhd \ldots \lhd K_n = K$ whose factors are abelian minimax and H/K is finite. An abelian minimax group A is an abelian group which has a finitely generated subgroup B such that A/B is a direct product of finitely many quasicyclic groups. They are described by [30, Lemma 10.31]rob and consequently soluble minimax groups are well known (see [30, Vol.II, Sections 10.3 and 10.4]).

Let $\mathfrak{Y}$ be one of the following classes of groups: finite groups, Chernikov groups, polycyclic-by-finite groups, (soluble minimax)-by-finite groups.

Note that $\mathfrak{Y}$ is closed with respect to forming subgroups, homomorphic images and extensions (see [8, Chapter 11] and [25]). This allows us to proceed as follows: given two $\mathfrak{Y}C$-elements x and y of G, both $G/C_G(\langle x\rangle^G)$ and $G/C_G(\langle y\rangle^G)$ satisfy $\mathfrak{Y}$, so the quotient group

$$G/(C_G(\langle x\rangle^G)\cap C_G(\langle y\rangle^G))$$

satisfies $\mathfrak{Y}$. But the intersection of $C_G(\langle x\rangle^G)$ with $C_G(\langle y\rangle^G)$ lies in $C_G(\langle xy^{-1}\rangle^G)$, so $G/C_G(\langle xy^{-1}\rangle^G)$ satisfies $\mathfrak{Y}$ and xy^{-1} is an $\mathfrak{Y}C$-element of G.

Hence the $\mathfrak{Y}C$-elements of G form a subgroup $Y(G)$ and $Y(G)$ is characteristic in G.

This simple remark allows us to define the series

$$1 = Y_0 \triangleleft Y_1 \triangleleft \ldots \triangleleft Y_\alpha \triangleleft Y_{\alpha+1} \triangleleft \ldots,$$

where $Y_1 = Y(G)$, the factor $Y_{\alpha+1}/Y_\alpha$ is the subgroup of G/Y_α generated by the YC-elements of G/Y_α and

$$Y_\lambda = \bigcup_{\alpha<\lambda} Y_\alpha,$$

with α ordinal and λ limit ordinal. This series is a characteristic ascending series of G and it is called *upper YC-central series* of G. The last term of the upper YC-central series of G is called *YC-hypercenter* of G and it is denoted by $Y_\lambda(G)$. If $G = Y_\beta$, for some ordinal β, we say that G is an *YC-hypercentral* group of type at most β and this is equivalent to say that $G = Y_\lambda(G)$.

The *YC-length* of an YC-hypercentral group is defined to be the least ordinal β such that $G = Y_\beta$, in particular when $G = Y_c$ for some positive integer c, we say that G is *YC-nilpotent* of length c. Certainly an YC-group is characterized to have YC-length ≤ 1.

In analogy with FC-groups, the first term $Y(G)$ of the upper YC-central series of G is said to be the *YC-center* of G and the α-th term of the upper YC-central series of G is said to be the *YC-center of length α* of G. Roughly speaking, the upper YC-central series of G measures the distance of G to be an YC-group. By definitions, it happens that $Z(G) \leq F(G) \leq Y(G)$, where $F(G)$ is the FC-center of G, that is, $F(G)$ is the subgroup of G which is generated by all elements which have only a finite number of conjugates.

For the class of finite groups we have FC-hypercentral groups, using the symbols F_α, F_β, F_λ, F_c instead of Y_α, Y_β, Y_λ, Y_c in the previous definitions. For the class of Chernikov groups we have CC-hypercentral groups, using the symbols C_α,C_β, C_λ, C_c instead of Y_α, Y_β, Y_λ,Y_c in the previous definitions. For the class of polycyclic-by-finite groups we have PC-hypercentral groups, using the symbols P_α, P_β, P_λ, P_c instead of Y_α, Y_β, Y_λ, Y_c in the previous definitions. For the class of (soluble minimax)-by-finite groups we have MC-hypercentral groups, using the symbols M_α, M_β, M_λ, M_c instead of Y_α, Y_β, Y_λ, Y_c in the previous definitions.

[1] and [22] investigate on PC-hypercentral and CC-hypercentral groups satisfying finiteness conditions. For instance [1, Theorems A and B], adapt some classical

McLain's Theorems on generalized nilpotent series [30, Theorems 4.37 and 4.38] with related finiteness conditions [30, Theorems 4.39, 4.39.1 and 4.39.2]. On the other hand the conditions of hypercentrality and YC-hypercentrality can be different in a same group as the consideration of the infinite dihedral group shows (see also [1,22]).

3. Some Results in Theory of Covering of Groups

Let G be a group and $\mathcal{A} = \{A_i : i \in I\}$ a collection of proper subgroups of G. The set $\mathcal{A}$ is said to be a cover (or a *covering*) of G if $G = \bigcup_{i \in I} A_i$. A cover $\Pi = \{A_i : i \in I\}$ of G is said to be a *partition* of G if $A_i \cap A_j = 1$ for each $i, j \in I$ with $i \neq j$.

Probably the first result on coverings is due to G.A.Miller (see [32]) who began to study partitions and characterized finite elementary abelian p-groups (p prime) as those abelian finite groups having a partition. Moreover he studied the connection between partitions of finite p-groups and what it is today called the Hughes-subgroup (see [15] and [32] for details).

It is an easy exercise to prove that no group can be covered by two proper subgroups. G.Scorza [31] was the first to analyze coverings by subgroups without restrictions on the intersection of the members of the covering and he gave a characterization of groups covered by 3 proper subgroups as follows.

Theorem 3.1. (G. Scorza) *A group G is the union of 3 proper subgroups if and only if $G/Z(G)$ is the 2-elementary abelian group of rank 2.*

The paper of G.Scorza was not well known in the mathematical community for long time and his result was rediscovered more than once.

In the period between 1951 and 1956, D.Greco, in a series of paper, gave a characterization of groups covered by 4 subgroups (see [12] and [13]). He obtained some partial results on groups covered by 5 subgroups.

Theorem 3.2. (D.Greco) *If the group G is the union of 4 or 5 proper subgroups, then $G/Z(G)$ is finite.*

However the main goals on coverings of groups have been proved in its generality by B.H.Neumann and P.Kontorovich (see [15, 16, 32]). In the finite case R.Baer, O.Kegel and M.Suzuki were able to describe all groups having a partition. Further details can be found in [32].

The connection of coverings with generalized FC-groups can be found in the following fact which has been noted by B.H.Neumann.

Proposition 3.3. *A periodic FC-group G can be covered by finite normal subgroups.*

Proof. G is locally-(normal and finite) from [30, Theorem 4.32], and this implies $G = \bigcup_{i \in I} S_i$, where S_i is a finite normal subgroup of G such that $S_i \leq S_j$ for each $i < j$.

We have also as follows.

Proposition 3.4. *Assume that a periodic group G satisfies one of the following conditions.*

[(a)] G is a CC-group.
[(b)] G is a PC-group.
[(c)] G is an MC-group.
Then G can be covered by Chernikov normal subgroups.

Proof. Periodic PC-groups and periodic MC-groups are obviously periodic CC-groups. A periodic CC-group is locally-(normal and Chernikov) as testified in [30, Theorem 4.36]rob. Then $G = \bigcup_{i \in I} C_i$, where C_i is a Chernikov normal subgroup of G such that $C_i \leq C_j$ for each $i < j$. The result follows.

Among infinite groups with a partition there are the groups of prime exponent p and the groups of Ol'shanskii type. In view of such constructions there is no general theory of infinite groups admitting a partition (see also [15]).

4. Statement of Results

From [24, Lemma 3, Proof, lines 1-2]s2, if H is a conjugately dense subgroup of a group G, then $G = \bigcup_{x \in G} H^x$. Therefore [6] is related with [23, 24, 39, 40], since the description of conjugately dense subgroups can be given from the following point of view:

$(*)$ Assume that G is a group and H is a subgroup of G such that $G = \bigcup_{x \in G} H^x$. When is it possible to deduce that $H = G$?

This simple remark allows us to state that the knowledge of the conjugately dense subgroups in [23, 24, 39, 40, 41] can be formulated as in $(*)$. Thanks to [6, Theorems 1-4], we may note that conjugately dense subgroups have been described for locally finite groups, locally supersoluble groups and groups which belongs to $\mathfrak{Z}$, where $\mathfrak{Z}$ denotes the class of groups defined as it follows: a group G belongs to $\mathfrak{Z}$ if it is not covered by conjugates of any proper subgroup.

The properties of the class $\mathfrak{Z}$ can be found in [6], [37], [38]. In particular, $\mathfrak{Z}$ is closed under extensions and (restricted) direct products (though not cartesian products), as well as containing all hypercentral and soluble groups. Furthermore,

$\mathfrak{Z}$ contains the class of all finite groups, but $\mathfrak{Z}$ contains neither all locally nilpotent periodic groups nor infinite transitive groups of finitary permutations (see [6, Section 1]).

It seems opportune to recall that the notion of *density* between subgroups has been studied by several authors in different contexts. [26] represents the first work which introduces the concept of dense subgroup. Following [26], a group G has *dense ascendant* (*subnormal*) *subgroups* if whenever $H < K \leq G$ and H is not maximal in K, there is an ascendant (subnormal) subgroup of G lying strictly between the subgroups H and K of G. If we point out on special families of subgroups of G, then the notion of density can be extended and many structural properties on the whole group G can be obtained. For instance [7] studies the dense subgroups which are related to the class of modular subgroups. Here we will say that a group G has *dense modular subgroups* if whenever $H < K \leq G$ and H is not maximal in K, there is a modular subgroup of G lying strictly between the subgroups H and K of G. In a similar way [11] is related to the class of infinite invariant abelian subgroups, [17] is related to the class of subnormal subgroups, [18] is related to the class of almost normal subgroups and [35] is related to the class of pronormal subgroups. The literature which is cited in [7, 11, 17, 18, 26, 35] shows that dense subgroups have interested many group-theoretical properties. In our case the reason of the terminology *dense* is due to the fact that a group G has conjugately dense subgroups if whenever $H < K \leq G$ and H is not maximal in K, there is a subgroup L of G such that for each element g of G the intersection of L with the conjugates of g in G is nonempty and L lies strictly between the subgroups H and K of G.

A common feature of [7, 11, 17, 18, 26, 35] can be found in the restrictions on the index of the dense subgroup which is discussed. Also the conjugately dense subgroups of a given group are subjected to restrictions of this type as we will see.

Notions of theory of covering of groups are naturally involved in our topic (see for instance [2, 3, 4, 5, 8, 30]), because the existence of conjugately dense subgroups can be formulated as in $(*)$ and because most of the results in [7, 11, 17, 18, 26, 35] furnishes information on classes of groups which can be covered by finitely many subgroups.

Our main results have been summarized as follows.

Theorem 4.1. *If G is a YC-group, then G has no proper conjugately dense subgroups.*

Theorem 4.2. *Assume that each $\mathfrak{Y}$-group is finitely generated. If G is a YC-hypercentral group, then G has no proper conjugately dense subgroups.*

Theorem 4.3. *Assume that G is a group which satisfies one of the following conditions:*

[(a)] G is the union of 3 proper subgroups.
[(b)] G is the union of 4 or 5 proper subgroups.
Then G has no proper conjugately dense subgroups.

Theorem 4.4. *Let G be an infinite group which has exactly three conjugacy classes of elements. Then either G has a proper conjugately dense subgroup or G is simple.*

5. Properties of Conjugately Dense Subgroups

As it often happens in Group Theory, a same result can be found by means of different approaches. This is the case of the following properties which have been communicated independently by [6] and by [23, 24, 39, 40].

Lemma 5.1. *Let H be a subgroup of a group G and n be a positive integer.*

(a) If H is conjugately dense in G, then $Z(G) \leq H$.

(b) If N is a normal subgroup of G such that $N \leq H \leq G$, then H is conjugately dense in G if and only if H/N is conjugately dense in G/N.

(c) If G is abelian or finite and H is conjugately dense in G, then $H = G$.

(d) If H is conjugately dense in G and N is a normal subgroup of G, then HN/N is conjugately dense in G/N and either $HN/N < G/N$ or $H \cap N < N$.

(e) If K is subgroup of G and H is conjugately dense in G, then $\langle H, K\rangle$ is a conjugately dense subgroup of G.

(f) The intersection of two conjugately dense subgroups of G can not be a conjugately dense subgroup of G.

(g) If H is conjugately dense in G, then the index of H in G is not finite.

(h) Assume that H is conjugately dense in G. If G is central-by-finite, then $H = G$.

(i) If H is conjugately dense in G and H is an extension of a perfect group by a soluble group of derived length n, then $G = HP$, where P is a perfect normal subgroup of G of derived length at most n. Moreover, if $H < G$, then $H \cap P < P$.

Proof. (a) and (b) are immediate consequence of the definitions. The abelian part of (c) follows from (a) and the finite case from [24, Lemma 3]. (d) follows from [6, Lemma 2]. (i) follows from [6, Lemma 6]. We will prove the statements (e), (f), (g), (h).

(e). For each element x in G we have that the conjugacy class x^G of x in G has $H \cap x^G \neq \varnothing$ and in particular, $\langle H, K\rangle \cap x^G \geq \langle H\rangle \cap x^G \neq \varnothing$. Therefore $\langle H, K\rangle$ is a conjugately dense subgroup of G.

(f). Let $G = PSL(2, \mathbb{Z})$ be the projective special linear group of dimension 2 over the ring of the integers. The subgroup U generated by the upper triangular matrices of G and the subgroup L generated by the lower triangular matrices of G are conjugately dense subgroups in G. The intersection $D = U \cap L$ is the diagonal subgroup of G and D is not conjugately dense in G.

(g). Suppose that $H \neq G$ is a conjugately dense subgroup of G with finite index $|G : H|$. Then $N_G(H) \geq H$ and $|G : N_G(H)|$ is finite. It follows that H has only finitely many conjugates $H^{g_1}, \ldots, H^{g_n}$ in G, where $g_1, \ldots, g_n \in G$ and n is a positive integer. Then $\bigcap_{i=1}^{n} H^{g_i} = K$ is normal in G and $|G : K|$ is finite. In particular G/K is a finite group and (b) implies that H/K is a conjugately dense subgroup of G/K. Now (c) implies that $H/K = G/K$ and so $G = H$ which is a contradiction.

(h). Suppose that $H \neq G$ is conjugately dense in G. $H \geq Z(G)$ from (a), therefore the index $|G : H|$ is finite and (g) gives a contradiction.

We recall for convenience of the reader some results of [6] which will be useful.

Lemma 5.2. *Let H be a conjugately dense subgroup of a group G.*

(a) If G is locally finite and H is finite, then $H = G$.

(b) Assume that G has a finite series whose factors are either locally supersoluble or belonging to $\mathfrak{Z}$. If H has finite abelian section rank, then $H = G$.

(c) If G is a residually-$\mathfrak{Z}$ group and H is soluble, then $H = G$.

(d) If G is locally soluble and H is soluble with max-n, then $H = G$.

Proof. (a), (b), (c) and (d) follow from Theorem 2, Theorem 4, Corollary 7 and Corollary 8 in [6], respectively.

Corollary 5.3. *If G is a soluble (hypercentral) group , then G has no proper conjugately dense subgroups.*

Proof. This follows by Lemma 5.2 (b).

Corollary 5.4. *Let H be a conjugately dense subgroup of a group G.*

(a) If G is central-by-Chernikov, then $H = G$.

(b) If G is central-by-(polycyclic-by-finite), then $H = G$.

(c) If G is central-by-(soluble minimax-by-finite), then $H = G$.

Proof. (a). In particular G is a central-by-soluble group and so G is a residually soluble group. Then G is a residually-$\mathfrak{Z}$ group. The result is true from Lemma 5.2 (c).

(b). We may argue as in (a) above.

(c). We may argue as in (a) above.

Corollary 5.5. *Assume that K is an arbitrary field, $GL(n, K)$ is a general linear group of dimension n over K, where n is a positive integer.*

(a) If G is a locally nilpotent subgroup of $GL(n, K)$, then G has no proper conjugately dense subgroups;

(b) If G is a locally supersoluble subgroup of $GL(n, K)$, then G has no proper conjugately dense subgroups.

Proof. (a). [30, Theorem 6.32 (iv)], implies that G is hypercentral, so the result follows by Corollary 5.3.

(b). This follows by Lemma 5.2 (b).

The following example shows that the usual operations of intersection, homomorphic images and conjugation are not respected by the subgroups which are conjugately dense in a given group.

Example 1. If G is the Tarski monster (see [14]), then it contains infinitely many proper conjugately dense subgroups H of G. In particular, each H is cyclic of order p, where p is a prime, and $G = \bigcup_{x \in G} H^x$. Clearly we may not state in this situation that $G = H$.

Example 2. In [14] it has been furnished a countable group H containing an element of 'big enough' (possibly infinite) order but none of order 2 such that it is possible to construct a simply 2-generated group $G = \bigcup_{x \in G} H^x$. Such group has clearly each H which is conjugately dense in G and $H \neq G$.

6. Proof of Main Theorems and Consequences

Proof of Theorem 4.1. Since G is a YC-group, $G/Z(G)$ is residually soluble and then residually $\mathfrak{Z}$-group. Lemma 5.2 (c) implies that $G/Z(G)$ has no proper conjugately dense subgroups. Now Lemma 5.1 (b) implies that G has no proper conjugately dense subgroups so the result follows.

A generalized FC-group can be neither locally nilpotent nor locally soluble a priori (see [9,19, 20, 28, 29, 30]). This fact allows us to note that generalized soluble groups, generalized nilpotent groups and generalized FC-groups can have a complicated structure. However the structure of generalized FC-groups which are either locally soluble or locally nilpotent is well-known. For instance locally soluble MC-groups are hyperabelian and locally nilpotent MC-groups are hypercentral thanks to [19, Theorems 3, 4]. Locally soluble FC-hypercentral, respectively CC-hypercentral, respectively PC-hypercentral, groups are hyperabelian by means of [10, Theorem 2.1] and [9, Theorem 3.2]. Moreover locally nilpotent FC-hypercentral, respectively CC-hypercentral, respectively PC-hypercentral, groups are hypercentral applying again [10, Theorem 2.1] and [9, Theorem 3.2].

Proof of Theorem 4.2. [22, Corollary 2.3] states that a YC-hypercentral group in which each $\mathfrak{Y}$-group is finitely generated is hypercentral. This is our case. Then G is a hypercentral group. Now the result follows from Corollary 5.3.

The terminology of the classes of generalized nilpotent groups which are listed in the following result can be found in [30, Chapter 6].

Corollary 5.3. *Let K be an arbitrary field and $GL(n,K)$ be a general linear group of dimension n over K. If G is a subgroup of $GL(n,K)$ and one of the following conditions holds*

(a) G is a Baer-nilpotent group,

(b) G is an Engel-group,

(c) G is a weakly nilpotent group,

(d) G is a V-group,

(e) G is an U-group,

(f) G is an $\tilde{N}$-groups,

(g) G is a $\bar{\bar{Z}}$-group,

then G has no proper conjugately dense subgroups.

Proof. A Baer-nilpotent linear group is hypercentral from [30, Vol.2, p.35]. The classes of generalized groups which are listed in the successive statements (b)-(g) are subclasses of the class of Baer-nilpotent groups (see [30, Vol.2, Chapter 6]). Then these groups are again hypercentral and the result follows by Lemma 5.2.

Proof of Theorem 4.3. If G satisfies the condition (a), then G is central-by-finite group from Theorem 3.1. Then the result follows from Lemma 5.1 (h). The same happens if G satisfies the condition (b). Now we use Theorem 3.2 and again Lemma 5.1 (h).

Proof of Theorem 4.4. Firstly, the following remark has to be done. If a given group K has only two conjugacy classes, then every nontrivial subgroup of K is a conjugately dense subgroup of K.

Now, assume that G has exactly three conjugacy classes, i.e. $G = e^G \cup x^G \cup y^G$ for some elements $e, x, y \in G$, and assume that G has no proper conjugately dense subgroups. We claim that G is simple.

Assume that G is not simple. There exists a normal subgroup $1 \neq N \neq G$ such that either $N = e^G \cup x^G$ or $N = e^G \cup y^G$. If $|x|$ denotes the order of x and $|y|$ that of y, then $exp(G) = 2$, whenever $|x| = |y| = 2$. Then we have that G is abelian, getting to a contradiction. From this, both $|x|$ and $|y|$ can not be 2. Thus either x or y should have order bigger than 2.

If $|y| > 2$, then either $y^2 \in x^G$ or $y^2 \in y^G$. If $y^2 \in x^G$, then $\langle x \rangle$ is a proper conjugately dense subgroup of G, getting to a contradiction. So $y^2 \in y^G$, hence there exists an element $g \in G$ such that $y^2 = y^g$, or equivalently $y = [y, g]$. If H is a subgroup of G, we can easily observe that H is a proper conjugately dense subgroup of G, whenever H/N is a proper subgroup of G/N. Thus G/N has no proper subgroups and, in particular, G/N is cyclic of prime order. Then G/N is abelian. So, $y \in G' \subseteq N$ and it implies that $N = e^G \cup y^G$.

If $|x| > 2$, then a similar method shows that $x \in G'$, getting to contradiction.

Hence $|x| = 2$ and we deduce that $|xa| = 2$ for every $a \in N$. Thus for every elements a and b in N, we have $(ab)^{-1} = (ab)^x = a^x b^x = a^{-1}b^{-1}$, and so N is abelian. Since N is abelian, the number of conjugacy classes of G is at least 4, getting to a contradiction. Therefore, G is simple, as claimed.

Finally, we announce a conjecture that generalizes Theorem 4.4.

Conjecture 6.2. Let n be a positive integer and G be an infinite group. If G has exactly n conjugacy classes of elements, then either G has a proper conjugately dense subgroup or the $(n-2)$-th derived subgroup of G, $G^{(n-2)}$ is simple.

Acknowledgement. The first author would like to thank Ferdowsi University of Mashhad for the research grant No. MP88096ERF.

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Ahmad Erfanian
Department of Pure Mathematics and
Centre of Excellence in Analysis on Algebraic Structures
Ferdowsi University of Mashhad
P.O.Box 1159, 91775 Mashhad, Iran
email: *erfanian@math.um.ac.ir*

Francesco G. Russo
Department of Mathematics
University of Naples Federico II
via Cinthia, 80120, Naples, Italy
email: *francesco.russo@dma.unina.it*