SĂLĂGEAN-TYPE HARMONIC MULTIVALENT FUNCTIONS

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Abstract. We define and investigate a new class of Sălăgean-type harmonic multivalent functions. we obtain coefficient inequalities, extreme points and distortion bounds for the functions in this class.

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1. Introduction

For fixed positive integer p, denote by $H(p)$ the set of all harmonic multivalent functions $f = h + \bar{g}$ which are sense-preserving in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ where h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \qquad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \qquad |b_p| < 1. \quad (1)$$

The differential operator D^m was introduced by Sălăgean [6]. For fixed positive integer m and for $f = h + \bar{g}$ given by (1) we define the modified Sălăgean operator $D^m f$ as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \;; \qquad p > m, \quad z \in \mathbb{U} \quad (2)$$

where

$$D^m h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{k+p-1}{p}\right)^m a_{k+p-1} z^{k+p-1}$$

and

$$D^m g(z) = \sum_{k=1}^{\infty} \left(\frac{k+p-1}{p}\right)^m b_{k+p-1} z^{k+p-1}.$$

It is known that, (see [3]), the harmonic function $f = h + \bar{g}$ is sense-preserving in U if $|g'| < |h'|$ in $\mathbb{U}$. The class $H(p)$ was studied by Ahuja and Jahangiri [1] and the class $H(p)$ for $p = 1$ was defined and studied by Jahangiri et al in [5].

For fixed positive integers m, n and p and for $0 \leq \alpha < 1$ we let $H_p(m, n, \alpha, \beta)$ denote the class of multivalent harmonic functions of the form (1) that satisfy the condition

$$Re\left\{\frac{D^m f(z)}{D^n f(z)}\right\} > \beta \left|\frac{D^m f(z)}{D^n f(z)} - 1\right| + \alpha. \tag{3}$$

The subclass $\overline{H_p}(m, n, \alpha, \beta)$ consists of function $f_m = h + \bar{g}_m$ in $H_p(m, n, \alpha, \beta)$ so that h and g are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \ |b_p| < 1. \tag{4}$$

The families $H_p(m, n, \alpha, \beta)$ and $\overline{H_p}(m, n, \alpha, \beta)$ include a variety of well-known classes of harmonic functions as well as many new ones. For example $\overline{H_1}(1, 0, \alpha, 0) \equiv HS(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in $\mathbb{U}$, $\overline{H_1}(2, 1, \alpha, 0) \equiv HK(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in $\mathbb{U}$ and $\overline{H_1}(n+1, n, \alpha, 0) \equiv \overline{H}(n, \alpha)$ is the class of Sălăgean-type harmonic univalent functions.

For the harmonic functions f of the form (1) with $b_1 = 0$, Avcı and Zlotkiewicz [2] showed that if

$$\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq 1,$$

then $f \in HS(0)$ and if

$$\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \leq 1,$$

then $f \in HK(0)$. Silverman [8] proved that the above two coefficient conditions are also necessary if $f = h + \overline{g}$ has negative coefficients. Later, Silverman and Silvia [9] improved the results of [5] and [6] to the case b_1 not necessarily zero.

For the harmonic functions f of the form (4) with $m = 1$, Jahangiri [4] showed that $f \in HS(\alpha)$ if and only if

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| + \sum_{k=1}^{\infty}(k+\alpha)|b_k| \leq 1-\alpha$$

and $f \in \overline{H_1}(2,1,\alpha,0)$ if and only of

$$\sum_{k=2}^{\infty}k(k-\alpha)|a_k| + \sum_{k=1}^{\infty}k(k+\alpha)|b_k| \leq 1-\alpha.$$

In this paper, the coefficient conditions for the classes $HS(\alpha)$ and $HK(\alpha)$ are extended to the class $H_p(m,n,\alpha,\beta)$,of the forms (3) above. Furthermore, we determine extreme points and distortion theorem for the functions in $\overline{H_p}(m,n,\alpha,\beta)$.

2. Main Results

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H_p(m,n,\alpha,\beta)$.

Theorem 1. *Let $f = h + \bar{g}$ be given by (1). Furthermore, let*

$$\sum_{k=1}^{\infty}\left\{\Psi(m,n,p,\alpha,\beta)\,|a_{k+p-1}| + \Theta(m,n,p,\alpha,\beta)\,|b_{k+p-1}|\right\} \leq 2 \tag{5}$$

where

$$\Psi(m,n,p,\alpha,\beta) = \frac{\mathcal{K}_{k,p}^m(1+\beta) - (\beta+\alpha)\mathcal{K}_{k,p}^n}{1-\alpha}$$

$$\Theta(m,n,p,\alpha,\beta) = \frac{\mathcal{K}_{k,p}^m(1+\beta) - (-1)^{m-n}\mathcal{K}_{k,p}^n(\beta+\alpha)}{1-\alpha},$$

$\mathcal{K}_{k,p} = \frac{k+p-1}{p}$, *$a_p = 1$, $\alpha(0 \leq \alpha < 1)$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $m > n$. Then $f \in H_p(m,n,\alpha,\beta)$.*

Proof. According to (2) and (3) we only need to show that

$$Re\left(\frac{D^m f(z) - \alpha D^n f(z) - \beta e^{i\theta}\left|D^m f(z) - D^n f(z)\right|}{D^n f(z)}\right) \geq 0$$

The case $r = 0$ is obvious. For $0 < r < 1$ it follows that

$$Re\left(\frac{D^m f(z) - \alpha D^n f(z) - \beta e^{i\theta}|D^m f(z) - D^n f(z)|}{D^n f(z)}\right)$$

$$= Re\{\frac{(1-\alpha)z^p + \sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^m - \alpha\mathcal{K}_{k,p}^n)a_{k+p-1}z^{k+p-1}}{z^p + \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k+p-1} + (-1)^n\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n \overline{b}_{k+p-1}\overline{z}^{k+p-1}}$$

$$+\frac{(-1)^m\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^m - (-1)^{m-n}\mathcal{K}_{k,p}^n\alpha)\overline{b}_{k+p-1}\overline{z}^{k+p-1}}{z^p + \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k+p-1} + (-1)^n\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n \overline{b}_{k+p-1}\overline{z}^{k+p-1}}$$

$$-\frac{\beta e^{i\theta}\left|\sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^m - \mathcal{K}_{k,p}^n)a_{k+p-1}z^{k+p-1} + (-1)^m\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^m - (-1)^{m-n}\mathcal{K}_{k,p}^n)\overline{b}_{k+p-1}\overline{z}^{k+p-1}\right|}{z^p + \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k+p-1} + (-1)^n\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n \overline{b}_{k+p-1}\overline{z}^{k+p-1}}\}$$

$$= Re\{\frac{(1-\alpha) + \sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^m - \alpha\mathcal{K}_{k,p}^n)a_{k+p-1}z^{k-1}}{1 + \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k-1} + (-1)^n\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n \overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}} +$$

$$+\frac{(-1)^m\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^m - (-1)^{m-n}\mathcal{K}_{k,p}^n\alpha)\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}{1 + \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k-1} + (-1)^n\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n \overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}$$

$$-\frac{\beta e^{i\theta}z^{-p}\left|\sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^{m}-\mathcal{K}_{k,p}^{n})a_{k+p-1}z^{k+p-1}+(-1)^{m}\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^{m}-(-1)^{m-n}\mathcal{K}_{k,p}^{n})\overline{b}_{k+p-1}\overline{z}^{k+p-1}\right|}{1+\sum_{k=2}^{\infty}\mathcal{K}_{k,p}^{n}a_{k+p-1}z^{k-1}+(-1)^{n}\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^{n}\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\}$$

$$=Re\left[\frac{(1-\alpha)+A(z)}{1+B(z)}\right].$$

For $z=re^{i\theta}$ we have

$$A(re^{i\theta})=\sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^{m}-\alpha\mathcal{K}_{k,p}^{n})a_{k+p-1}r^{k-1}e^{(k-1)\theta i}$$

$$+(-1)^{m}\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^{m}-(-1)^{m-n}\mathcal{K}_{k,p}^{n}\alpha)\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}-\beta e^{-(p-1)i\theta}T(m,n,p,\alpha)$$

where

$$T(m,n,p,\alpha)=\left|\sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^{m}-\mathcal{K}_{k,p}^{n})a_{k+p-1}r^{k-1}e^{-(k+p-1)i\theta}+\right.$$

$$\left.+(-1)^{m}\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^{m}-(-1)^{m-n}\mathcal{K}_{k,p}^{n})\overline{b}_{k+p-1}r^{k-1}e^{-(k+p-1)i\theta}\right|$$

and

$$B(re^{i\theta})=\sum_{k=2}^{\infty}\mathcal{K}_{k,p}^{n}a_{k+p-1}r^{k-1}e^{(k-1)\theta i}+(-1)^{n}\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^{n}\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}.$$

Setting

$$\frac{(1-\alpha)+A(z)}{1+B(z)}=(1-\alpha)\frac{1+w(z)}{1-w(z)}$$

the proof will be complete if we can show that $|w(z)| \leq r < 1$. This is the case since, by the condition (5), we can write

$$|w(z)| = \left| \frac{A(z) - (1-\alpha)B(z)}{A(z) + (1-\alpha)B(z) + 2(1-\alpha)} \right|$$

$$\leq \frac{\sum_{k=1}^{\infty} \left[(1+\beta)(\mathcal{K}_{k,p}^m - \mathcal{K}_{k,p}^n)|a_{k+p-1}| + (1+\beta)(\mathcal{K}_{k,p}^m - (-1)^{m-n}\mathcal{K}_{k,p}^n)|b_{k+p-1}|\right] r^{k-1}}{4(1-\alpha) - \sum_{k=1}^{\infty} \left\{[\mathcal{K}_{k,p}^m(1+\beta) - \Lambda\mathcal{K}_{k,p}^n]|a_{k+p-1}| + [\mathcal{K}_{k,p}^m(1+\beta) - (-1)^{m-n}\Lambda\mathcal{K}_{k,p}^n]|b_{k+p-1}|\right\} r^{k-1}}$$

$$< \frac{\sum_{k=1}^{\infty}(1+\beta)(\mathcal{K}_{k,p}^m - \mathcal{K}_{k,p}^n)|a_{k+p-1}| + (\mathcal{K}_{k,p}^m - (-1)^{m-n}\mathcal{K}_{k,p}^n)(1+\beta)|b_{k+p-1}|}{4(1-\alpha) - \left\{\sum_{k=1}^{\infty}[\mathcal{K}_{k,p}^m(1+\beta) - \Lambda\mathcal{K}_{k,p}^n]|a_{k+p-1}| + [\mathcal{K}_{k,p}^m(1+\beta) - (-1)^{m-n}\Lambda\mathcal{K}_{k,p}^n]|b_{k+p-1}|\right\}}$$

$\leq 1,$

where $\Lambda = \beta + 2\alpha - 1$. The harmonic univalent functions

$$f(z) = z^p + \sum_{k=2}^{\infty} \frac{1}{\Psi(m,n,p,\alpha,\beta)} x_k z^{k+p-1} + \sum_{k=1}^{\infty} \frac{1}{\Theta(m,n,p,\alpha,\beta)} \overline{y_k z^{k+p-1}} \quad (6)$$

where $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m \geq n$ $0 \leq \alpha < 1$, $\beta \geq 0$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (5) is sharp. The functions of the form (6) are in $H_p(m,n,\alpha,\beta)$ because

$$\sum_{k=1}^{\infty} \left[\Psi(m,n,p,\alpha,\beta)\,|a_{k+p-1}| + \Theta(m,n,p,\alpha,\beta)\,|b_{k+p-1}|\right] = 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2.$$

In the following theorem it is shown that the condition (5) is also necessary for functions $f_m = h + \overline{g_m}$ where h and g_m are of the form (4).

Theorem 2. *Let $f_m = h + \overline{g_m}$ be given by (4). Then $f_m \in \overline{H}_p(m,n,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty} \left[\Psi(m,n,p,\alpha,\beta) a_{k+p-1} + \Theta(m,n,p,\alpha,\beta) b_{k+p-1}\right] \leq 2 \quad (7)$$

where $a_p = 1$, $0 \leq \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ *and* $m > n$.

Proof. Since $\overline{H}_p(m,n,\alpha,\beta) \subset H_p(m,n,\alpha,\beta)$, we only need to prove the "only if" part of the theorem. For functions f_m of the form (4), we note that the condition

$$Re\left\{\frac{D^m f(z)}{D^n f(z)}\right\} > \beta\left|\frac{D^m f(z)}{D^n f(z)} - 1\right| + \alpha$$

is equivalent to

$$Re\{\frac{(1-\alpha)z^p - \sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^m - \alpha\mathcal{K}_{k,p}^n)a_{k+p-1}z^{k+p-1}}{z^p - \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k+p-1} + (-1)^{m+n-1}\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n b_{k+p-1}\overline{z}^{k+p-1}}$$

$$+\frac{(-1)^{2m-1}\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^m - (-1)^{m-n}\mathcal{K}_{k,p}^n\alpha)b_{k+p-1}\overline{z}^{k+p-1}}{z^p - \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k+p-1} + (-1)^{m+n-1}\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n b_{k+p-1}\overline{z}^{k+p-1}} \tag{8}$$

$$-\frac{\beta e^{i\theta}\left|-\sum_{k=2}^{\infty}(\mathcal{K}_{k,p}^m - \mathcal{K}_{k,p}^n)a_{k+p-1}z^{k+p-1} + (-1)^{2m-1}\sum_{k=1}^{\infty}(\mathcal{K}_{k,p}^m - (-1)^{m-n}\mathcal{K}_{k,p}^n)\overline{b}_{k+p-1}\overline{z}^{k+p-1}\right|}{z^p - \sum_{k=2}^{\infty}\mathcal{K}_{k,p}^n a_{k+p-1}z^{k+p-1} + (-1)^{m+n-1}\sum_{k=1}^{\infty}\mathcal{K}_{k,p}^n b_{k+p-1}\overline{z}^{k+p-1}}\}$$

$$\geq 0$$

where $\mathcal{K}_{k,p} = \frac{k+p-1}{p}$.

The above required condition (8) must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[\mathcal{K}^m_{k,p}(1+\beta)-(\beta+\alpha)\mathcal{K}^n_{k,p}\right]a_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\mathcal{K}^n_{k,p}a_{k+p-1}r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}\mathcal{K}^n_{k,p}b_{k+p-1}r^{k+p-1}}$$
$$+\frac{-\sum_{k=1}^{\infty}\left[\mathcal{K}^m_{k,p}(1+\beta)-(-1)^{m-n}\mathcal{K}^n_{k,p}(\beta+\alpha)\right]b_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\mathcal{K}^n_{k,p}a_{k+p-1}r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}\mathcal{K}^n_{k,p}b_{k+p-1}r^{k-1}}\geq 0. \tag{9}$$

If the condition (8) does not hold, then the expression in (9) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0,1) for which the quotient in (9) is negative. This contradicts the required condition for $f_m \in \overline{H}_p(m,n,\alpha,\beta)$. And so the proof is complete.

Next we determine the extreme points of the closed convex hull of $\overline{H}_p(m,n,\alpha,\beta)$, denoted by $clco\overline{H}_p(m,n,\alpha,\beta)$.

Theorem 3. *Let f_m be given by (4). Then $f_m \in \overline{H}_p(m,n,\alpha,\beta)$ if and only if*

$$f_m(z)=\sum_{k=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z)+y_{k+p-1}g_{m_{k+p-1}}(z)\right]$$

where

$$h_p(z)=z^p,\qquad h_{k+p-1}(z)=z^p-\frac{1}{\Psi(m,n,p,\alpha,\beta)}z^{k+p-1}\,;\qquad (k=2,3,\dots)$$

and

$$g_{m_{k+p-1}}(z)=z^p+(-1)^{m-1}\frac{1}{\Theta(m,n,p,\alpha,\beta)}\overline{z}^{k+p-1}\,;\qquad (k=1,2,3,\dots)$$

$x_{k+p-1}\geq 0$, $y_{k+p-1}\geq 0$, $x_p=1-\sum_{k=2}^{\infty}x_{k+p-1}-\sum_{k=1}^{\infty}y_{k+p-1}$. *In particular, the extreme points of $\overline{H}_p(m,n,\alpha,\beta)$ are $\{h_{k+p-1}\}$ and $\{g_{k+p-1}\}$.*

Proof. For functions f_m of the form (4)

$$f_m(z) = \sum_{k=1}^{\infty} \left[x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{m_{k+p-1}}(z) \right]$$

$$= \sum_{k=1}^{\infty} (x_{k+p-1} + y_{k+p-1}) z^p - \sum_{k=2}^{\infty} \frac{1}{\Psi(m,n,p,\alpha,\beta)} x_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{m-1} \sum_{k=1}^{\infty} \frac{1}{\Theta(m,n,p,\alpha,\beta)} y_{k+p-1} \overline{z}^{k+p-1}.$$

Then

$$\sum_{k=2}^{\infty} \Psi(m,n,p,\alpha,\beta) \left(\frac{1}{\Psi(m,n,p,\alpha,\beta)} x_{k+p-1} \right)$$

$$+ \sum_{k=1}^{\infty} \Theta(m,n,p,\alpha,\beta) \left(\frac{1}{\Theta(m,n,p,\alpha,\beta)} y_{k+p-1} \right)$$

$$= \sum_{k=2}^{\infty} x_{k+p-1} + \sum_{k=1}^{\infty} y_{k+p-1} = 1 - x_p \leq 1$$

and so $f_m(z) \in clco\overline{H}_p(m,n,\alpha,\beta)$.

Conversely, suppose that $f_m(z) \in clco\overline{H}_p(m,n,\alpha,\beta)$. Set

$$x_{k+p-1} = \Psi(m,n,p,\alpha,\beta) a_{k+p-1}, \quad (k = 2,3,\dots)$$

$$y_{k+p-1} = \Theta(m,n,p,\alpha,\beta) b_{k+p-1}, \quad (k = 1,2,3,\dots)$$

and

$$x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}.$$

Then, as required, we obtain

$$f_m(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \frac{1}{\Psi(m,n,p,\alpha,\beta)} x_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} \frac{1}{\Theta(m,n,p,\alpha,\beta)} y_{k+p-1} \overline{z^{k+p-1}}$$

$$= z^p - \sum_{k=2}^{\infty} \left[z^p - h_{k+p-1}(z)\right] x_{k+p-1} - \sum_{k=1}^{\infty} \left[z^p - g_{m_{k+p-1}}(z)\right] y_{k+p-1}$$

$$= \left[1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}\right] z^p + \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z) + \sum_{k=1}^{\infty} y_{k+p-1} g_{m_{k+p-1}}(z)$$

$$= \sum_{k=1}^{\infty} \left[x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{m_{k+p-1}}(z)\right].$$

The following theorem gives the distortion bounds for functions in $\overline{H}_p(m,n,\alpha,\beta)$ which yields a covering results for this class.

Theorem 4. *Let $f_m \in \overline{H}_p(m,n,\alpha,\beta)$. Then for $|z| = r < 1$ we have*

$$|f_m(z)| \leq (1 + b_p) r^p + \left[\Phi(m,n,p,\alpha,\beta) - \Omega(m,n,p,\alpha,\beta) b_p\right] r^{n+p}$$

and

$$|f_m(z)| \geq (1 - b_p) r^p - \left\{\Phi(m,n,p,\alpha,\beta) - \Omega(m,n,p,\alpha,\beta) b_p\right\} r^{n+p}$$

where,

$$\Phi(m,n,p,\alpha,\beta) = \frac{1-\alpha}{\left(\frac{p+1}{p}\right)^m (1+\beta) - \left(\frac{p+1}{p}\right)^n (\beta+\alpha)},$$

$$\Omega(m,n,p,\alpha,\beta) = \frac{(1+\beta) - (-1)^{m-n}(\alpha+\beta)}{\left(\frac{p+1}{p}\right)^m (1+\beta) - \left(\frac{p+1}{p}\right)^n (\beta+\alpha)}.$$

Proof. We prove the right hand side inequality for $|f_m|$. The proof for the left hand inequality can be done using similar arguments. Let $f_m \in \overline{H}_p(m,n,\alpha,\beta)$. Taking the absolute value of f_m then by Theorem 2, we obtain:

$$|f_m(z)| = \left| z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1} \right|$$

$$\leq r^p+\sum_{k=2}^{\infty}a_{k+p-1}r^{k+p-1}+\sum_{k=1}^{\infty}b_{k+p-1}r^{k+p-1}$$

$$= r^p+b_pr^p+\sum_{k=2}^{\infty}(a_{k+p-1}+b_{k+p-1})r^{k+p-1}$$

$$\leq r^p+b_pr^p+\sum_{k=2}^{\infty}(a_{k+p-1}+b_{k+p-1})r^{p+1}$$

$$= (1+b_p)r^p+\Phi(m,n,p,\alpha,\beta)\sum_{k=2}^{\infty}\frac{1}{\Phi(m,n,p,\alpha,\beta)}\left(a_{k+p-1}+b_{k+p-1}\right)r^{p+1}$$

$$\leq (1+b_p)r^p+\Phi(m,n,p,\alpha,\beta)r^{n+p}\left[\sum_{k=2}^{\infty}\Psi(m,n,p,\alpha,\beta)a_{k+p-1}+\Theta(m,n,p,\alpha,\beta)b_{k+p-1}\right]$$

$$\leq (1+b_p)r^p+\left[\Phi(m,n,p,\alpha,\beta)-\Omega(m,n,p,\alpha,\beta)b_p\right]r^{n+p}.$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary 1. *Let $f_m \in \overline{H}_p(m,n,\alpha,\beta)$, then for $|z| = r < 1$ we have*

$$\{w : |w| < 1 - b_p - [\Phi(m,n,p,\alpha,\beta) - \Omega(m,n,p,\alpha,\beta)b_p] \subset f_m(\mathbb{U})\}.$$

Remark 1. If we take $m = n+1$, $\beta = 0$ and $p = 1$, then the above covering result given in [5]. Furthermore, the results of this paper, for $p = 1$ and $\beta = 0$ coincide with the results in [10].

Remark 2. For $p = 1$, we obtain the results given in [7].

References

[1] Ahuja O.P, Jahangiri J.M., *Multivalent harmonic starlike functions*, Ann. Univ. Marie Crie-Sklodowska Sect.A, **LV 1**(2001), 1-13.

[2] Avcı Y., Zlotkiewicz E., *On harmonic Univalent mappings*, Ann. Univ. Marie Crie-Sklodowska Sect.A, **44**(1991), 1-7.

[3] Clunie J.,Sheil-Small T., *Harmonic Univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math, **9**(1984), 3-25.

[4] Jahangiri J.M., *Harmonic functions starlike in the unit disc*, J. Math. Anal. Appl., **235**(1999), 470-477.

[5] Jahangiri J.M., Murugusundaramoorthy G. and Vijaya K., *Sălăgean-type harmonic univalent functions*, South. J. Pure Appl.Math., **2**(2002), 77-82.

[6] Sălăgean G.S., *Subclass of univalent functions*, Complex analysis-Fifth Romanian Finish Seminar, Bucharest,**1**(1983), 362-372.

[7] Seker B., Sümer Eker S., *On Sălăgean-Type Harmonic Univalent Functions* (to appear)

[8] Silverman H., *Harmonic univalent functions with negative coefficients*, J. Math.Anal.Appl. **220**(1998), 283-289.

[9] Silverman H. and Silvia E.M. *Subclasses of harmonic univalent functions*, New Zealand J. Math. **28**(1999), 275-284.

[10] Yalçın S., *A new class of Sălăgean-type harmonic univalent functions*, Applied Mathematics Letters, Vol.18, **2**(2005), 191-198.

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