ON VECTOR-BUNDLE VALUED COHOMOLOGY ON COMPLEX FINSLER MANIFOLDS

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Abstract. In this paper we define a complex adapted connection of type Bott on vertical bundle of a complex Finsler manifold. When this connection is flat we get a vertical vector valued cohomology. The notions are introduced by analogy with the real case for foliations. Finally, using the partial Bott connection we give a characterization of strongly Kähler-Finsler manifolds.

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1. Introduction and preliminaries notions

In the first section of this paper, following [2], [3], [7], we recall briefly some notions on complex Finsler manifolds, concerning to the Chern-Finsler linear connection, the canonical linear connection and the Levi-Civita connection associated with the Hermitian metric structure on holomorphic tangent bundle given by Sasaki lift of the fundamental tensor $g_{i\bar{j}}$. In the second section, by analogy with the real case for foliations (see [4], [9], [11]), we define an adapted vertical complex connection of Bott type on vertical bundle $V_C(T^{'}M)$, denoted by D^v. In the third section we assume that the connection D^v is flat, then the natural exterior derivative d_{D^v} associated with D^v on $V_C(T^{'}M)$-vector valued differential forms has the property $d^2_{D^v} = 0$. Thus we can think a cohomology of $V_C(T^{'}M)$-vector valued differential forms $H^*(T^{'}M, V_C(T^{'}M))$. Finally, using the partial Bott connection [2], we give a characterization of strongly Kähler-Finsler manifolds.

Let $\pi : T^{'}M \longrightarrow M$ be the holomorphic tangent bundle of a complex manifold M, $\dim_C M = n$. Denote by $(\pi^{-1}(U),\ (z^k, \eta^k))_{k=\overline{1,n}}$ the induced complex local coordinates on $T^{'}M$, where (U, z^k) is a local chart domain of M.

At local change charts on $T^{'}M$, the transformation rules of these coordinates are given by:

$$z^{'k} = z^{'k}(z);\ \eta^{'k} = \frac{\partial z^{'k}}{\partial z^j}\eta^j \tag{1}$$

where $\frac{\partial z^{'k}}{\partial z^j}$ are holomorphic functions on z and $det(\frac{\partial z^{'k}}{\partial z^j}) \neq 0$

It is well known the fact that $T^{'}M$ has a structure of $2n$-dimensional complex manifold, because the transition functions $\frac{\partial z^{'k}}{\partial z^j}$ are holomorphic.

Consider $T_C(T^{'}M) = T^{'}(T^{'}M) \oplus T^{''}(T^{'}M)$ the complexified tangent bundle of $T^{'}M$ where $T^{'}(T^{'}M)$ and $T^{''}(T^{'}M) = \overline{T^{'}(T^{'}M)}$ are the holomorphic and antiholomorphic tangent bundles of $T^{'}M$.

On $T^{'}M$ we fix an arbitrary complex nonlinear connection, briefly (c.n.c) having the local coefficients $N_k^j(z,\eta)$, which determines the following decomposition:

$$T^{'}(T^{'}M) = H^{'}(T^{'}M) \oplus V^{'}(T^{'}M) \tag{2}$$

By conjugation over all, we get a decomposition of the complexified tangent bundle, namely:

$$T_C(T^{'}M) = H^{'}(T^{'}M) \oplus V^{'}(T^{'}M) \oplus H^{''}(T^{'}M) \oplus V^{''}(T^{'}M) \tag{3}$$

The adapted frames with respect to this (c.n.c) are given by,

$$\{\delta_k = \partial_k - N_k^j\ \dot{\partial}_j;\ \dot{\partial}_k;\ \delta_{\overline{k}} = \partial_{\overline{k}} - N_{\overline{k}}^{\overline{j}}\ \dot{\partial}_{\overline{j}};\ \dot{\partial}_{\overline{k}}\} \tag{4}$$

where $\partial_k = \frac{\partial}{\partial z^k}$; $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and the adapted coframes, are given by

$$\{dz^k;\ \delta\eta^k = d\eta^k + N_j^k dz^j;\ d\overline{z}^k;\ \delta\overline{\eta}^k = d\overline{\eta}^k + N_{\overline{j}}^{\overline{k}} d\overline{z}^j\} \tag{5}$$

Definition 1 *A strictly pseudoconvex complex Finsler metric on M, is a continuous function $F : T^{'}M \to R$ satisfying:*

(i) $L := F^2$ is C^∞-smooth on $T^{'}M - \{0\}$;

(ii) $L(z,\eta) \geq 0$ and $L(z,\eta) = 0 \Leftrightarrow \eta = 0$;

(iii) $L(z,\lambda\eta) = |\lambda|^2 L(z,\eta)$, $\forall\ \lambda \in C$;

(iv) the following Hermitian matrix ($g_{i\overline{j}} = \dot{\partial}_i \dot{\partial}_{\overline{j}}\ (L)$) is positive defined on $T^{'}M - \{0\}$ and defines a Hermitian metric on vertical bundle.

Definition 2 *The pair (M,F) is called a complex Finsler manifold.*

Proposition 1 *A (c.n.c) on (M, F) depending only on the complex Finsler metric F is the Chern-Finsler (c.n.c) given by:*

$$\overset{CF}{N_k^j} = g^{\overline{m}j}\partial_k \dot{\partial}_{\overline{m}} (L) \tag{6}$$

Proposition 2 *The Lie brackets of the adapted frames from $T_C(T^{'}M)$, with repect to the Chern-Finsler (c.n.c) are,*

$$[\delta_j, \delta_k] = (\delta_k \overset{CF}{N_j^i} - \delta_j \overset{CF}{N_k^i}) \dot{\partial}_i = 0; [\delta_j, \delta_{\overline{k}}] = (\delta_{\overline{k}} \overset{CF}{N_j^i}) \dot{\partial}_i - (\delta_j \overset{CF}{N_{\overline{k}}^{\overline{i}}}) \dot{\partial}_{\overline{i}};$$

$$[\delta_j, \dot{\partial}_k] = (\dot{\partial}_k \overset{CF}{N_j^i}) \dot{\partial}_i; \ [\delta_j, \dot{\partial}_{\overline{k}}] = (\dot{\partial}_{\overline{k}} \overset{CF}{N_j^i}) \dot{\partial}_i; \ [\dot{\partial}_j, \dot{\partial}_k] = [\dot{\partial}_j, \dot{\partial}_{\overline{k}}] = 0$$

and their conjugates.

In the sequel we will consider the adapted frames and adapted coframes with respect to the Chern-Finsler (c.n.c) and the Hermitian metric structure G on $T^{'}M$ given by the Sasaki lift of fundamental tensor $g_{i\bar{j}}$:

$$G = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\overline{\eta}^j \tag{7}$$

Also we recall the Chern-Finsler linear connection $\overset{CF}{D}\Gamma = (\overset{CF}{N_j^i}, \overset{CF}{L_{jk}^i}, \overset{CF}{C_{jk}^i})$ and the canonical linear connection $\overset{c}{D}\Gamma = (\overset{CF}{N_j^i}, \overset{c}{L_{jk}^i}, \overset{c}{C_{jk}^i}, \overset{c}{L_{\bar{j}k}^{\bar{i}}}, \overset{c}{C_{\bar{j}k}^{\bar{i}}})$, where

$$\begin{aligned}
\overset{CF}{L_{jk}^i} &= g^{\overline{m}i} \delta_k g_{j\overline{m}}; \ \overset{CF}{C_{jk}^i} = g^{\overline{m}i} \dot{\partial}_k g_{j\overline{m}}; \\
\overset{c}{L_{jk}^i} &= \frac{1}{2} g^{\overline{m}i} (\delta_k g_{j\overline{m}} + \delta_j g_{k\overline{m}}); \ \overset{c}{C_{jk}^i} = \frac{1}{2} g^{\overline{m}i} (\dot{\partial}_k g_{j\overline{m}} + \dot{\partial}_j g_{k\overline{m}}); \\
\overset{c}{L_{\bar{j}k}^{\bar{i}}} &= \frac{1}{2} g^{\bar{i}m} (\delta_k g_{m\bar{j}} - \delta_m g_{k\bar{j}}); \ \overset{c}{C_{\bar{j}k}^{\bar{i}}} = \frac{1}{2} g^{\bar{i}m} (\dot{\partial}_k g_{m\bar{j}} - \dot{\partial}_m g_{k\bar{j}})
\end{aligned}$$

for details (see [7] p.51, p.61). By homogenity conditions of complex Finsler metric F, we note that

$$\overset{CF}{L_{jk}^i} = \dot{\partial}_j \overset{CF}{N_k^i}; \ \overset{CF}{C_{jk}^i} = \overset{CF}{C_{kj}^i} \tag{8}$$

We denote by ∇ the Levi-Civita connection associated to G, i.e. $\nabla G = 0$ and the torsion $T_\nabla = 0$. According to [7] p.52 and [3] p.93, the local expression of

∇ is given by:

$$\begin{aligned}
\nabla_{\delta_k}\delta_j &= \overset{c}{L}{}^i_{jk}\,\delta_i;\ \nabla_{\delta_k}\,\dot{\partial}_j = B^i_{jk}\delta_i + \overset{CF}{L}{}^i_{jk}\dot{\partial}_i;\\
\nabla_{\delta_k}\delta_{\bar{j}} &= \overset{c}{L}{}^i_{k\bar{j}}\,\delta_i + D^i_{\bar{j}k}\,\dot{\partial}_i + \overset{c}{L}{}^{\bar{i}}_{\bar{j}k}\,\delta_{\bar{i}} + E^{\bar{i}}_{\bar{j}k}\,\dot{\partial}_{\bar{i}};\ \nabla_{\delta_k}\,\dot{\partial}_{\bar{j}} = F^i_{\bar{j}k}\delta_i;\\
\nabla_{\dot{\partial}_k}\delta_j &= B^i_{kj}\delta_i;\ \nabla_{\dot{\partial}_k}\,\dot{\partial}_j = G^i_{jk}\delta_i + \overset{CF}{C}{}^i_{jk}\dot{\partial}_i;\ \nabla_{\dot{\partial}_k}\delta_{\bar{j}} = F^{\bar{i}}_{k\bar{j}}\delta_{\bar{i}} + H^{\bar{i}}_{\bar{j}k}\,\dot{\partial}_{\bar{i}}
\end{aligned}$$

where

$$\begin{aligned}
B^i_{jk} = \frac{1}{2}g^{\bar{l}i}(g_{j\bar{h}}\delta_k\,\overset{CF}{N}{}^{\bar{h}}_{\bar{l}} + \dot{\partial}_j\,g_{k\bar{l}});\ D^i_{\bar{j}k} &= \frac{1}{2}g^{\bar{l}i}(g_{h\bar{l}}\delta_{\bar{j}}\,\overset{CF}{N}{}^h_k - \dot{\partial}_{\bar{l}}\,g_{k\bar{j}});\\
E^{\bar{i}}_{\bar{j}k} = -\frac{1}{2}g^{\bar{i}l}(g_{l\bar{h}}\delta_k\,\overset{CF}{N}{}^{\bar{h}}_{\bar{j}} + \dot{\partial}_l\,g_{k\bar{j}});\ F^i_{\bar{j}k} &= -\frac{1}{2}g^{\bar{l}i}(g_{h\bar{j}}\delta_{\bar{l}}\,\overset{CF}{N}{}^h_k - \dot{\partial}_{\bar{j}}\,g_{k\bar{l}});\\
G^i_{jk} = g^{\bar{l}i}g_{j\bar{h}}\,\dot{\partial}_k\overset{CF}{N}{}^{\bar{h}}_{\bar{l}};\ ;\ H^{\bar{i}}_{\bar{j}k} = -\,\dot{\partial}_k\overset{CF}{N}{}^{\bar{i}}_{\bar{j}} &
\end{aligned}$$

and their conjugates.

We remark that the Levi-Civita connection ∇ is not compatible with the natural complex structure J on $T^{'}M$, defined by:

$$J(\delta_k) = i\delta_k;\ J(\delta_{\overline{k}}) = -i\delta_{\overline{k}};\ J(\dot{\partial}_k) = i\,\dot{\partial}_k;\ J(\dot{\partial}_{\overline{k}}) = -i\,\dot{\partial}_{\overline{k}} \tag{9}$$

Imposing the condition that ∇ to be compatible with the complex structure J, namely:

$$(\nabla_{\xi_1}J)\xi_2 = \nabla_{\xi_1}(J\xi_2) - J(\nabla_{\xi_1}\xi_2) = 0;\ \forall\,\xi_1,\xi_2 \in \Gamma(T_C(T^{'}M)) \tag{10}$$

we get the conditions,

$$\delta_i g_{j\overline{k}} = \delta_j g_{i\overline{k}};\ g^{\bar{i}l}\,\dot{\partial}_{\overline{k}}\,g_{j\bar{i}} = \delta_{\overline{k}}(\overset{CF}{N}{}^l_j) \tag{11}$$

and in this case we call the metric structure *G-total Kähler*.

2. The vertical Bott type complex connection

In the similar manner with the real case for foliations, (see [4], [9], [11]), for the complex vertical vector fields V, V_1, $V_2 \in \Gamma(V_C(T^{'}M))$ and a complex horizontal vector field $X \in \Gamma(H_C(T^{'}M))$, we define on vector bundle $V_C(T^{'}M)$ a connection D^v, as follows:

$$D^v_{V_1}V_2 = (v^{'} + v^{''})\nabla_{V_1}V_2;\ D^v_X V = (v^{'} + v^{''})[X, V] \qquad (12)$$

where $v^{'}$ and $v^{''}$ are the complex vertical projectors, and ∇ is the Levi-Civita connection.

Definition 3 *The connection D^v from (12) is called vertical Bott type complex connection.*

The local expression of the connection D^v is given by,

$$D^v_{\dot{\partial}_k}\dot{\partial}_j = (v^{'} + v^{''})\nabla_{\dot{\partial}_k}\dot{\partial}_j = \overset{CF}{C^i_{jk}}\dot{\partial}_i;\ D^v_{\delta_k}\dot{\partial}_j = (v^{'} + v^{''})[\delta_k, \dot{\partial}_j] = \overset{CF}{L^i_{jk}}\dot{\partial}_i$$

$$D^v_{\dot{\partial}_k}\dot{\partial}_{\bar{j}} = (v^{'} + v^{''})\nabla_{\dot{\partial}_k}\dot{\partial}_{\bar{j}} = 0;\ D^v_{\delta_k}\dot{\partial}_{\bar{j}} = (v^{'} + v^{''})[\delta_k, \dot{\partial}_{\bar{j}}] = (\dot{\partial}_{\bar{j}}\overset{CF}{N^i_k})\,\dot{\partial}_i$$

and their conjugates, since $\overline{D^v_\xi V} = D^v_{\bar{\xi}}\overline{V}, \forall\, \xi \in \Gamma(T_C(T^{'}M)),\ V \in \Gamma(V_C(T^{'}M))$.

Let R^{D^v} and $R^{v\nabla}$ be the curvature tensors on $V_C(T^{'}M)$ induced by D^v and ∇, where $v = v^{'} + v^{''}$. For the complex horizontal vector fields X, Y and the complex vertical vector fields U, V, W we have,

Proposition 3

$$(i)\ R^{D^v}_{X,Y}V = -v\nabla_V v[X, Y]$$

$$(ii)\ R^{D^v}_{X,U}V = (\mathcal{L}_X v\nabla)_U V$$

$$(iii)\ R^{D^v}_{U,W}V = R^{v\nabla}_{U,W}V$$

where $\mathcal{L}_X$ denotes the Lie derivative.

Proof. (i) $R^{D^v}_{X,Y}V = D^v_X D^v_Y V - D^v_Y D^v_X V - D^v_{[X,Y]}V = v[X, v[Y, V]] -$

$$v[Y, v[X, V]] - v[h[X, Y], V] - v\nabla_{v[X,Y]}V = [X, [Y, V]] - [Y, [X, V]] -$$

$$v[[X, Y] - v[X, Y], V] - v\nabla_{v[X,Y]}V = [X, [Y, V]] - [Y, [X, V]] - [[X, Y], V] +$$

$[v[X,Y],V] - v\nabla_{v[X,Y]}V = [v[X,Y],V] - v\nabla_{v[X,Y]}V$. From $T_\nabla = 0$ we have,

$$[v[X,Y],V] = v[v[X,Y],V] = v\nabla_{v[X,Y]}V - v\nabla_V v[X,Y].$$

So,

$$R^{D^v}_{X,Y}V = v\nabla_{v[X,Y]}V - v\nabla_V v[X,Y] - v\nabla_{v[X,Y]}V = -v\nabla_V v[X,Y]$$

The relations (ii) and (iii) follows in a similar manner. *Q.e.d*

Remark 1 *Proposition 2.1 shows that the curvature of the vertical Bott type connection D^v, is related only in terms of the induced Levi-Civita connection on $V_C(T^{'}M)$.*

Taking all combination of X, Y, U, V, W in local frames $\{\delta_k;\ \dot{\partial}_k;\ \delta_{\overline{k}};\ \dot{\partial}_{\overline{k}}\}$ a direct calculus leads to the following nonzero curvature of the vertical Bott type complex connection D^v:

$$\begin{aligned}
v^{'}R^{D^v}_{\delta_k,\delta_{\overline{i}}}\,\dot{\partial}_i &= -\delta_{\overline{j}}(\overset{CF}{L^l_{ik}})\,\dot{\partial}_l - \delta_{\overline{j}}(\overset{CF}{N^m_k})\,\overset{CF}{C^l_{mi}}\dot{\partial}_l = \overset{CF}{R^l_{i,\overline{j}k}}\dot{\partial}_l \\
v^{''}R^{D^v}_{\delta_k,\delta_{\overline{i}}}\,\dot{\partial}_i &= \dot{\partial}_i\,\delta_k(\overset{CF}{N^{\overline{l}}_{\overline{j}}})\,\dot{\partial}_{\overline{l}} = \widetilde{R^{\overline{l}}_{i,\overline{j}k}}\,\dot{\partial}_{\overline{l}} \\
v^{'}R^{D^v}_{\delta_k,\dot{\partial}_j}\,\dot{\partial}_{\overline{i}} &= -\,\dot{\partial}_{\overline{i}}\,(\overset{CF}{L^l_{jk}})\,\dot{\partial}_l - \dot{\partial}_{\overline{i}}\,(\overset{CF}{N^m_k})\,\overset{CF}{C^l_{mj}}\dot{\partial}_l = \overset{CF}{Q^l_{\overline{i},jk}}\dot{\partial}_l \\
v^{'}R^{D^v}_{\delta_{\overline{k}},\dot{\partial}_j}\,\dot{\partial}_i &= \delta_{\overline{k}}(\overset{CF}{C^l_{ij}})\,\dot{\partial}_l = \overset{CF}{P^l_{i,j\overline{k}}}\dot{\partial}_l \\
v^{''}R^{D^v}_{\delta_{\overline{k}},\dot{\partial}_j}\,\dot{\partial}_i &= \dot{\partial}_m\,(\overset{CF}{N^{\overline{l}}_{\overline{k}}})\,\overset{CF}{C^m_{ij}}\dot{\partial}_{\overline{l}} - \dot{\partial}_j\dot{\partial}_i\,(\overset{CF}{N^{\overline{l}}_{\overline{k}}})\,\dot{\partial}_{\overline{l}} = \widetilde{P^{\overline{l}}_{i,j\overline{k}}}\,\dot{\partial}_{\overline{l}} \\
v^{'}R^{D^v}_{\dot{\partial}_k,\dot{\partial}_{\overline{j}}}\,\dot{\partial}_i &= -\,\dot{\partial}_{\overline{j}}\,(\overset{CF}{C^l_{ik}})\,\dot{\partial}_l = \overset{CF}{S^l_{i,\overline{j}k}}\dot{\partial}_l
\end{aligned}$$

and their conjugates.

Remark 2 *The curvatures of the vertical Bott type complex connection differs from the curvatures of the Chern-Finsler connection by two components, namely $\widetilde{R^{\overline{l}}_{i,\overline{j}k}}$ and $\widetilde{P^{\overline{l}}_{i,j\overline{k}}}$.*

Proposition 4 *If the complex Finsler metric is locally Minkowski and the Hermitian metric G is vertical Kahler, i.e. it satisfy the second condition of (11), then the vertical Bott type conection D^v is flat.*

Proof: If the complex Finsler metric is locally Minkowski, namely $L = L(\eta)$ [2], then $g_{i\bar{j}} = g_{i\bar{j}}(\eta)$ and $\overset{CF}{N_k^j} = 0$. Thus all curvatures of the vertical Bott type connection except $\overset{CF}{S^l_{i,\bar{j}k}}$ are vanish. Imposing the condition $g^{\bar{i}l}\ \dot{\partial}_{\bar{k}}\ g_{j\bar{i}} = 0$ we get $\overset{CF}{S^l_{i,\bar{j}k}} = 0$. *Q.e.d*

3. Cohomology with vertical vector values

Let $\Omega^p(V_C(T^{'}M))$ be the set of all V_C-vector valued differential p-forms on $T^{'}M$ and $\Omega(V_C(T^{'}M)) = \sum\limits_{p=0}^{4n} \Omega^p(V_C(T^{'}M))$. We note that $\Omega^0(V_C(T^{'}M)) = \Gamma(V_C(T^{'}M))$ and for every $\phi \in \Omega^p(V_C(T^{'}M))$ we have,

$$\phi(\xi_1, ..., \xi_p) \in \Gamma(V_C(T^{'}M)), \forall\, \xi_1, ..., \xi_p \in \Gamma(T_C(T^{'}M)) \tag{13}$$

By analogy with the real case for foliations, we define the following exterior differential with respect to the complex connection D^v :

$$d_{D^v} : \Omega^p(V_C(T^{'}M)) \longrightarrow \Omega^{p+1}(V_C(T^{'}M)) \tag{14}$$

where

$$d_{D^v}\phi(\xi_0, \xi_1, ..., \xi_p) = \sum_{j=0}^{p}(-1)^j D^v_{\xi_j}(\phi(\xi_0, ..., \widehat{\xi_j}, ..., \xi_p)) +$$
$$+ \sum_{0\leq i<j\leq p}(-1)^{i+j}\phi([\xi_i, \xi_j], \xi_0, ..., \widehat{\xi_i}, ..., \widehat{\xi_j}, ..., \xi_p) \tag{15}$$

Proposition 5

$$d^2_{D^v}\phi(\xi_0, \xi_1, \xi_2) = \sum_{cicl} R^{D^v}_{\xi_i,\xi_j}\phi(\xi_k) \ \ on\ \Omega^1(V_C(T^{'}M))$$

Proof: We have $d_{D^v}\phi(\xi_0, \xi_1) = D^v_{\xi_0}\phi(\xi_1) - D^v_{\xi_1}\phi(\xi_0) - \phi([\xi_0, \xi_1])$ and directly we get $d^2_{D^v}\phi(\xi_0, \xi_1, \xi_2) = R^{D^v}_{\xi_0,\xi_1}\phi(\xi_2) + R^{D^v}_{\xi_1,\xi_2}\phi(\xi_0) + R^{D^v}_{\xi_2,\xi_0}\phi(\xi_1)$. *Q.e.d*

More general on $\Omega^p(V_C(T^{'}M))$ we have,

$$d^2_{D^v}\phi(\xi_0, \xi_1, ..., \xi_{p+1}) = \sum_{cicl} R^{D^v}_{\xi_i,\xi_j}\phi(\xi_0, ..., \widehat{\xi_i}, ..., \widehat{\xi_j}, ..., \xi_{p+1}) \tag{16}$$

Thus we get a complex,

$$\Omega^0(V_C(T^{'}M)) \xrightarrow{d_{D^v}} \Omega^1(V_C(T^{'}M)) \xrightarrow{d_{D^v}} ... \xrightarrow{d_{D^v}} \Omega^p(V_C(T^{'}M)) \xrightarrow{d_{D^v}} ... \tag{17}$$

From the above discussion we have,

Theorem 1 *Let (M,F) be a complex Finsler manifold. If the vertical Bott type complex connection D^v is flat i.e., $R^{D^v}_{\xi_1,\xi_2}=0,\ \forall\,\xi_1,\xi_2\in\Gamma(T_C(T^{'}M))$, then,*

$$d^2_{D^v}=0 \tag{18}$$

In this case d_{D^v} determines a cohomology

$$H^*(T^{'}M,V_C(T^{'}M))=\sum_{p=0}^{4n}H^p(T^{'}M,V_C(T^{'}M))$$

where

$$H^p(T^{'}M,V_C(T^{'}M))=\frac{Ker\{d_{D^v}:\Omega^p(V_C(T^{'}M))\to\Omega^{p+1}(V_C(T^{'}M))\}}{Im\{d_{D^v}:\Omega^{p-1}(V_C(T^{'}M))\to\Omega^p(V_C(T^{'}M))\}}$$

In the sequel for every complex vector fields $\xi,\xi_1,\xi_2\in\Gamma(T_C(T^{'}M))$ we define

$$\omega(\xi)=v\xi;\Theta(\xi_1,\xi_2)=v[h\xi_1,h\xi_2] \tag{19}$$

where $v=v^{'}+v^{''}$ and $h=h^{'}+h^{''}$. Then, $\omega\in\Omega^1(V_C(T^{'}M))$ and $\Theta\in\Omega^2(V_C(T^{'}M))$.

We have,

Theorem 2

$$d_{D^v}\omega=-\Theta \tag{20}$$

Proof: It sufficient to verify the relation (20) for every two complex horizontal and vertical vector fields. Let X,Y be horizontal vector fields and U,V be vertical vector fields. We have:

$$d_{D^v}\omega(X,Y)=D^v_X\omega(Y)-D^v_Y\omega(X)-\omega([X,Y])=0-0-v[X,Y]=-\Theta(X,Y);$$

$$d_{D^v}\omega(X,V)=D^v_X\omega(V)-D^v_V\omega(X)-\omega([X,V])=D^v_XV-0-v[X,V]=$$
$$v[X,V]-v[X,V]=0=-v[hX,hV]=-\Theta(X,V);$$

$$d_{D^v}\omega(U,V)=D^v_U\omega(V)-D^v_V\omega(U)-\omega([U,V])=v\nabla_UV-v\nabla_VU-v[U,V]=$$
$$T^{vv}_{\nabla}=0=-v[hU,hV]=-\Theta(U,V).$$ *Q.e.d*

Theorem 3 *Let (M,F) be a complex Finsler manifold. Then $H_C(T^{'}M)$ is integrable if and only if $d_{D^v}\omega=0$.*

Proof: If $H_C(T^{'}M)$ is integrable then, $v[X,Y]=0, \forall X,Y\in\Gamma(H_C(T^{'}M))$ and the Theorem 3.2 leads to $\Theta=0$ and, so $d_{D^v}\omega=0$. Conversely, if $d_{D^v}\omega=0$ we obtain that $v[X,Y]=0, \forall X,Y\in\Gamma(H_C(T^{'}M))$, so $H_C(T^{'}M)$ is integrable. *Q.e.d*

Proposition 6 $d_{D^v}\Theta=0$ *provided* D^v *is flat.*

According to (8) and (9) we have,

Proposition 7 *Let* (M,F) *be a complex Finsler manifold. Then,*

$$(d_{D^v}J)(U,V)=0,\ \forall\, U,V\in\Gamma(V_C(T^{'}M))$$

Finally, we give a characterization of strongly Kähler-Finsler manifolds. According to [2] the *partial Bott connection* is defined by,

$$\overset{B}{D}_X V=v^{'}[X,V],\ \forall\, X\in\Gamma(H^{'}(T^{'}M)),\ V\in\Gamma(V^{'}(T^{'}M)) \tag{21}$$

Locally, we have $\overset{B}{D}_{\delta_k}\dot{\partial}_j=\overset{B}{L}{}^i_{jk}\dot{\partial}_i$, where $\overset{B}{L}{}^i_{jk}=\dot{\partial}_j\overset{CF}{N}{}^i_k=\overset{CF}{L}{}^i_{jk}$.

Definition 4 *(Cf. [1]) The complex Finsler manifold* (M,F) *is called strongly Kähler if* $\overset{CF}{L}{}^i_{jk}=\overset{CF}{L}{}^i_{kj}$.

If we consider $\Omega^p(H^{'}(T^{'}M);V^{'}(T^{'}M))$ the set of all horizontal $p-$ differentials forms with vertical valued, then the exterior derivative associated to the partial Bott connection is given by

$$d_B:\Omega^p(H^{'}(T^{'}M);V^{'}(T^{'}M))\to\Omega^{p+1}(H^{'}(T^{'}M);V^{'}(T^{'}M))$$

where

$$(d_B\phi)(X_0,...,X_p)=\sum_{j=0}^{p}(-1)^j\ \overset{B}{D}_{X_j}\ \phi(X_0,...,\widehat{X_j},...,X_p)$$

$\forall\,\phi\in\Omega^p(H^{'}(T^{'}M);V^{'}(T^{'}M)),\ \forall\, X_0,...,X_p\in\Gamma(H^{'}(T^{'}M))$.

Let S be the tangent structure [7] locally defined by,

$$S(\partial_k)=\dot{\partial}_k,\ S(\dot{\partial}_k)=0,\ S(\partial_{\overline{k}})=\dot{\partial}_{\overline{k}},\ S(\dot{\partial}_{\overline{k}})=0 \tag{22}$$

In [8] is proved that S is a global defined and integrable structure. We have $S(\delta_k)=\dot{\partial}_k$ and we can consider $S|_{H^{'}(T^{'}M)}\in\Omega^1(H^{'}(T^{'}M);V^{'}(T^{'}M))$. Then,

Proposition 8 *The complex Finsler manifold* (M, F) *is strongly Kähler if and only if* $(d_B S)(X, Y) = 0$, $\forall\, X, Y \in \Gamma(H^{'}(T^{'}M))$.

Proof: Follows by definitions of S and d_B. *Q.e.d*

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