CERTAIN COEFFICIENT INEQUALITIES FOR SAKAGUCHI TYPE FUNCTIONS AND APPLICATIONS TO FRACTIONAL DERIVATIVE OPERATOR

S. P. Goyal, Pranay Goswami

Abstract. In the present paper, sharp upper bounds of $|a_3 - \mu a_2^2|$ for the functions $f(z) = z + a_2z^2 + a_3z^3 + ...$ belonging to a new subclass of Sakaguchi type functions are obtained. Also, application of our results for subclass of functions defined by convolution with a normalized analytic function are given. In particular, Fekete-Szegö inequalities for certain classes of functions defined through fractional derivatives are obtained.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \Delta := \{z \in C | \, |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $univalent functions$. For two functions $f, \ g \in \mathcal{A}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in Δ and write $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that $f(z) = g(w(z)), (z \in \Delta)$. In particular, if the function g is univalent in Δ, the above subordination is *equivalent* to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Recently Owa et al. [7,8] introduced and studied a Sakaguchi type class $S^*(\alpha, t)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $S^*(\alpha, t)$ if it satisfies

$$\mathrm{Re}\left\{\frac{(1-t)zf'(z)}{f(z) - f(zt)}\right\} > \alpha, \quad |t| \leq 1, \ t \neq 1 \tag{1.2}$$

for some $\alpha \in [0, 1)$ and for all $z \in \Delta$. For $\alpha = 0$ and $t = -1$, we get the class the class $S^*(0, -1)$ studied be Sakaguchi [9]. A function $f(z) \in S^*(\alpha, -1)$ is called Sakaguchi function of order α.

In this paper, we define the following class $S^*(\phi, t)$, which is generalization of the class $S^*(\alpha, t)$.

Definition 1.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ be univalent starlike function with respect to '1' which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. Then function $f \in \mathcal{A}$ is in the class $S^*(\phi, t)$ if*

$$\frac{(1-t)zf'(z)}{f(z) - f(zt)} \prec \phi(z), \qquad |t| \leq 1, t \neq 1 \tag{1.5}$$

Again $\mathcal{T}(\phi, t)$ denote the subclass of $\mathcal{A}$ consisting functions $f(z)$ such that $zf'(z) \in \mathcal{S}^*(\phi, t)$.

When $\phi(z) = (1 + Az)/(1 + Bz), (-1 \leq B < A \leq 1)$, we denote the subclasses $S^*(\phi, t)$ and $T(\phi, t)$ by $\mathcal{S}^*[A, B, t]$ and $\mathcal{T}[A, B, t]$ respectively.

Obviously $S^*(\phi, 0) \equiv S^*(\phi)$. When $t = -1$, then $S^*(\phi, -1) \equiv S^*_s(\phi)$, which is a known class studied by Shanmugam et al. [10]. For $t = 0$ and $\phi(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$, the subclass $S^*(\phi, t)$ reduces to the class $S^*[A, B]$ studied by Janowski [3]. For $0 \leq \alpha < 1$ let $S^*(\alpha, t) := S^*[1 - 2\alpha, -1; t]$, which is a known class studied by Owa et al. [8]. Also, for $t = -1$ and $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$, our class reduces to a known class $S(\alpha, -1)$ studied by Cho et al. ([1], see also [8]).

In the present paper, we obtain the Fekete-Szegö inequality for the functions in the subclass $S^*(\phi, t)$. We also give application of our results to certain functions defined through convolution (or Hadamard product) and in particular, we consider the class $S^\lambda(\phi, t)$ defined by fractional derivatives.

To prove our main results, we need the following lemma:

Lemma 1.2. **([4])** *If $p(z) = 1 + c_1 z + c_2 z^2 ...$ is an analytic function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & if \quad v \leq 0, \\ 2 & if \quad 0 \leq v \leq 1, \\ 4v - 2 & if \quad v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $(1 + z)/(1 - z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p(z)$ is $(1 + z^2)/(1 - z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \qquad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \qquad (1/2 < v \leq 1).$$

2. Main Results

Our main result is contained in the following Theorem :

Theorem 2.1. *If $f(z)$ given by (1.1) belongs to $S^*(\phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(t+2)(1-t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right) - \mu B_1^2\left(\frac{2+t}{1-t}\right)\right] & if \quad \mu \leq \sigma_1, \\ \frac{B_1}{(t+2)(1-t)} & if\ \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{(t+2)(1-t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right) - \mu B_1^2\left(\frac{2+t}{1-t}\right)\right] & if \quad \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(1-t)}{B_1(2+t)}\left[-1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right],$$

and

$$\sigma_2 := \frac{(1-t)}{B_1(2+t)}\left[1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right].$$

The result is sharp.

Proof. Let $f \in S^*(\phi, t)$. Then there exists a $Schwarz$ function $w(z) \in \mathcal{A}$ such that

$$\frac{(1-t)zf'(z)}{f(z) - f(zt)} = \phi(w(z)) \quad (z \in \Delta; \quad |t| \leq 1, t \neq 1) \tag{2.1}$$

If $p_1(z)$ is analytic and has positive real part in Δ and $p_1(0) = 1$, then

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + ... \qquad (z \in \Delta). \tag{2.2}$$

From (2.2), we obtain

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + ... \tag{2.3}$$

Let

$$p(z) = \frac{(1-t)zf'(z)}{f(z) - f(tz)} = 1 + b_1 z + b_2 z^2 + ... \qquad (z \in \Delta), \tag{2.4}$$

which gives

$$b_1 = (1-t)a_2 \text{ and } b_2 = (t^2-1)a_2^2 + (2-t-t^2)a_3. \tag{2.5}$$

Since $\phi(z)$ is univalent and $p \prec \phi$, therefore using (2.3), we obtain

$$p(z) = \phi(w(z)) = 1 + \frac{B_1 c_1}{2} z + \left\{ \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2 \right\} z^2 + \ldots \quad (z \in \Delta), \tag{2.6}$$

Now from (2.4), (2.5) and (2.6), we have

$$(1-t)a_2 = \frac{B_1 c_1}{2}, \quad \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2 = (t^2-1)a_2^2 + (2-t-t^2)a_3, \ |t| \le 1, \ t \neq 1.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(1-t)(t+2)} \left[c_2 - v c_1^2\right], \tag{2.7}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} - B_1\left(\frac{1+t}{1-t}\right) + \mu B_1 \left(\frac{2+t}{1-t}\right)\right]$$

Our result now follows by an application of Lemma 1.2. To shows that these bounds are sharp, we define the functions K_{ϕ_n} $(n = 2, 3...)$ by

$$\frac{(1-t)zK'_{\phi_n}(z)}{K_{\phi_n}(z) - K'_{\phi_n}(tz)} = \phi(z^{n-1}), \quad K_{\phi_n}(0) = 0 = [K_{\phi_n}]'(0) - 1$$

and the function F^λ and G^λ $(0 \le \lambda \le 1)$ by

$$\frac{(1-t)zF'_\lambda(z)}{F_\lambda(z) - F_\lambda(tz)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F_\lambda(0) = 0 = [F_\lambda]'(0) - 1$$

and

$$\frac{(1-t)zG'_\lambda(z)}{G_\lambda(z) - G_\lambda(tz)} = \phi\left(\frac{-z(z+\lambda)}{1+\lambda z}\right), \quad G_\lambda(0) = 0 = [G_\lambda]'(0) - 1.$$

Obviously the functions $K_{\phi_n}, F_\lambda, G_\lambda \in S^*(\phi, t)$. Also we write $K_\phi := K_{\phi_2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality hods if and only if f is K_{ϕ_3} or one of its rotations. $\mu = \sigma_1$ then equality holds if and only if f is F_λ or one of its rotations. $\mu = \sigma_2$ then equality holds if and only if f is G_λ or one of its rotations.

If $\sigma_1 \le \mu \le \sigma_2$, in view of Lemma 1.2, Theorem 2.1 can be improved.

Theorem 2.2. *Let $f(z)$ given by (1.1) belongs to $S^*(\phi, t)$. and σ_3 be given by*

$$\sigma_3 := \frac{1}{B_1}\left(\frac{1-t}{2+t}\right)\left[\frac{B_2}{B_1} - B_1\left(\frac{1+t}{1-t}\right)\right].$$

If $\sigma_1 < \mu \leq \sigma_3$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{B_1^2}\left[(B_1 - B_2)\left(\frac{1-t}{2+t}\right) - B_1^2\left(\frac{1+t}{2+t}\right) + \mu B_1^2\right]|a_2|^2 \leq \frac{B_1}{(1-t)(t+2)}.$$

If $\sigma_3 < \mu \leq \sigma_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{B_1^2}\left[(B_1 + B_2)\left(\frac{1-t}{2+t}\right) + B_1^2\left(\frac{1+t}{2+t}\right) + \mu B_1^2\right]|a_2|^2 \leq \frac{B_1}{(1-t)(t+2)}$$

Example 2.3. Let $-1 \leq B < A \leq 1$. If $f(z) \in \mathcal{A}$ belongs to $S^*[A, B; t]$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B-A}{(2+t)(1-t)}\left[B - (A-B)\left(\frac{1+t}{1-t}\right) + \mu(A-B)\left(\frac{2+t}{1-t}\right)\right] & if \quad \mu \leq \sigma_1^*, \\ \frac{A-B}{(2+t)(1-t)} & if \quad \sigma_1^* \leq \mu \leq \sigma_2^*, \\ \frac{A-B}{(t+2)(1-t)}\left[B - (A-B)\left(\frac{1+t}{1-t}\right) + \mu(A-B)\left(\frac{2+t}{1-t}\right)\right] & if \quad \mu \leq \sigma_2^*, \end{cases}$$

where

$$\sigma_1^* = \frac{(1-t)}{(2+t)}\left[-\frac{1+B}{(A-B)} + \left(\frac{1+t}{1-t}\right)\right],$$

and

$$\sigma_2^* = \frac{(1-t)}{(2+t)}\left[\frac{1-B}{(A-B)} + \left(\frac{1+t}{1-t}\right)\right].$$

In particular, if $f \in S(\alpha, t)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(1-\alpha)}{(2+t)(1-t)}\left[1 + 2(1-\alpha)\left(\frac{1+t}{1-t}\right) - 2\mu(1-\alpha)\left(\frac{2+t}{1-t}\right)\right] & if \quad \mu \leq \sigma_1^{**}, \\ \frac{2(1-\alpha)}{(2+t)(1-t)} & if \quad \sigma_1^{**} \leq \mu \leq \sigma_2^{**}, \\ 2\frac{1-\alpha}{(t+2)(1-t)}\left[(1-2\alpha) - \alpha\left(\frac{1+t}{1-t}\right) - \mu\alpha\left(\frac{2+t}{1-t}\right)\right] & if \quad \mu \geq \sigma_2^{**}, \end{cases}$$

where

$$\sigma_1^{**} = -\left(\frac{1+t}{2+t}\right)$$

and

$$\sigma_2^{**} = \frac{(1-t)}{(2+t)}\left[\frac{1}{(1-\alpha)} - \left(\frac{1+t}{1-t}\right)\right]$$

The results are sharp.

3. Applications to functions defined by fractional derivatives

For two analytic functions $f(z) = z + \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=0}^{\infty} g_n z^n$, their convolution (or Hadamard product) is defined to be the function $(f * g)(z) = z + \sum\limits_{n=0}^{\infty} a_n g_n z^n$. For a fixed $g \in \mathcal{A}$, let $S^g(\phi, t)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in S^*(\phi, t)$.

Definition 3.1. *Let $f(z)$ be analytic in a simply connected region of the z-plane containing origin. The fractional derivative of f of order λ is defined by*

$$ {}_0D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int\limits_0^z (z-\zeta)^{-\lambda} f(\zeta) d\zeta \quad (0 \leq \lambda < 1), \tag{3.1}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring that $log(z-\zeta)$ is real for $(z-\zeta) > 0$

Using definition 3.1, Owa and Srivastava (see [5],[6]; see also [11], [12]) introduced a fractional derivative operator $\Omega^{\lambda} : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$(\Omega^{\lambda} f)(z) = \Gamma(2-\lambda) z^{\lambda} {}_0D_z^{\lambda} f(z), \qquad (\lambda \neq 2, 3, 4, ...)$$

The class $S^{\lambda}(\phi, t)$ consists of the functions $f \in \mathcal{A}$ for which $\Omega^{\lambda} f \in S^*(\phi, t)$. The class $S^{\lambda}(\phi, t)$ is a special case of the class $S^g(\phi, t)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n \qquad (z \in \Delta).$$

Now applying Theorem 2.1 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + ...$, we get following theorem after an obvious change of the parameter μ:

Theorem 3.2 *Let $g(z) = z + \sum\limits_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$. If $f(z)$ is given by (1.1) belongs to $S^g(\phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{g_3(2+t)(1-t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) - \mu \frac{g_3}{g_2^2} B_1^2 \left(\frac{2+t}{1-t}\right)\right] & if\ \mu \leq \eta_1, \\ \frac{B_1}{g_3(2+t)(1-t)} & if\ \ \eta_1 \leq \mu \leq \eta_2, \\ -\frac{1}{g_3(2+t)(1-t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) - \mu \frac{g_3}{g_2^2} B_1^2 \left(\frac{2+t}{1-t}\right)\right] & if\ \mu \geq \eta_2, \end{cases}$$

where

$$\eta_1 := \frac{g_2^2}{g_3 B_1} \left(\frac{1-t}{2+t}\right) \left[-1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right)\right]$$
$$\eta_2 := \frac{g_2^2}{g_3 B_1} \left(\frac{1-t}{2+t}\right) \left[1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right)\right]$$

The result is sharp.
Since

$$\Omega^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n,$$

We have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{3.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

For g_2, g_3 given by (3.2) and (3.3) respectively, Theorem 3.2 reduces to the following :

Theorem 3.3. *Let $\lambda < 2$. If $f(z)$ given by (1.1) belongs to $S^\lambda(\phi, t)$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\lambda)(3-\lambda)}{6(2+t)(1-t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) - \frac{3}{2}\mu \left(\frac{3-\lambda}{2-\lambda}\right) B_1^2 \left(\frac{2+t}{1-t}\right)\right] & if\ \mu \leq \eta_1^*, \\ \frac{(2-\lambda)(3-\lambda)}{6(2+t)(1-t)} B_1 & if\ \eta_1^* \leq \mu \leq \eta_2^*, \\ -\frac{(2-\lambda)(3-\lambda)}{6(2+t)(1-t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) - \frac{3}{2}\mu \left(\frac{3-\lambda}{2-\lambda}\right) B_1^2 \left(\frac{2+t}{1-t}\right)\right] & if\ \mu \geq \eta_1^*, \end{cases}$$

where

$$\begin{aligned} \eta_1 &:= \tfrac{(3-\lambda)}{(2-\lambda)B_1} \left(\tfrac{1-t}{2+t}\right) \left[-1 + \tfrac{B_2}{B_1} + B_1 \left(\tfrac{1+t}{1-t}\right)\right] \\ \eta_2 &:= \tfrac{(3-\lambda)}{(2-\lambda)B_1} \left(\tfrac{1-t}{2+t}\right) \left[1 + \tfrac{B_2}{B_1} + B_1 \left(\tfrac{1+t}{1-t}\right)\right]. \end{aligned}$$

The result is sharp.

Remark. For $t = -1$ in aforementioned Theorems 2.1, 2.2, 3.2, 3.3 and example 2.3, we arrive at the results obtained recently by Shanmugham et al. [10].

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S. P. Goyal
Department of Mathematics
University of Rajasthan
Jaipur (INDIA) - 302055
email:*somprg@gmail.com*

Pranay Goswami
Department of Mathematics
Amity University Rajasthan
Jaipur (INDIA) - 302002
email:*pranaygoswami83@gmail.com*