SEMISYMMETRIC CUBIC GRAPHS OF ORDER $4p^n$

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Abstract. An regular graph is said to be semisymmetric if its full automorphism group acts transitively on its edge set but not on its vertex set. In this paper we prove that for every prime $p(\neq 5)$, there is no semisymmetric cubic graph of order $4p^n$, where $n \geq 1$.

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1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For a graph X, we denote by $V(X)$, $E(X), A(X)$ and $\mathrm{Aut}(X)$ its vertex set, edge set, arc set and automorphism group, respectively. For u, $v \in V(X)$, denote by uv the edge incident to u and v in X, and by $N_X(u)$ the *neighborhood* of u in X, that is, the set of vertices adjacent to u in X. A graph is *vertex-transitive, edge-transitive* and *arc-transitive* if its automorphism group acts transitively on the vertices, edges and arcs, respectively. An arc-transitive graph is called *symmetric.*

Let N be a subgroup of $Aut(X)$. The *quotient graph* X/N or X_N of X relative to N is defined as the graph such that the set Σ of N-orbits in $V(X)$ is the vertex set of X/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$.

A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$, if p is a surjection from $V(\widetilde{X})$ to $V(X)$ such that $p\mid_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The *fibre* of an edge or a vertex is its preimage under p. If $\widetilde{X}$ is connected, then any two vertex or edge fibres are of the same cardinality n. This number is called the *fold number* of the covering, and we say that p is an n-fold covering. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h.

Covering techniques have long been known as a powerful tool in topology and graph theory. The study of semisymmetric graphs was initiated by Folkman [6]. Semisymmetric graphs of order $2pq$ and semisymmetric cubic graphs of orders $2p^3$ and $6p^2$ are classified in [5, 8, 7], and also in [1, 2] it is proved that every edge-transitive cubic graph of orders $4p^2$ or $8p^2$ is vertex-transitive. In [4], it is given an overview of known families of semisymmetric cubic graphs.
In this paper, we investigate semisymmetric cubic graphs of order $4p^n$, where $n \geq 1$ and $p(\neq 5)$ is a prime. The following is the main result of this paper.

Theorem 1.1. Let $p(\neq 5)$ be a prime. Then there is no semisymmetric cubic graph of order $4p^n$, where $n \geq 1$.

2. Primary Analysis

The following proposition is a special case of [7, Lemma 3.2].

Proposition 2.1. *Let X be a connected semisymmetric cubic graph with bipartition sets $U(X)$ and $W(X)$. Moreover, suppose that N is a normal subgroup of $A :=Aut(X)$. If N is intransitive on bipartition sets, then N acts semiregularly on both $U(X)$ and $W(X)$, and X is an N-regular covering of a A/N-semisymmetric graph.*

We quote the following propositions.

Proposition 2.2. [8, Proposition 2.4] *The vertex stabilizers of a connected G-edge-transitive cubic graph X have order $2^r \cdot 3$, $r \geq 0$. Moreover, if u and v are two adjacent vertices, then $|G : \langle G_u, G_v\rangle| \leq 2$, and the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.*

Proposition 2.3. [11] *Every both edge-transitive and vertex-transitive cubic graph is symmetric.*

Proposition 2.4. [3] *If $\tilde{X}$ is a bipartite covering of a non-bipartite graph X; then the fold number is even.*

3. Proof of Theorem 1.1

We intend to study (non)existence of semisymmetric cubic graphs of order $4p^n$, where p is a prime and n is a positive integer. First, we investigate semisymmetric cubic graphs of order $4p$, where p ia an odd prime.

Lemma 3.1. *Suppose that X is a semisymmetric cubic graph of order $4p$, where $p > 7$ is an odd prime. Set $A :=Aut(X)$, moreover suppose that $Q := O_p(A)$ be the maximal normal p-subgroup of A. Then $|Q| = p$.*

Proof. Let X be a cubic graph satisfying the above assumptions. Then X is a bipartite graph. Denote the bipartition sets of X by $U(X)$ and $W(X)$, where $|U(X)| = |W(X)| = 2p$. By Proposition 2.2, we have $|A| = 2^{r+1}3p$ $(r \geq 0)$. Let $Q = O_p(A)$ be the maximal normal p-subgroup of A. Then we will show that $|Q| = p$.
Now suppose that $|Q| = 1$. Let N be a minimal normal subgroup of A. Then we deduce that N is elementary abelian. Because if not, then by the classification of $\{2, 3, p\}$-simple groups, $N \cong A_5$ or $PSL(2, 7)$. But on the other hand, by Proposition 2.1, N is semiregular on $U(X)$ (also on $W(X)$). However this is impossible because $2p \neq 60k$ or $168k$, where k is the number of N-orbits on $U(X)$. Thus, N is elementary abelian. By Proposition 2.1, $|N| = 2$ or p. But, since $|Q| = 1$, we have only $|N| = 2$. Now we consider the quotient graph $X_N = X/N$ of X relative to N, where A/N is semisymmetric on X_N. Suppose that M/N be a minimal subgroup of A/N. Then similarly as previous, M/N is elementary abelian and we must have $|M/N| = p$. Therefore, M is a normal subgroup of A of order $2p$. Let $P \in Syl_p(M)$. Then we can easily see that P is normal in M and also characteristic in M. Then, A has a normal subgroup of order p, a contradiction to $|Q| = 1$. Therefore $|Q| = p$, as claimed. ■

We point out that by [4] there is no semisymmetric cubic graph of order $4p$, where $p \leq 7$ is a prime. On the other hand, for $p > 7$, by Lemma 3.1 and Proposition 2.1, every semisymmetric cubic graph of order $4p$ is a Q-regular covering of A/Q-semisymmetric graph X_Q, where X_Q is edge-transitive cubic graph of order 4. By [2], the quotient graph X_Q must be vertex-transitive. By Proposition 2.3, X_Q is symmetric cubic graph of order 4. Then X_Q is the complete graph K_4. Since X is bipartite and K_4 is non-bipartite, the fold number p must be even, which is a contradiction to Proposition 2.4. So, we have the following corollary.
Corollary 3.2. *Let p be a prime. Then, there is no connected semisymmetric cubic graph of order $4p$.*

Remark. One of the results of Theorem 5.2 [10] is that for a given prime $p \geq 7$ and a positive integer d, there exists at most one semisymmetric graph of order 4p and valency d. More precisely, such a graph exists (and is unique) if and only if d is an even number dividing $p - 1$. So, there is no semisymmetric cubic graph of order $4p$. One might derive the following corollary from [10]. But, we have given its (simpler) proof in Corollary 3.2.

In [2], it is proved that there is no connected semisymmetric cubic graph of order $4p^2$, where p be a prime. From now on, we assume that $n \geq 3$. Now, let X be a semisymmetric cubic graph of order $4p^n$, where $p = 2$. It is obvious that $Aut(X)$ is a solvable edge transitive group. By Corollary 4.5 [9], X is either a $\mathbb{Z}_2^{n+1}$-covering of the 3-dipole Dip_3 or a $\mathbb{Z}_2^n$-covering of the complete graph K_4. Then, $n \leq 1$ or $n \leq 3$ respectively and in both cases, we can get a contradiction. Therefore, we can assume that $p \neq 5$ is an odd prime and $n \geq 3$.

Lemma 3.3. *Suppose that X is a semisymmetric cubic graph of order $4p^n$, where $n \geq 3$ and $p \neq 5$ is an odd prime. Set $A :=Aut(X)$, moreover suppose that $Q := O_p(A)$ be the maximal normal p-subgroup of A. Then $|Q| = p^n$.*

Proof. Let X be a semisymmetric cubic graph of order $4p^n$ and set $A :=\mathrm{Aut}(X)$. Then X is bipartite graph. Denote by $U(X)$ and $W(X)$ the bipartition sets of X, where $|U(X)| = |W(X)| = 2p^n$. By Proposition 2.2, $|A| = 2^r 3p^n$, where $r \geq 1$. We claim that A is solvable. Otherwise, by the classification of finite simple groups its composition factors would have to be a A_5 or $PSL(2, 7)$, a contradiction to order of A. Let $Q := O_p(A)$ be the maximal normal p-subgroup of A. We will show that $|Q| = p^n$.

First suppose that $|Q| = 1$. Let N be a minimal normal subgroup of A. By solvability of A, N is solvable. It is easy to see that N is not transitive on the bipartition sets $U(X)$ and $W(X)$, and hence by Proposition 2.1, N acts semiregularly on $U(X)$ (also on $W(X)$). Therefore, $|N| = 2, p, \ldots, p^{n-1}$ or p^n. Because $|Q| = 1$, N must be isomorphic to $\mathbb{Z}_2$. Now we consider the quotient graph $X_N = X/N$ of X relative to N, where X_N is A/N-semisymmetric. We have $|U(X_N)| = |W(X_N)| = p^n$. Now let M/N is a minimal normal subgroup of A/N. As before, M/N is elementary abelian. If M/N is transitive on $U(X_N)$, then $|M/N| = p^n$. Therefore, M is a normal subgroup of A of order $2p^n$. Let $P \in \mathrm{Syl}_p(M)$. It is easy to see that P is a normal and hence characteristic subgroup in M. Thus A has a normal subgroup of order p^n, a contradiction to our assumption. On the other side, If M/N is intransitive on $U(X_N)$ (also on $W(X_N)$), then $|M/N| = p, \ldots, p^{n-2}$ or p^{n-1}. As same argument, A has a normal subgroup of order $p, p^2, \ldots, p^{n-2}$ or p^{n-1},which is a contradiction. Thus $|Q| \neq 1$.

Now suppose that $|Q| = p^s$ $(1 \leq s \leq n-2)$. It is obvious that Q is intransitive on bipartition sets $U(X)$ and $W(X)$ and hence by Proposition 2.1, it is semiregular on $U(X)$ (also on $W(X)$). Let X_Q be the quotient graph of X relative to Q, where X_Q is A/Q-semisymmetric. We have $|U(X_Q)| = |W(X_Q)| = 2p^{n-s}$. Suppose N/Q be a minimal normal subgroup of A/N. Since N/Q is elementary abelian, $2p^{n-s} \nmid |N/Q|$. Hence by Proposition 2.1, N/Q is semiregular on bipartition sets $U(X_Q)$ and $W(X_Q)$ and hence $|N/Q| = 2, p, \ldots, p^{n-s}$. Because $|Q| = p^s$, we must have $|N/Q| = 2$. Now suppose that the quotient graph X_N with $|U(X_N)| = |W(X_N)| = p^{n-s}$, where X_N is A/N-semisymmetric. Let M/N be a minimal normal subgroup of A/N. Now similarly as previous, we must have $|M/N| = p, \ldots, p^{n-s-1}$ or p^{n-s} and hence M is a normal subgroup of A of order $2p^{s+1}, \ldots, 2p^{n-1}$ or $2p^n$. Suppose that $P \in \mathrm{Syl}_p(M)$. Then it is easy to see that P is normal and hence characteristic in M. Therefore, A has a normal subgroup of order $p^{s+1}, \ldots, p^{n-1}$ or p^n. Now we can get a contradiction. Finally, suppose that $|Q| = p^{n-1}$. Let X_Q be the quotient graph of X relative to Q. We have $|X_Q| = 4p$. For $p = 7$, there is no any semisymmetric or symmetric cubic graph of order 28 and for the remained cases, with the similar argument as before, one can get a contradiction and the result now follows. ■

Proof of Theorem 1.1 By [2, Theorem 1.1], Corollary 3.2 and Lemma 3.3, the proof of main theorem is complete. ■

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