DIFFERENTIAL SUBORDINATION FOR NEW GENERALISED DERIVATIVE OPERATOR

Ma'moun Harayzeh Al-Abbadi and Maslina Darus

Abstract. In this work, a new generalised derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$ is introduced. This operator generalised many well-known operators studied earlier by many authors. Using the technique of differential subordination, we shall study some of the properties of differential subordination. In addition we investigate several interesting properties of the new generalised derivative operator.

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \quad \text{is complex number} \tag{1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let S, $S^*(\alpha)$, $C(\alpha)\,(0 \le \alpha < 1)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are univalent, starlike of order α and convex of order α in U, respectively. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex functions in U, respectively. And a function $f \in C(\alpha)$ if $\mathrm{Re}(1 + \frac{zf''}{f'}) > \alpha$. Furthermore a function f analytic in U is said to be convex if it is univalent and $f(U)$ is convex. Suppose $\mathcal{H}(U)$ be the class of holomorphic function in unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots , \quad (z \in U)\},$$

with $\mathcal{A}_1 = \mathcal{A}$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = z + a_n z^n + a_{n+1}z^{n+1} + \dots , \quad (z \in U)\}.$$

Let be given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Then the Hadamard product (or convolution) $f * g$ of two functions f, g is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \ .$$

Next, we state basic ideas on subordination. If f and g are analytic in U, then the function f is said to be subordinate to g, and can be written as

$$f \prec g \quad \text{and} \quad f(z) \prec g(z), \quad (z \in U),$$

if and only if there exists the Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $(z \in U)$.

Furthermore if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$, [see [10], p.36].

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad (z \in U). \tag{2}$$

Then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (2) is said to be the best dominant of (2). (Note that the best dominant is unique up to a rotation of U).

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0,\ x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)(x+2)...(x+k-1) & \text{for } k \in \mathbb{N} = \{1,2,3,...\} \text{and } x \in \mathbb{C}. \end{cases}$$

In order to derive our new generalised derivative operator, we define the analytic function

$$\phi^m_{\lambda_1,\lambda_2}(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\ z^k, \tag{3}$$

where $m \in \mathbb{N}_0 = \{0,1,2,\dots\}$ and $\lambda_2 \geq \lambda_1 \geq 0$. Now, we introduce the new generalised derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ as the following:

Definition 1. *For $f \in \mathcal{A}$ the operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ is defined by $\mu^{n,m}_{\lambda_1,\lambda_2} : \mathcal{A} \to \mathcal{A}$*

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = \phi^m_{\lambda_1,\lambda_2}(z) * R^n f(z), \quad (z \in U), \tag{4}$$

where $n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$ and $R^n f(z)$ denotes the Ruscheweyh derivative operator [12], and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k, \quad (n \in \mathbb{N}_0, z \in U),$$

where $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

If f is given by (1), then we easily find from the equality (4) that

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\ c(n,k) a_k z^k, \quad (z \in U),$$

where $n, m \in \mathbb{N}_0 = \{0,1,2...\}$, $\lambda_2 \geq \lambda_1 \geq 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes the Ruscheweyh derivative operator in the cases $\mu^{n,1}_{\lambda_1,0} \equiv \mu^{n,m}_{0,0} \equiv \mu^{n,0}_{0,\lambda_2} \equiv R^n$ [12], the Salagean derivative operator $\mu^{0,m+1}_{1,0} \equiv S^n$ [13], the generalised Ruscheweyh derivative operator $\mu^{n,2}_{\lambda_1,0} \equiv R^n_\lambda$ [2], the generalised Salagean derivative operator introduced by Al-Oboudi $\mu^{0,m+1}_{\lambda_1,0} \equiv S^n_\beta$ [1], and the generalised Al-Shaqsi and Darus derivative operator $\mu^{n,m+1}_{\lambda_1,0} \equiv D^n_{\lambda,\beta}$ [3]. Now, let we remind the well known Carlson-Shaffer operator $L(a,c)$ [4] associated with the incomplete beta function $\phi(a,c;z)$, defined by

$$\begin{aligned} L(a,c) \quad &: \quad \mathcal{A} \to \mathcal{A}, \\ L(a,c)f(z) \quad &:= \quad \phi(a,c;z) * f(z), \quad (z \in U), \end{aligned}$$

where

$$\phi(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k,$$

a is any real number and $c \notin z_0^-$; $z_0^- = \{0,-1,-2,...\}$.

It is easily seen that

$$\begin{aligned} \mu_{\lambda_1,0}^{0,1} f(z) \quad &= \quad \mu_{0,0}^{0,m} f(z) = \mu_{0,\lambda_2}^{0,0} f(z) = L(a,a)f(z) = f(z), \\ \mu_{\lambda_1,0}^{1,1} f(z) \quad &= \quad \mu_{0,0}^{1,m} f(z) = \mu_{0,\lambda_2}^{1,0} f(z) = L(2,1)f(z) = zf'(z), \end{aligned}$$

and also

$$\mu_{\lambda_1,0}^{a-1,1} f(z) = \mu_{0,\lambda_2}^{a-1,0} f(z) = \mu_{0,0}^{a-1,m} f(z) = L(a,1)f(z),$$

where $a = 1,2,3,...$.

For $n,m \in \mathbb{N}_0 = \{0,1,2...\}$, and $\lambda_2 \geq \lambda_1 \geq 0$, we need the following equality to prove our results.

$$\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z) = (1-\lambda_1)\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_1,\lambda_2}^{1}(z)\right] + \lambda_1 z \left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_1,\lambda_2}^{1}(z)\right]'. \quad (5)$$

Where $(z \in U)$ and $\phi_{\lambda_1,\lambda_2}^{1}(z)$ analytic function and from (3) given by

$$\phi_{\lambda_1,\lambda_2}^{1}(z) = z + \sum_{k=2}^{\infty} \frac{z^k}{1+\lambda_2(k-1)}.$$

Next, we give another definition as follows:

Definition 2. *For $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, we let $R_{\lambda_1,\lambda_2}^{n,m}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition*

$$\mathrm{Re}\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' > \alpha, \quad (z \in U). \quad (6)$$

Also let $K_{\lambda_1,\lambda_2}^{n,m}(\delta)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\mathrm{Re}\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_1,\lambda_2}^{1}(z)\right)' > \delta, \quad (z \in U).$$

It is clear that the classes $R^{0,1}_{\lambda_1,0}(\alpha) \equiv R(\lambda_1, \alpha)$ *the class of function* $f \in \mathcal{A}$ *satisfying*

$$\mathrm{Re}(\lambda_1 z f''(z) + f'(z)) > \alpha, \quad (z \in U).$$

Studied by Ponnusamy [11] and others.

In the present paper, we shall use the method of differential subordination to derive certain properties of generalisation derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2} f(z)$. Not that differential subordination has been studied by various authors, and here we recognised this work done by Oros [8] and Oros and Oros [9].

2. Main Results

In proving our main results, we need the following Lemmas.

Lemma 1 ([5],p.71]). *Let h be analytic, univalent, convex in U, with $h(0) = a$, $\gamma \neq 0$ and Re $\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \qquad (z \in U),$$

then

$$p(z) \prec q(z) \prec h(z), \qquad (z \in U),$$

where

$$q(z) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t)\, t^{\left(\frac{\gamma}{n}\right)-1} dt, \qquad (z \in U).$$

The function q is convex and is the best (a, n)-dominant.

Lemma 2 ([6]). *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z),$$

where $\alpha > 0$ and n is a positive integer.
If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots ., \quad (z \in U),$$

is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z), \qquad (z \in U),$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Lemma 3 ([7]). *Let $f \in \mathcal{A}$,*
if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2},$$

then

$$\frac{2}{z}\int_0^z f(t)dt, \qquad (z \in U \text{ and } z \neq 0),$$

belongs to the convex functions.

Now we begin with the first result as the following:

Theorem 1. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)\,z}{1 + z}, \qquad (z \in U),$$

be convex in U, with $h(0)=1$ and $0 \le \alpha < 1$. If $n, m \in \mathbb{N}_0$, $\lambda_2 \ge \lambda_1 \ge 0$, and the differential subordination.

$$(\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z))' \prec h(z), \qquad (z \in U), \tag{7}$$

then

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_1,\lambda_2}^{1}(z)\right)' \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{\frac{1}{\lambda_1}}}\sigma(\frac{1}{\lambda_1}).$$

Where σ is given by

$$\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t}dt, \quad (z \in U). \tag{8}$$

The function q is convex and is the best dominant.

Proof. By differentiating (5), with respect to z, we obtain

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z)\right)' = \left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_1,\lambda_2}^{1}(z)\right]' + \lambda_1 z\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_1,\lambda_2}^{1}(z)\right]''. \tag{9}$$

Using (9) in (7), differential subordination (7) becomes

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^1_{\lambda_1,\lambda_2}(z)\right]' + \lambda_1 z \left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^1_{\lambda_1,\lambda_2}(z)\right]'' \prec h(z) = \frac{1+(2\alpha-1)\,z}{1+z}. \tag{10}$$

Let

$$p(z) = \left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^1_{\lambda_1,\lambda_2}(z)\right]' = \left[z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^{m+1}}\, c(n,k) a_k z^k\right]' \tag{11}$$
$$= 1 + p_1 z + p_2 z^2 + \ldots, \qquad (p \in \mathcal{H}[1,1],\ z \in U).$$

Using (11) in (10), the differential subordination becomes:

$$p(z) + \lambda_1 z p'(z) \prec h(z) = \frac{1+(2\alpha-1)\,z}{1+z}.$$

By using Lemma 1, we have

$$\begin{aligned} p(z) \prec q(z) &= \frac{1}{\lambda_1 z^{\frac{1}{\lambda_1}}} \int_0^z h(t)\, t^{\left(\frac{1}{\lambda_1}\right)-1} dt, \\ &= \frac{1}{\lambda_1 z^{\frac{1}{\lambda_1}}} \int_0^z \left(\frac{1+(2\alpha-1)\,t}{1+t}\right) t^{\left(\frac{1}{\lambda_1}\right)-1} dt, \\ &= 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{\frac{1}{\lambda_1}}} \sigma(\frac{1}{\lambda_1}). \end{aligned}$$

Where σ is given by (8), so we get

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^1_{\lambda_1,\lambda_2}(z)\right]' \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{\frac{1}{\lambda_1}}} \sigma(\frac{1}{\lambda_1}).$$

The functions q is convex and is the best dominant. The proof is complete. $\square$

Theorem 2. *If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, then we have*

$$R^{n,m+1}_{\lambda_1,\lambda_2}(\alpha) \subset K^{n,m}_{\lambda_1,\lambda_2}(\delta),$$

where

$$\delta = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1} \sigma(\frac{1}{\lambda_1}),$$

and σ given by (8).

Proof. Let $f \in R^{n,m+1}_{\lambda_1,\lambda_2}(\alpha)$, then from (6) we have

$$\mathrm{Re}(\mu^{n,m+1}_{\lambda_1,\lambda_2} f(z))' > \alpha, \qquad (z \in U).$$

Which is equivalent to

$$(\mu^{n,m+1}_{\lambda_1,\lambda_2} f(z))' \prec h(z) = \frac{1+(2\alpha-1)\,z}{1+z}.$$

Using Theorem 1, we have

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^{1}_{\lambda_1,\lambda_2}(z)\right]' \prec \; q(z) = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{\frac{1}{\lambda_1}}}\sigma(\frac{1}{\lambda_1}).$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned}\mathrm{Re}\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^{1}_{\lambda_1,\lambda_2}(z)\right]' > \mathrm{Re}\; q(1) &= \delta = \delta(\alpha, \lambda_1)\\ &= 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1}\sigma(\frac{1}{\lambda_1}).\end{aligned}$$

For which we deduce $R^{n,m+1}_{\lambda_1,\lambda_2}(\alpha) \subset K^{n,m}_{\lambda_1,\lambda_2}(\delta)$. This completes the proof of Theorem 2.

Remark 1. Special case of Theorem 2 with $\lambda_2 = 0$ was given earlier in [3].

Theorem 3. *Let q be a convex function in U, with $q(0) = 1$ and let*

$$h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

If $n, m \in \mathbb{N}_0,\ \lambda_2 \geq \lambda_1 \geq 0,\ f \in \mathcal{A}$ and satisfies the differential subordination

$$(\mu^{n,m+1}_{\lambda_1,\lambda_2} f(z))' \prec h(z), \quad (z \in U), \tag{12}$$

then

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^{1}_{\lambda_1,\lambda_2}(z)\right]' \prec \; q(z), \qquad (z \in U),$$

and this result is sharp.

Proof. Using (11) in (9), differential subordination (12) becomes

$$p(z) + \lambda_1 z p'(z) \prec h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

Using Lemma 2, we obtain

$$p(z) \prec q(z), \qquad (z \in U).$$

Hence

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi^1_{\lambda_1,\lambda_2}(z)\right]' \prec \; q(z), \quad (z \in U).$$

And the result is sharp. This completes the proof of the theorem. □

We give a simple application for Theorem 3.

Example 1. For $n = 0$, $m = 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $q(z) = \frac{1+z}{1-z}$, $f \in \mathcal{A}$ and $z \in U$ and applying Theorem 3, we have

$$h(z) = \frac{1+z}{1-z} + \lambda_1 z \left(\frac{1+z}{1-z}\right)' = \frac{1+2\lambda_1 z - z^2}{(1-z)^2}.$$

After that we find

$$\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi^1_{\lambda_1,\lambda_2}(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{(1+\lambda_2(k-1))^2} z^k.$$

Now,

$$\begin{aligned}
\left(\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi^1_{\lambda_1,\lambda_2}(z)\right)' &= 1 + \sum_{k=2}^{\infty} \frac{k a_k}{(1+\lambda_2(k-1))^2} z^{k-1}, \\
&= f'(z) * \left[\frac{\left(\phi^1_{\lambda_1,\lambda_2} * \phi^1_{\lambda_1,\lambda_2}\right)(z)}{z}\right].
\end{aligned}$$

And similarly we find

$$\mu^{0,2}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))a_k}{(1+\lambda_2(k-1))^2} z^k,$$

then

$$\begin{aligned}
\left(\mu^{0,2}_{\lambda_1,\lambda_2} f(z)\right)' &= 1 + \sum_{k=2}^{\infty} \frac{k(1+\lambda_1(k-1))a_k}{(1+\lambda_2(k-1))^2} z^{k-1}, \\
&= f'(z) * \left[\frac{\phi^2_{\lambda_1,\lambda_2}(z)}{z}\right].
\end{aligned}$$

From Theorem 3 we deduce

$$f'(z) * \left[\frac{\phi^2_{\lambda_1,\lambda_2}(z)}{z} \right] \prec \frac{1 + 2\lambda_1 z - z^2}{(1-z)^2},$$

implies

$$f'(z) * \left[\frac{\left(\phi^1_{\lambda_1,\lambda_2} * \phi^1_{\lambda_1,\lambda_2}\right)(z)}{z} \right] \prec \frac{1+z}{1-z}, \quad (z \in U).$$

Theorem 4. *Let q be a convex function in U, with $q(0) = 1$ and let*

$$h(z) = q(z) + zq'(z), \quad (z \in U).$$

If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z), \tag{13}$$

then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z), \qquad (z \in U).$$

And the result is sharp.

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \qquad (14) \\ &= \frac{z + \sum\limits_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k) a_k z^k}{z}, \\ &= 1 + p_1 z + p_2 z^2 + ..., \qquad (p \in \mathcal{H}[1,1], \ z \in U). \end{aligned}$$

Differentiating (14), with respect to z, we obtain

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' = p(z) + zp'(z), \quad (z \in U). \tag{15}$$

Using (15), (13), becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + zq'(z),$$

using Lemma 2, we deduce

$$p(z) \prec q(z), \qquad (z \in U),$$

and using (14), we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z), \qquad (z \in U).$$

This proves Theorem 4. □

We give a simple application for Theorem 4.

Example 2. For $n = 0, m = 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $q(z) = \frac{1}{1-z}$, $f \in \mathcal{A}$ and $z \in U$ and applying Theorem 4, we have

$$h(z) = \frac{1}{1-z} + z\left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}.$$

After that we find

$$\begin{aligned} \mu^{0,1}_{\lambda_1,\lambda_2} f(z) &= z + \sum_{k=2}^{\infty} \frac{a_k}{(1+\lambda_2(k-1))} z^k, \\ &= f(z) * \phi^1_{\lambda_1,\lambda_2}(z). \end{aligned}$$

Now

$$\begin{aligned} (\mu^{0,1}_{\lambda_1,\lambda_2} f(z))' &= 1 + \sum_{k=2}^{\infty} \frac{k a_k}{(1+\lambda_2(k-1))} z^{k-1}, \\ &= f'(z) * \frac{\phi^1_{\lambda_1,\lambda_2}(z)}{z}. \end{aligned}$$

From Theorem 4 we deduce

$$f'(z) * \frac{\phi^1_{\lambda_1,\lambda_2}(z)}{z} \prec \frac{1}{(1-z)^2},$$

implies

$$\frac{f(z) * \phi^1_{\lambda_1,\lambda_2}(z)}{z} \prec \frac{1}{1-z}.$$

Theorem 5. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)\, z}{1 + z}, \quad (z \in U).$$

Be convex in U*, with* $h(0) = 1$ *and* $0 \leq \alpha < 1$*. If* $n, m \in \mathbb{N}_0$*,* $\lambda_2 \geq \lambda_1 \geq 0$*,* $f \in \mathcal{A}$ *and the differential subordination*

$$(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z), \tag{16}$$

then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec \quad q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}.$$

The function q *is convex and is the best dominant.*

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \qquad (17) \\ &= \frac{z + \sum\limits_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k) a_k z^k}{z}, \\ &= 1 + p_1 z + p_2 z^2 + ..., \qquad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned}$$

Differentiating (17), with respect to z, we obtain

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' = p(z) + zp'(z), \quad (z \in U). \tag{18}$$

Using (18), the differential subordination (16) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1 + (2\alpha - 1)\, z}{1 + z}, \quad (z \in U).$$

From Lemma 1, we deduce

$$\begin{aligned}
p(z) \prec q(z) &= \frac{1}{z}\int_0^z h(t)\,dt, \\
&= \frac{1}{z}\int_0^z \left(\frac{1+(2\alpha-1)\,t}{1+t}\right)\,dt, \\
&= \frac{1}{z}\left[\int_0^z \frac{1}{1+t}dt + (2\alpha-1)\int_0^z \frac{t}{1+t}dt\right], \\
&= 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}.
\end{aligned}$$

Using (17), we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec \quad q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}.$$

The proof is complete. $\square$

From Theorem 5, we deduce the following Corollary:

Corollary. *If* $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, *then*

$$\mathrm{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > (2\alpha-1) + 2(1-\alpha)\ln 2, \qquad (z \in U).$$

Proof. Since $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, from Definition 2

$$\mathrm{Re}\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > \alpha, \quad (z \in U),$$

which is equivalent to

$$(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z) = \frac{1+(2\alpha-1)\,z}{1+z}.$$

Using Theorem 5, we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = (2\alpha-1) + 2(1-\alpha)\frac{\ln(1+z)}{z}.$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > \operatorname{Re}\, q(1) = (2\alpha-1) + 2(1-\alpha)\ln 2, \qquad (z \in U).$$

Theorem 6. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$ which satisfy the inequality*

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad (z \in U).$$

If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' \prec h(z), \quad (z \in U), \tag{19}$$

then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \\ &= \frac{z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k) a_k z^k}{z}, \\ &= 1 + p_1 z + p_2 z^2 + \ldots., \qquad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{20}$$

Differentiating (20), with respect to z, we have

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' = p(z) + zp'(z), \quad (z \in U). \tag{21}$$

Using (21), the differential subordination (19) becomes

$$p(z) + zp'(z) \prec h(z), \quad (z \in U).$$

From Lemma 1, we deduce

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)\, dt,$$

with (20), we obtain

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

From Lemma 3, we have that the function q is convex, and from Lemma 1, q is the best dominant for subordination (19). This completes the proof of Theorem 6. □

3. Conclusion

We remark that several subclasses of analytic univalent functions can be derived using the operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ and studied their properties.

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Ma'moun Harayzeh Al-Abbadi and Maslina Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia.
Email:
mamoun_nn@yahoo.com
maslina@ukm.my