A CERTAIN CLASS OF QUADRATURES WITH HAT FUNCTION AS A WEIGHT FUNCTION

Zlatko Udovičić

Abstract. We consider the quadrature rules of "practical type" (with five knots) for approximately computation of the integral

$$\int_{-1}^{1} (1 - |x|)\, f(x) dx.$$

We proved that maximal algebraic degree of exactness for this type of formulas is equal to five. At the end we gave numerical result.

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1. Introduction

Central role in the process of construction of the continuous, piecewise linear approximation of the given function $f(\cdot)$ plays the so called hat function. Hat function is defined in the following way:

$$h(x) = \begin{cases} 1 - |x|, & x \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Thereat, the problem of calculation of the integral

$$\int_{-1}^{1} h(x) f(x) dx.$$

is unavoidable. In this paper we are investigating a certain class of quadratures (the so called quadratures of "practical type") for approximate computation of the previous integral. The paper was motivated by results recently published in [1] and [2], where the same class of quadratures was considered, but without weight function.

We say that quadrature formula

$$\int_{-1}^{1} h(x)f(x)dx = \sum_{i=1}^{5} A_i f(x_i) + R[f] \tag{1}$$

is of "practical type" if the following conditions hold:

1. coefficients $A_k, 1 \leq k \leq 5$ are symmetric, i.e. $A_1 = A_5$ and $A_2 = A_4$.
2. nodes $x_k, 1 \leq k \leq 5$ are symmetric and rational numbers from the interval $[-1, 1]$, i.e. $x_1 = -r_1, x_2 = -r_2, x_3 = 0, x_4 = r_2$ and $x_5 = r_1$, for some $r_1, r_2 \in (0, 1] \cap \mathbb{Q}, r_2 < r_1$ (as usuall, $\mathbb{Q}$ denotes the set of rational numbers).

Hence, quadratures of "practical type" have the following form:

$$\begin{aligned} \int_{-1}^{1} h(x)f(x)dx &= A(f(-r_1) + f(r_1)) + \\ & \quad B(f(-r_2) + f(r_2)) + Cf(0) + R[f], \end{aligned} \tag{2}$$

for some $r_1, r_2 \in (0, 1] \cap \mathbb{Q}, r_2 < r_1$.

Quadrature rule (1) has algebraic degree of exactness equal to $m, m \in \mathbb{N}$, if and only if $R[p] = 0$ whenever $p(\cdot)$ is a polynomial of degree not greater than m and there exists the polynomial $q(\cdot)$, of degree $m + 1$, such that $R[q] \neq 0$. Our aim is construction of the quadrature rules of "practical type" with maximal algebraic degree of exactness.

We will finish this section with some well known facts from the theory of numerical integration.

Lemma 1 *Quadrature rule* (1) *(i.e.* (2)*) has algebraic degree of exactness equal to* $m, m \in \mathbb{N}$ *if and only if* $R[x^k] = 0$ *for all* $k \in \{0, 1, \ldots, m - 1\}$.

Lemma 2 *Quadrature rule* (2) *is exact for every odd function* $f(\cdot)$ *(i.e.* $R[f] = 0$ *for every odd function* $f(\cdot)$*).*

From the previous lemmas follows that algebraic degree of exactness of the formula (2) have to be odd.

Finally, with the choice

$$x_{1245} = \pm\sqrt{\frac{350 \pm \sqrt{35518}}{798}}, x_3 = 0$$

(expressions for the coefficients $A_k, 1 \leq k \leq 5$, are much more complicated, so we omit them here) formula (1) attains maximal algebraic degree of exactness (which is equal to nine), but this formula obviously is not of "practical type". Hence, algebraic degree of exactness of the formula (2) can not be greater than seven.

2. Main result

Let us determine the coefficients A, B and C such that formula (2) has maximal algebraic degree of exactness.

From the condition that formula is exact for $f(x) = 1$ (i.e. exact for any polynomial of zero degree) simply follows

$$C = 1 - 2(A + B).$$

Then formula (2) becomes

$$\begin{aligned} \int_{-1}^{1} h(x)f(x)dx &= A\left(f(-r_1) - 2f(0) + f(r_1)\right) + \\ & B\left(f(-r_2) - 2f(0) + f(r_2)\right) + \\ & f(0) + R[f] \end{aligned} \tag{3}$$

and in accordance with the previous, this formula has algebraic degree of exactness equal to one. Furthermore, conditions that the last formula is exact for $f(x) = x^2$ and $f(x) = x^4$ give the following system of linear equations

$$\begin{aligned} 2r_1^2 A + 2r_2^2 B &= \frac{1}{6} \\ 2r_1^4 A + 2r_2^4 B &= \frac{1}{15}. \end{aligned}$$

which has the solution

$$A = \frac{2 - 5r_2^2}{60r_1^2(r_1^2 - r_2^2)} \text{ and } B = \frac{2 - 5r_1^2}{60r_2^2(r_2^2 - r_1^2)}. \tag{4}$$

Hence, with this choice of the coefficients A and B formula (3) has algebraic degree of exactness equal to five. Therein (in formula (3))

$$R[x^6] = \frac{1}{28} - \frac{2r_1^2 - 5r_1^2 r_2^2 + 2r_2^2}{30}.$$

It is natural to ask is it possible to choose rational nodes r_1 and r_2 such that formula (3) has algebraic degree of exactness equal to six, i.e. seven. Negative answer to this question gives the following lemma.

Lemma 3 *There is no numbers $r_1, r_2 \in (0, 1] \cap \mathbb{Q}$ such that*

$$\frac{2r_1^2 - 5r_1^2 r_2^2 + 2r_2^2}{30} = \frac{1}{28}. \tag{5}$$

Proof: Let us assume contrary, i.e. that $r_1 = \frac{a}{b}$ and $r_2 = \frac{c}{d}$, for some $a, b, c, d \in \mathbb{N}$ such that $(a, b) = 1$ and $(c, d) = 1$. Putting this in equality (5), after simplification, gives

$$14(2a^2d^2 - 5a^2c^2 + 2b^2c^2) = 15b^2d^2, \tag{6}$$

from which follows that $b^2d^2 \equiv 0(mod\, 14)$, i.e. $bd \equiv 0(mod\, 14)$. There are four cases which can occur.

The case $b \equiv 0(mod\, 14)$. Putting $b = 14k$ for some $k \in \mathbb{N}$, after simplification, equality (6) becomes

$$a^2(2d^2 - 5c^2) = 14k^2(15d^2 - 28c^2).$$

Since $(a, b) = 1$, it has to be $2d^2 - 5c^2 \equiv 0(mod\, 14)$, so it also has to be $2d^2 - 5c^2 \equiv 0(mod\, 7)$. Furthermore, because of $2d^2 - 5c^2 = 7d^2 - 5(c^2 + d^2)$, it also has to be $c^2 + d^2 \equiv 0(mod\, 7)$, and direct checking verify that the last relation is impossible unless $c \equiv 0(mod\, 7)$ and $d \equiv 0(mod\, 7)$, which together with an assumption $(c, d) = 1$. gives the contradiction.

The case $b \equiv 0(mod\, 7) \wedge d \equiv 0(mod\, 2)$. Putting $b = 7k_1$ and $d = 2k_2$ for some $k_1, k_2 \in \mathbb{N}$, after simplification, equality (6) becomes

$$5a^2c^2 = 2(4a^2k_2^2 - 105k_1^2k_2^2 + 49c^2k_1^2),$$

and from this follows that it has to be $a \equiv 0(mod\, 2)$ (it can not be $c \equiv 0(mod\, 2)$ because of $d \equiv 0(mod\, 2)$). Furthermore, putting $a = 2l$ for some $l \in \mathbb{N}$ the last equality boecomes

$$2l^2(5c^2 - 8k_2^2) = 7(7k_1^2c^2 - 15k_1^2k_2^2).$$

It can not be $l \equiv 0(mod\, 7)$ (because of $b \equiv 0(mod\, 7), a = 2l$ and $(a, b) = 1$), so it has to be $5c^2 - 8k_2^2 \equiv 0(mod\, 7)$. Direct checking verify that the last relation is impossible unless $c \equiv 0(mod\, 7)$ and $k_2 \equiv 0(mod\, 7)$. This fact, together with $d = 2k_2$ and $(c, d) = 1$ gives the contradiction.

The cases $b \equiv 0(mod\, 2) \wedge d \equiv 0(mod\, 7)$ and $d \equiv 0(mod\, 14)$, because of symmetry of the relation (6) can be proved analogue. This completes the proof. ▶

Let us estimate the error of the formula (3), under assumption that the coefficients A and B are given by the equalities (4). Let $H_5(\cdot)$ be Hermite's interpolating polynomial which interpolates the function $f(\cdot)$ through the points $\pm r_1, \pm r_2$ and 0, where the node 0 has multiplicity two. Then (see for example [3], p. 55),

$$f(x) - H_5(x) = \frac{f^{(vi)}(\xi(x))}{6!} x^2(x^2 - r_1^2)(x^2 - r_2^2),$$

and the error of the formula (3) is given by

$$\begin{aligned} R[f] &= \int_{-1}^{1} h(x) \frac{f^{(vi)}(\xi(x))}{6!} x^2 (x^2 - r_1^2)(x^2 - r_2^2) dx \\ &= \frac{1}{6 \cdot 6!} f^{(vi)}(\eta)(\eta^2 - r_1^2)(\eta^2 - r_2^2), \end{aligned}$$

for some $\eta \in [-1, 1]$, assuming $f(\cdot) \in C^6[-1, 1]$. Let

$$\Phi(\eta) = (\eta^2 - r_1^2)(\eta^2 - r_2^2).$$

It is easy to check that

$$\begin{aligned} \max_{\eta \in [-1,1]} |\Phi(\eta)| &= \max \left\{ |\Phi(0)|, \left| \Phi(\sqrt{\frac{r_1^2 + r_2^2}{2}}) \right|, |\Phi(1)| \right\} \\ &= \max \left\{ r_1^2 r_2^2, \frac{(r_1^2 - r_2^2)^2}{4}, (1 - r_1^2)(1 - r_2^2) \right\}, \end{aligned}$$

so the error of the formula (3) can be estimated in the following way

$$|R[f]| \leq \frac{M_6}{6 \cdot 6!} \max \left\{ r_1^2 r_2^2, \frac{(r_1^2 - r_2^2)^2}{4}, (1 - r_1^2)(1 - r_2^2) \right\}, \tag{7}$$

where $M_6 = \max_{x \in [-1,1]} \left| f^{(vi)}(x) \right|$.

3. Numerical result

Estimation (7) naturally imposes the following problem

$$F(r_1, r_2) = \max \left\{ r_1^2 r_2^2, \frac{(r_1^2 - r_2^2)^2}{4}, (1 - r_1^2)(1 - r_2^2) \right\} \to \min., \tag{8}$$

where $r_1, r_2 \in (0, 1] \cap \mathbb{Q}, r_2 < r_1$. It is obvious that, for fixed $r_1 \in (0, 1] \cap \mathbb{Q}$, the function $F(\cdot, \cdot)$ attains its minimum in one of the intersection points among three curves $r_1^2 r_2^2$, $\frac{(r_1^2 - r_2^2)^2}{4}$ and $(1 - r_1^2)(1 - r_2^2)$.

1. Curves $r_1^2 r_2^2$ and $\frac{(r_1^2 - r_2^2)^2}{4}$ (r_1 is fixed) intersect each other at $r_2 = \pm(1 \pm \sqrt{2})r_1$, and since $r_2 \notin \mathbb{Q}$ we will not consider this case.
2. Similarly, curves $\frac{(r_1^2 - r_2^2)^2}{4}$ and $(1 - r_1^2)(1 - r_2^2)$ (r_1 is fixed) intersect each other at $r_2 = \pm\sqrt{3r_1^2 - 2 \pm 2\sqrt{2}(r_1^2 - 1)}$, and again because of $r_2 \notin \mathbb{Q}$ we will not consider this case.

3. Finally, curves $r_1^2 r_2^2$ and $(1-r_1^2)(1-r_2^2)$ (r_1 is still fixed) intersect each other at $r_2 = \sqrt{1-r_1^2}$, and we will look for the nodes r_1 and r_2 among "rational points" from the unit circle.

In the following table we give some admissible values of the nodes r_1 and r_2 for which the function $F(\cdot,\cdot)$ attains its local minimums. The corresponding rational numbers are round off to the six decimal places.

r_1	r_2	$F(r_1, r_2)$
$\frac{4}{5} = 0.8$	$\frac{3}{5} = 0.6$	0.230400
$\frac{21}{29} = 0.724138$	$\frac{20}{29} = 0.689655$	0.249406
$\frac{55}{73} = 0.753425$	$\frac{48}{73} = 0.657534$	0.245424
$\frac{72}{97} = 0.742268$	$\frac{65}{97} = 0.670103$	0.247403
$\frac{377}{505} = 0.746535$	$\frac{336}{505} = 0.665347$	0.246715
$\frac{987}{1325} = 0.744906$	$\frac{884}{1325} = 0.667170$	0.246988
$\frac{1292}{1733} = 0.745528$	$\frac{1155}{1733} = 0.666474$	0.246885
$\frac{6765}{9077} = 0.745290$	$\frac{6052}{9077} = 0.666740$	0.246924
$\frac{17711}{23761} = 0.745381$	$\frac{15840}{23761} = 0.666639$	0.246909
$\frac{23184}{31105} = 0.745346$	$\frac{20737}{31105} = 0.666677$	0.246915

At the end, let us say that, by using any of the given choices for the nodes r_1 and r_2, the error (7) can be estimated in the following way

$$|R[f]| \leq 0.6 \cdot 10^{-4} \cdot M_6.$$

References

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Zlatko Udovičić
Faculty of Sciences, Department of Mathematics
University of Sarajevo
Zmaja od Bosne 33-35, 71000 Sarajevo, Bosnia and Herzegovina
email: *zzlatko@pmf.unsa.ba*