ON THE FIBONACCI Q-MATRICES OF THE ORDER m

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Abstract. In this paper, the Fibonacci Q-matrices of the order m and their properties are considered. In addition to this, we introduce the Fibonacci Q-matrices of the order a negative real number m and the pure imaginary m. Further, we examine some properties of these matrices.

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1. Introduction

In the last decades the theory of Fibonacci numbers was complemented by the theory of the so-called Fibonacci Q-matrix (see [1], [2], [5]). This 2×2 square matrix is defined in [5] as follows.

$$Q=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

It is well known that, (see [7]), the n^{th} power of the Q-matrix is

$$Q^n=\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \tag{2}$$

where $n=0,\pm1,\pm2,\cdots$, and F_n is $n\ th$ Fibonacci number. Q^n matrix is expressed by the formula

$$Q^n=\begin{bmatrix} F_n+F_{n-1} & F_{n-1}+F_{n-2} \\ F_{n-1}+F_{n-2} & F_{n-2}+F_{n-3} \end{bmatrix}=\begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}+\begin{bmatrix} F_{n-1} & F_{n-2} \\ F_{n-2} & F_{n-3} \end{bmatrix}.$$

The Fibonacci Q-matrices of the order m are considered by Stakhov in [1] . These matrices are defined by Stakhov as follows, for $m\in R^+$

$$G_m=\begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

It is well known that the Fibonacci numbers of order m is defined by the following recurrence relation for $n \in Z$,

$$F_m(n+2) = mF_m(n+1) + F_m(n) \tag{4}$$

where $F_m(0) = 0$ and $F_m(1) = 1$. The entries of these matrices of order m can be represented by Fibonacci numbers of order m as follows,

$$G_m^n = \begin{bmatrix} F_m(n+1) & F_m(n) \\ F_m(n) & F_m(n-1) \end{bmatrix}, \tag{5}$$

where $m \in R^+$and $n \in Z$. If we consider the determinants of the G_m^n matrices of the order m, then we can see that

$$Det\left(G_m^n\right) = \begin{vmatrix} F_m(n+1) & F_m(n) \\ F_m(n) & F_m(n-1) \end{vmatrix} = (-1)^n. \tag{6}$$

It can be seen that the Cassini Formula for the Fibonacci numbers of the order m is

$$F_m(n+1)F_m(n-1) - \left[F_m(n)\right]^2 = (-1)^n. \tag{7}$$

Theorem 1. *For $m \in R^+$ and $n \in Z$,*

$$G_m^n = mG_m^{n-1} + G_m^{n-2}. \tag{8}$$

Proof. Since,

$$G_m^{n-1} = \begin{bmatrix} F_m(n) & F_m(n-1) \\ F_m(n-1) & F_m(n-2) \end{bmatrix} \tag{9}$$

and

$$G_m^{n-2} = \begin{bmatrix} F_m(n-1) & F_m(n-2) \\ F_m(n-2) & F_m(n-3) \end{bmatrix}, \tag{10}$$

one can write

$$mG_m^{n-1} + G_m^{n-2} = \begin{bmatrix} mF_m(n) + F_m(n-1) & mF_m(n-1) + F_m(n-2) \\ mF_m(n-1) + F_m(n-2) & mF_m(n-2) + F_m(n-3) \end{bmatrix} \tag{11}$$

$$mG_m^{n-1} + G_m^{n-2} = \begin{bmatrix} F_m(n+1) & F_m(n) \\ F_m(n) & F_m(n-1) \end{bmatrix}. \tag{12}$$

Thus, the proof is completed.

2. The Fibonacci Q-Matrices of the Order m

For $m \in R^+$, the Fibonacci Q-matrices of order $-m$ can be defined by

$$G_{-m} = \begin{bmatrix} -m & 1 \\ 1 & 0 \end{bmatrix}. \tag{13}$$

Since, the Fibonacci numbers of order $-m$ can be defined by the following recurrence relation , for $m \in R^+$, $n \in Z$

$$F_{-m}(n+2) = -mF_{-m}(n+1) + F_{-m}(n) \tag{14}$$

where $F_{-m}(0) = 0$, $F_{-m}(1) = 1$.The following theorem sets a connection of the G_{-m} matrix with the generalized Fibonacci numbers of order $-m$.

Theorem 2. *For $m \in R^+$ and $n \in Z$,*

$$G_{-m}^n = \begin{bmatrix} F_{-m}(n+1) & F_{-m}(n) \\ F_{-m}(n) & F_{-m}(n-1) \end{bmatrix}. \tag{15}$$

Proof. Since, $F_{-m}(0) = 0$, $F_{-m}(1) = 1$ and

$F_{-m}(2) = -m$,
$F_{-m}(3) = m^2 + 1$,
$F_{-m}(4) = -m^3 - 2m$,
$F_{-m}(5) = m^4 + 3m^2 + 1, ...$
we can write

$$G_{-m}^1 = \begin{bmatrix} -m & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{-m}(2) & F_{-m}(1) \\ F_{-m}(1) & F_{-m}(0) \end{bmatrix}, \tag{16}$$

$$G_{-m}^2 = \begin{bmatrix} -m & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -m & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} m^2+1 & -m \\ -m & 1 \end{bmatrix} = \begin{bmatrix} F_{-m}(3) & F_{-m}(2) \\ F_{-m}(2) & F_{-m}(1) \end{bmatrix}, \tag{17}$$

$$G_{-m}^3 = \begin{bmatrix} -m^3-2m & m^2+1 \\ m^2+1 & -m \end{bmatrix} = \begin{bmatrix} F_{-m}(5) & F_{-m}(4) \\ F_{-m}(4) & F_{-m}(3) \end{bmatrix}. \tag{18}$$

By the inductive method, we can easily verify that

$$G_{-m}^n = \begin{bmatrix} F_{-m}(n+1) & F_{-m}(n) \\ F_{-m}(n) & F_{-m}(n-1) \end{bmatrix}. \tag{19}$$

Theorem 3. *For a given integer n and a positive real m number we have that*

$$Det\left(G^{n}_{-m}\right)=(-1)^{n}. \tag{20}$$

Proof. Using general properties of the determinants the proof can be easily seen. It is clear that the above equation is a generalized of the famous Cassini formula.

Theorem 4. *For a given integer n and a positive real number m we have that*

$$G^{n}_{-m}=-mG^{n-1}_{-m}+G^{n-2}_{-m}. \tag{21}$$

Proof. From the recursive relation $F_{-m}(n+2)=-mF_{-m}(n+1)+F_{-m}(n)$ we can write

$$-mG^{n-1}_{-m}+G^{n-2}_{-m}=\begin{bmatrix} -mF_{-m}(n) & -mF_{-m}(n-1) \\ -mF_{-m}(n-1) & -mF_{-m}(n-2) \end{bmatrix}+\begin{bmatrix} F_{-m}(n-1) & F_{-m}(n-2) \\ F_{-m}(n-2) & F_{-m}(n-3) \end{bmatrix}, \tag{22}$$

$$-mG^{n-1}_{-m}+G^{n-2}_{-m}=\begin{bmatrix} -mF_{-m}(n)+F_{-m}(n-1) & -mF_{-m}(n-1)+F_{-m}(n-2) \\ -mF_{-m}(n-1)+F_{-m}(n-2) & -mF_{-m}(n-2)+F_{-m}(n-3) \end{bmatrix}=G^{n}_{-m}. \tag{23}$$

3. The Fibonacci Q-Matrices of the Order $\sqrt{-m}$

The Fibonacci Q-matrices with the order $\sqrt{-m}$ can be defined by the following expression

$$G_{\sqrt{-m}}=\begin{bmatrix} \sqrt{-m} & 1 \\ 1 & 0 \end{bmatrix}, \tag{24}$$

where m is a positive number. The Fibonacci numbers with order $\sqrt{-m}$ is defined by the following recurrence relation,

$$F_{\sqrt{-m}}(n+2)=\sqrt{-m}F_{\sqrt{-m}}(n+1)+F_{\sqrt{-m}}(n), \tag{25}$$

$F_{\sqrt{-m}}(0)=0$, $F_{\sqrt{-m}}(1)=1$ and $m\in R^{+}, n\in Z$.

Theorem 5. *For a given integer n and a positive real m numberwe have that*

$$G^{n}_{\sqrt{-m}}=\begin{bmatrix} F_{\sqrt{-m}}(n+1) & F_{\sqrt{-m}}(n) \\ F_{\sqrt{-m}}(n) & F_{\sqrt{-m}}(n-1) \end{bmatrix}. \tag{26}$$

Proof. Since, $F_{\sqrt{-m}}(2) = \sqrt{m}i$, $F_{\sqrt{-m}}(3) = -m+1$, $F_{\sqrt{-m}}(4) = -m\sqrt{m}i + 2\sqrt{m}i, \cdots$ one can write

$$G^{1}_{\sqrt{-m}} = \begin{bmatrix} \sqrt{m}i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{\sqrt{-m}}(2) & F_{\sqrt{-m}}(1) \\ F_{\sqrt{-m}}(1) & F_{\sqrt{-m}}(0) \end{bmatrix} \tag{27}$$

and

$$G^{4}_{\sqrt{-m}} = \begin{bmatrix} m^2-3m+1 & -m\sqrt{m}i+2\sqrt{m}i \\ -m\sqrt{m}i+2\sqrt{m}i & -m+1 \end{bmatrix} = \begin{bmatrix} F_{\sqrt{-m}}(5) & F_{\sqrt{-m}}(4) \\ F_{\sqrt{-m}}(4) & F_{\sqrt{-m}}(3) \end{bmatrix}. \tag{28}$$

Thus, by the inductive method we obtain that

$$G^{n}_{\sqrt{-m}} = \begin{bmatrix} F_{\sqrt{-m}}(n+1) & F_{\sqrt{-m}}(n) \\ F_{\sqrt{-m}}(n) & F_{\sqrt{-m}}(n-1) \end{bmatrix}. \tag{29}$$

Taking into account the Cassini formula we can write

$$Det\left(G^{n}_{\sqrt{-m}}\right) = (-1)^n . \tag{30}$$

Thus, we can also write

$$F_{\sqrt{-m}}(n+1)\, F_{\sqrt{-m}}(n-1) - \left[F_{\sqrt{-m}}(n)\right]^2 = (-1)^n . \tag{31}$$

This equation can be called Cassini Formula of Fibonacci numbers of the order $\sqrt{-m}$.

Using the recursive relation $F_{\sqrt{-m}}(n+2) = \sqrt{-m}F_{\sqrt{-m}}(n+1) + F_{\sqrt{-m}}(n)$ we write the following theorem.

Theorem 6.

$$G^{n}_{\sqrt{-m}} = \sqrt{-m}G^{n-1}_{\sqrt{-m}} + G^{n-2}_{\sqrt{-m}}. \tag{32}$$

Proof. The proof is immediately follows from the definition $G^{n}_{\sqrt{-m}}$.

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