CONVERGENCE THEOREMS FOR UNIFORMLY L-LIPSCHITZIAN ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}, \{c_n\}_{n\geq 0}$ be real sequences in $[0, 1]$ satisfying the following conditions: (i) $a_n + b_n + c_n = 1$; (ii) $\sum_{n\geq 0} b_n = \infty$; (iii) $c_n = o(b_n)$; (iv) $\lim\limits_{n\to\infty} b_n = 0$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \ n \geq 0,$$

where $\{u_n\}_{n\geq 0}$ is a bounded sequence of error terms in K. Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - p, j(x - p)\rangle \leq k_n ||x - p||^2 - \psi(||x - p||), \ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T)$.

The results proved in this paper significantly improve the results of Ofoedu [11]. The remark 3 is important.

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1.Introduction

Let E be a real normed space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = ||x||^2 \ \text{ and } \ ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in E.

Definition 1 *The mapping T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$*

$$\|T^n x - T^n y\| \leq L \|x - y\| .$$

Definition 2 *T is said to be nonexpansive if for all x, $y \in D(T)$, the following inequality holds:*

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x,\ y \in D(T).$$

Definition 3 *T is said to be asymptotically nonexpansive [6], if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x,\ y \in D(T),\ n \geq 1.$$

Definition 4 *T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and there exists $j(x - y) \in J(x - y)$ such that*

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n \|x - y\|^2 \text{ for all } x,\ y \in D(T),\ n \geq 1.$$

Remark 1 *1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian.*

2. If T is asymptotically nonexpansive mapping then for all x, $y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\begin{aligned} \langle T^n x - T^n y, j(x - y)\rangle &\leq \|T^n x - T^n y\| \|x - y\| \\ &\leq k_n \|x - y\|^2,\ n \geq 1. \end{aligned}$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

3. Rhoades in [12] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The asymptotically pseudocontractive mappings were introduced by Schu [13] who proved the following theorem:

Theorem 1 *Let K be a nonempty bounded closed convex subset of a Hilbert space H, $T : K \to K$ a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1, \infty)$; $q_n = 2k_n - 1$, $\forall n \in N$; $\sum(q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$; $\epsilon < \alpha_n < \beta_n \leq b$, $\forall n \in N$, and some $\epsilon > 0$ and some $b \in (0, L^{-2}[(1 + L^2)^{\frac{1}{2}} - 1])$; $x_1 \in K$ for all $n \in N$, define*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ converges to some fixed point of T.

The recursion formula of theorem 1 is a modification of the well-known Mann iteration process (see [9]).

Recently, Chang [1] extended Theorem 1 to real uniformly smooth Banach space; in fact, he proved the following theorem:

Theorem 2 *Let K be a nonempty bounded closed convex subset of a real uniformly smooth Banach space E, $T : K \to K$ an asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, and $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\} \subset [0, 1]$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n,\ \ n \geq 0.$$

If there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||),\ \ \forall n \in N,$$

then $x_n \to x^* \in F(T)$.

Remark 2 *Theorem 2, as stated is a modification of Theorem 2.4 of Chang [1] who actually included error terms in his algorithm.*

In [11], E. U. Ofoedu proved the following results.

Theorem 3 *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n,\ \ n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||),\ \ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ is bounded.

Theorem 4 *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K :$*

$Tx = x\}$. *Let* $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ *be such that* $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ *and* $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. *For arbitrary* $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n, \; n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||), \; \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ *converges strongly to* $x^* \in F(T)$.

Theorem 5 *Let* K *be a nonempty closed convex subset of a real Banach space* E, $T : K \to K$ *a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence* $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim\limits_{n\to\infty} k_n = 1$ *such that* $x^* \in F(T) = \{x \in K : Tx = x\}$. *Let* $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}, \{c_n\}_{n\geq 0}$ *be real sequences in* $[0,1]$ *satisfying the following conditions:*

i) $a_n + b_n + c_n = 1$;

ii) $\sum_{n\geq 0}(b_n + c_n) = \infty$;

iii) $\sum_{n\geq 0}(b_n + c_n)^2 < \infty$;

iv) $\sum_{n\geq 0}(b_n + c_n)(k_n - 1) < \infty$; *and*

v) $\sum_{n\geq 0} c_n < \infty$.

For arbitrary $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be iteratively defined by*

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \; n \geq 0,$$

where $\{u_n\}_{n\geq 0}$ *is a bounded sequence of error terms in* K. *Suppose there exists a strictly increasing function* $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||), \; \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ *converges strongly to* $x^* \in F(T)$.

Remark 3 *One can easily see that if we take in theorems 3 and 4,* $\alpha_n = \frac{1}{n^\sigma}$; $0 < \sigma < 1$, *then* $\sum \alpha_n = \infty$, *but* $\sum \alpha_n^2 = \infty$. *Hence the conclusions of theorems 3, 4 and 5 can be improved. The same argument can be applied on the results of [5].*

In this paper our purpose is to improve the results of Ofoedu [11] in a significantly more general context by removing the conditions $\sum_{n\geq 0}\alpha_n^2<\infty$ and $\sum_{n\geq 0}\alpha_n(k_n-1)<\infty$ from the theorems $3-4$. We also significantly extend theorem 2 from uniformly smooth Banach space to arbitrary real Banach space. The boundedness assumption imposed on K in the theorem is also dispensed with.

2.Main Results

The following lemmas are now well known.

Lemma 6 *[14] Let $J: E\to 2^E$ be the normalized duality mapping. Then for any $x,y\in E$, we have*

$$||x+y||^2\leq ||x||^2+2\langle y,j(x+y)\rangle,\quad \forall j(x+y)\in J(x+y).$$

Suppose there exists a strictly increasing function $\psi:[0,\infty)\to[0,\infty)$ with $\psi(0)=0$.

Lemma 7 *[10] Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, $\{\lambda_n\}$ be a real sequence satisfying*

$$0\leq\lambda_n\leq 1,\ \sum_{n=0}^{\infty}\lambda_n=\infty$$

and let $\psi\in\Psi$. If there exists a positive integer n_0 such that

$$\theta_{n+1}^2\leq\theta_n^2-\lambda_n\psi(\theta_{n+1})+\sigma_n,$$

for all $n\geq n_0$, with $\sigma_n\geq 0$, $\forall n\in\mathbb{N}$, and $\sigma_n=0(\lambda_n)$, then $\lim_{n\to\infty}\theta_n=0$.

Theorem 8 *Let K be a nonempty closed convex subset of a real Banach space E, $T:K\to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0}\subset[1,\infty)$, $\lim\limits_{n\to\infty}k_n=1$ such that $p\in F(T)=\{x\in K: Tx=x\}$. Let $\{a_n\}_{n\geq 0},\{b_n\}_{n\geq 0},\{c_n\}_{n\geq 0}$ be real sequences in $[0,1]$ satisfying the following conditions:*

i) $a_n+b_n+c_n=1$;

ii) $\sum_{n\geq 0}b_n=\infty$;

iii) $c_n=o(b_n)$;

iv) $\lim\limits_{n\to\infty}b_n=0$.

For arbitrary $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be iteratively defined by*

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \ n \geq 0, \tag{2.1}$$

where $\{u_n\}_{n\geq 0}$ *is a bounded sequence of error terms in* K. *Suppose there exists a strictly increasing function* $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ *such that*

$$\langle T^n x - p, j(x-p)\rangle \leq k_n ||x-p||^2 - \psi(||x-p||), \ \forall x \in K. \tag{2.2}$$

Then $\{x_n\}_{n\geq 0}$ *converges strongly to* $p \in F(T)$.

Proof. From the condition $c_n = o(b_n)$ implies $c_n = t_n b_n$, where $t_n \to 0$ as $n \to \infty$. Since p is a fixed point of T, then the set of fixed points $F(T)$ of T is nonempty.

Since the sequence $\{u_n\}_{n\geq 0}$ is bounded, we set

$$M = \sup_{n\geq 0} \|u_n - p\| .$$

By $\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} t_n$ imply there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $b_n \leq \delta$;

$$\begin{aligned} 0 \quad < \quad \delta = \min\Bigg\{ & \frac{1}{18[\phi^{-1}(a_0)]^2}, \frac{\phi^{-1}(a_0)}{2(2+L)\phi^{-1}(a_0) + M}, \\ & \frac{\phi(2\phi^{-1}(a_0))}{12(1+L)\phi^{-1}(a_0)\left[2(2+L)\phi^{-1}(a_0) + M\right]} \Bigg\}, \end{aligned}$$

and

$$t_n \leq \frac{\phi(2\phi^{-1}(a_0))}{12\phi^{-1}(a_0)(M + 2\phi^{-1}(a_0))}.$$

Define $a_0 := \|x_{n_0} - T^{n_0} x_{n_0}\| \|x_{n_0} - p\| + (k_{n_0} - 1)\|x_{n_0} - p\|^2$. Then from (2.2), we obtain that $\|x_{n_0} - p\| \leq \phi^{-1}(a_0)$.

CLAIM. $\|x_n - p\| \leq 2\phi^{-1}(a_0) \ \forall n \geq n_0$.

The proof is by induction. Clearly, the claim holds for $n = n_0$. Suppose it holds for some $n \geq n_0$, i.e., $\|x_n - p\| \leq 2\phi^{-1}(a_0)$. We prove that $\|x_{n+1} - p\| \leq 2\phi^{-1}(a_0)$. Suppose that this is not true. Then $\|x_{n+1} - p\| > 2\phi^{-1}(a_0))$, so that $\phi(\|x_{n+1} - p\|) > \phi(2\phi^{-1}(a_0))$. Using the recursion formula (2.1), we have the following estimates

$$\begin{aligned} \|x_n - T^n x_n\| &\leq \|x_n - p\| + \|p - T^n x_n\| \\ &\leq (1+L)\|x_n - p\| \\ &\leq 2(1+L)\phi^{-1}(a_0), \end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-p\| &= \|a_n x_n + b_n T^n x_n + c_n u_n - p\| \\
&= \|x_n - p - b_n(x_n - T^n x_n) + c_n(u_n - x_n)\| \\
&\leq \|x_n - p\| + b_n\|x_n - T^n x_n\| + c_n\|u_n - x_n\| \\
&\leq 2\phi^{-1}(a_0) + 2(1+L)\phi^{-1}(a_0)b_n + (M + 2\phi^{-1}(a_0))c_n \\
&\leq 2\phi^{-1}(a_0) + [2(2+L)\phi^{-1}(a_0) + M]b_n \\
&\leq 3\phi^{-1}(a_0).
\end{aligned}$$

With these estimates and again using the recursion formula (2.1), we obtain by lemma 1 that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|a_n x_n + b_n T^n x_n + c_n u_n - p\|^2 \\
&= \|x_n - p - b_n(x_n - T^n x_n) + c_n(u_n - x_n)\| \\
&\leq \|x_n - p\|^2 - 2b_n\langle x_n - T^n x_n, j(x_{n+1}-p)\rangle \\
&\quad +2c_n\langle u_n - x_n, j(x_{n+1}-p)\rangle \\
&= \|x_n - p\|^2 + 2b_n\langle T^n x_{n+1} - p, j(x_{n+1}-p)\rangle \\
&\quad -2b_n\langle x_{n+1} - p, j(x_{n+1}-p)\rangle \\
&\quad +2b_n\langle T^n x_n - T^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad +2b_n\langle x_{n+1} - x_n, j(x_{n+1}-p)\rangle \\
&\quad +2c_n\langle u_n - x_n, j(x_{n+1}-p)\rangle \\
&\leq \|x_n - p\|^2 + 2b_n\left(k_n||x_{n+1}-p||^2 - \phi(||x_{n+1}-p||)\right) \\
&\quad -2b_n\|x_{n+1}-p\|^2 + 2b_n\|T^n x_n - T^n x_{n+1}\|\,||x_{n+1}-p|| \\
&\quad +2b_n\|x_{n+1}-x_n\|\,||x_{n+1}-p|| \\
&\quad +2c_n(M + \|x_n - p\|)\|x_{n+1}-p\| \\
&\leq \|x_n - p\|^2 + 2b_n(k_n - 1)||x_{n+1}-p||^2 - 2b_n\phi(||x_{n+1}-p||) \\
&\quad +2b_n(1+L)\|x_{n+1}-x_n\|\,||x_{n+1}-p|| \\
&\quad +2c_n(M + \|x_n - p\|)\|x_{n+1}-p\|, \qquad (2.3)
\end{aligned}$$

where

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|a_n x_n + b_n T^n x_n + c_n u_n - x_n\| \\
&= \|b_n(T^n x_n - x_n) + c_n(u_n - x_n)\| \\
&\leq b_n\|x_n - T^n x_n\| + c_n\|u_n - x_n\| \\
&\leq 2(1+L)\phi^{-1}(a_0)b_n + (M + 2\phi^{-1}(a_0))c_n \\
&\leq [2(2+L)\phi^{-1}(a_0) + M]b_n. \qquad (2.4)
\end{aligned}$$

Substituting (2.4) in (2.3), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2 - 2b_n\phi(2\phi^{-1}(a_0)) \\ &\quad +18[\phi^{-1}(a_0)]^2(k_n-1)b_n \\ &\quad +6(1+L)\phi^{-1}(a_0)[2(2+L)\phi^{-1}(a_0)+M]b_n^2 \\ &\quad +6\phi^{-1}(a_0)(M+2\phi^{-1}(a_0))c_n \\ &\leq \|x_n-p\|^2+(k_n-1)-\phi(2\phi^{-1}(a_0))b_n.\end{aligned}$$

Thus

$$\phi(2\phi^{-1}(a_0))b_n \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+(k_n-1),$$

implies

$$\begin{aligned}\phi(2\phi^{-1}(a_0))\sum_{n=n_0}^{j} b_n &\leq \sum_{n=n_0}^{j}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)+\sum_{n=n_0}^{j}(k_n-1) \\ &= \|x_{n_0}-p\|^2+\sum_{n=n_0}^{j}(k_n-1),\end{aligned}$$

so that as $j\to\infty$ we have

$$\phi(2\phi^{-1}(a_0))\sum_{n=n_0}^{\infty} b_n \leq \|x_{n_0}-p\|^2+\sum_{n=n_0}^{j}(k_n-1)<\infty,$$

which implies that $\sum b_n<\infty$, a contradiction. Hence, $\|x_{n+1}-x^*\|\leq 2\phi^{-1}(a_0)$; thus $\{x_n\}$ is bounded.

Now with the help (2.4) and the condition $c_n=o(b_n)$, (2.3) takes the form

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq ||x_n-p||^2-2b_n\phi(\|x_{n+1}-p\|) \\ &\quad +2b_n[4[\phi^{-1}(a_0)]^2(k_n-1) \\ &\quad +2(1+L)\phi^{-1}(a_0)[2(2+L)\phi^{-1}(a_0)+M]b_n \\ &\quad +2\phi^{-1}(a_0)(M+2\phi^{-1}(a_0))t_n].\end{aligned} \tag{2.5}$$

Denote

$$\begin{aligned}\theta_n &= ||x_n-p||, \\ \lambda_n &= 2b_n, \\ \sigma_n &= 2b_n[4[\phi^{-1}(a_0)]^2(k_n-1) \\ &\quad +2(1+L)\phi^{-1}(a_0)[2(2+L)\phi^{-1}(a_0)+M]b_n \\ &\quad +2\phi^{-1}(a_0)(M+2\phi^{-1}(a_0))t_n].\end{aligned}$$

Condition $\lim_{n\to\infty} b_n = 0$ assures the existence of a rank $n_0 \in \mathbb{N}$ such that $\lambda_n = 2b_n \leq 1$, for all $n \geq n_0$. Now with the help of $\sum_{n\geq 0} b_n = \infty$, $c_n = o(b_n)$, $\lim_{n\to\infty} b_n = 0$ and Lemma 2, we obtain from (2.5) that

$$\lim_{n\to\infty} ||x_n - p|| = 0,$$

completing the proof. ■

Corollary 9 *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \ n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - p, j(x - p)\rangle \leq k_n ||x - p||^2 - \psi(||x - p||), \ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T)$.

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