ON CERTAIN SUBCLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS INVOLVING CONVOLUTION OPERATORS

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Abstract. Let $A(\omega)$ denote the class of functions of the form $f(z) = (z-\omega) + a_2(z-\omega)^2 + ...$ normalized by $f(\omega) = 0$ and $f'(\omega) - 1 = 0$ and which are anal;ytic in the unit disk $E = \{z : |z| < 1\}$. We define operators $L^{\tau}_{\omega,n} : A(\omega) \to A(\omega)$ using the convolution $*$ and we use this operators to define and study a refinement by way of new generalization of some subclasses of analytic and univalent functions and properties such as inclusion, distortion, closure under certain integral transformations and coefficient inequalities are obtained.

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1. Introduction

Recently, precisely in 1999 Kanas and Ronning [1] introduced the classes of functions starlike and convex, which are normalized with $f(\omega) = f'(\omega) - 1 = 0$ and ω is a fixed point in E. Later on, in 2005 Acu and Owa [2] further the study of the functions of these classes by defining the classes of close-to-convex and α- convex functions normalized the same way and they obtained some useful results concerning these classes. This class of normalized analytic functions is denoted by $A(\omega)$ and of the form

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k, z \in E = z : |z| < 1. \tag{1}$$

Also, we let $P(\omega)$ denote the class of functions of the form

$$p_\omega(z) = 1 + \sum_{k=1}^{\infty} B_k (z-\omega)^k, z \in E \tag{2}$$

where
$$|B_k| \leq \frac{2}{(1+d)(1-d)^k}, d = |\omega|, k \geq 1(see[1,2,3])$$
which are also analytic in the unit disk E, and satisfy $p_\omega(\omega) = 1$, $Rep_\omega(z) > 0$ and ω is a fixed point in E.

We want to note here that $A(0) \equiv A$ and $P(0) \equiv P$ which are the well known classes of analytic and Caratheodory functions respectively. Moreover, let $P(\omega, \beta)$ denote the subclass of $P(\omega)$ which consist of the analytic functions of the form

$$p_{\omega,\beta}(z) = \beta + (1-\beta)p_\omega(z), p_\omega \in P(\omega), 0 \leq \beta < 1 \tag{3}$$

We also note that a function $f \in A(\omega)$ belongs to the class$S_{\omega,0}(\beta)$ if $\frac{f(z)}{(z-\omega)} \in P(\omega, \beta)$ which is of ω-bounded turning of order β if $f'(z) \in P(\omega, \beta)$ and this class of ω-bounded turning shall be denoted by $R_\omega(\beta)$ and it shall consist only of univalent functions in the unit disk. At $\omega = 0$ various generalzation of them were in print see [5,6,8,11,12,13].

It is our interest here in this work to use this concept of analytic functions of the form (1) to obtain a refinement in terms of new generalizations for some existing classes and define some new ones under certain family of integral transforms. Now let

$$g(z) = (z-\omega) + \sum_{k=2}^{\infty} b_k (z-\omega)^k \tag{4}$$

We define the convolution (or Hadamard product) of f and g defined in (1) and (4) respectively written as

$$(f * g)(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k b_k (z-\omega)^k \tag{5}$$

Let τ be a fixed number and $n \in N$, we define

$$\zeta_{\omega,\tau,n}(z) = \frac{z-\omega}{(1-(z-\omega))^{\tau-(n-1)}}, \tau - (n-1) > 0$$

and $\zeta^{-1}_{\omega,\tau,n}$ such that

$$\left(\zeta_{\omega,\tau,n} * \zeta^{-1}_{\omega,\tau,n}\right)(z) = \frac{z-\omega}{(1-(z-\omega))}$$

which is the refinement analogue of the one defined in [7]. For $n = 0$ we simply write $\zeta_{\omega,\tau}$ and $\zeta^{-1}_{\omega,\tau}$ respectively.

Furthermore, let $f \in A(\omega)$, we define the operator $D_\omega^\tau : A(\omega) \to A(\omega) by D_\omega^\tau f(z) = (\zeta_{\omega,\tau} * f)(z)$, and here the operator D_ω^τ is the ω-modified of Ruscheweyh operators used in [7,14,16] and $\tau - \omega$ modified integral operator is defined as follows $I_\tau^\omega : A(\omega) \to A(\omega)$ is given by

$$I_\tau^\omega f(z) = \left(\zeta_{\omega,\tau}^{-1} * f\right)(z).$$

These two operators, that is, D_ω^τ and I_τ^ω will be used to define some classes. We want to remark here that at $\omega = 0$ we have the earlier two operators reduce to D_0^τ and I_τ^0 respectively and which are equivalent to D^τ and I_τ and these have been severally used by various authors to define different subclasses of analytic functions (see[4,5,6,11,12,14]).

The author here wish to give the following definitions:

Definition 1.1 *Let $f \in A(\omega)$ as defined in (1), we define the operators $L_{\omega,n}^\tau : A(\omega) \to A(\omega)$ as*

$$L_{\omega,n}^\tau f(z) = \left(\zeta_{\omega,n} * \zeta_{\omega,\tau,n}^{-1} * f\right)(z).$$

Definition 1.2 *Let $f \in A(\omega)$, we define the operator $L_{\omega,n}^\tau : A(\omega) \to A(\omega)$ as*

$$L_{\omega,n}^\tau f(z) = \left(\zeta_{\omega,\tau}^{-1} * \zeta_{\omega,\tau,n} * f\right)(z).$$

The above definitions are the refinement analogue by extending the ones given by Babalola in [7]. We also note that $L_{\omega,0}^\tau f(z) = L_{\omega,0}^0 f(z) = f(z), L_{\omega,1}^1 f(z) = (z-\omega) f'(z)$, and also $L_{\omega,n}^n f(z) = D_\omega^n f(z)$ and $L_{\omega,n}^0 f(z) = I_n^\omega f(z)$, and similarly $l_{\omega,0}^\tau f(z) = l_{\omega,0} f(z) = f(z), l_{\omega,1}^1 f(z) = (z-\omega) f'(z), l_{\omega,n}^n f(z) = I_n^\omega f(z)$ and $l_{\omega,n}^0 f(z) = D_\omega^n f(z)$.

Remark 1.1 Let $f \in A(\omega)$, then

$$L_{\omega,n}^\tau \left(l_{\omega,n}^\tau f(z)\right) = l_{\omega,n}^\tau \left(L_{\omega,n}^\tau f(z)\right) = f(z).$$

For the case $\tau = n$ we shall write $L_{\omega,n} f(z) = D_\omega^n f(z)$ and $l_{\omega,n} f(z) = I_n^\omega f(z)$ instead of $L_{\omega,n}^n f(z)$ and $l_{\omega,n}^n f(z)$ respectively.

Definition 1.3 *Let $f \in A(\omega)$, and let τ be any real number satisfying $\tau - (n-1) > 0$ for $n \in N$. Then for $0 \le \beta < 1$ a function $f \in A(\omega)$ is said to be in the class $B_{\omega,n}^\tau(\beta)$ if and only if*

$$Re \frac{L_{\omega,n}^\tau f(z)}{z-\omega} > \beta, z \in E \tag{6}$$

and ω is a fixed point in E. For $\tau = n$ we write $B_{\omega,n}(\beta)$ instead of $B^{\tau}_{\omega,n}(\beta)$. We also note the following equivalence classes. $B_{\omega,0}(\beta) \equiv S_{\omega,0}(\beta), B_{\omega,1}(\beta) \equiv R_{\omega}(\beta), B_{0,0}(\beta) \equiv S_{0,0}(\beta) \equiv S_0(\beta)$ and $B_{0,1}(\beta) \equiv R_0(\beta) \equiv R(\beta)$.

Here we want to refine by extension through (1)and (2) the classes defined by Goel and Sohi [12] and Babalola [7].

Definition 1.4 *Let $f \in A(\omega)$ and denote by $M_{\omega,n}(\beta)$ the classes of functions satisfying*

$$Re\frac{D^{n+1}_{\omega}f(z)}{z-\omega} > \beta, z \in E \tag{7}$$

and ω is a fixed point in E. At $\omega = 0$ these classes coincide with classes of functions defined by Goel [12] and Babalola [7]

From (6) and Remark 1.1, functions in the classes $B^{\tau}_{\omega,n}(\beta)$ can be represented in terms of functions in $P(\omega,\beta)$ as $f(z) = l^{\tau}_{\omega,n}\left[(z-\omega)p_{\omega,\beta}(z)\right]$.

We shall investigate this class of functions in the last section of this work.

2.Two-Parameter Integral Iteration of the class $P(\omega)$

Oladipo, A.T. Two-parameter integral iteration of the class $P(\omega)$ We shall start this section with the following definitions

Definition 2.1 *Let $p_{\omega} \in P(\omega)$ and $\alpha > 0$ be real, the nth iterated integral transform of $p_{\omega}(z), z \in E$ and ω a fixed point in E is defined as*

$$p_{\omega,n}(z) = \frac{\alpha}{(z-\omega)^{\alpha}}\int_{\omega}^{z}(t-\omega)^{\alpha-1}p_{\omega,n-1}(t)dt, n \geq 1 \tag{8}$$

with $p_{\omega,0}(z) = p_{\omega}(z)$.Thus transformation, which is denoted by $P_n(\omega)$ arose from our thinking of extending (by refinement) the classes $T^{\alpha}_{n}(\beta)$ consisting of functions defined by the geometric condition

$$Re\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}} > \beta,$$

where $\alpha > 0$ is real, $0 \leq \beta < 1$ and $D^n (n \in N)$ is the Salagean derivative operator defined as $D^0 f(z) = f(z)$ and $D^n f(z) = z(D^{n-1}f(z))'$ see [4,5,6,7,11].

By extension we mean the classes $T^{\alpha}_{n}(\omega,\beta)$ consisting of functions defined by the goemetric conditions

$$Re\frac{D^n_{\omega} f(z)^{\alpha}}{\alpha^n (z-\omega)^{\alpha}} > \beta,$$

where α and β are as earlier defined and D_ω^n in this case is the ω-modified Salageaan derivative operator defined as $D_\omega^0 f(z) = f(z)$ and $D_\omega^n f(z) = (z-\omega)(D_\omega^{n-1} f(z))'$. For $n \geq 1$ the classes $T_n^\alpha(\omega,\beta)$ consisting of univalent functions in the unit disk.

Through the help of the next Lemma, relationship between the classes $P_n(\omega)$ and $T_n^\alpha(\omega,\beta)$ shall be established.

Lemma A. *Let $f \in A(\omega)$, and $\alpha, \beta, D_\omega^n$ be as defined earlier. Then the following are equivalent*

$$(i) f \in T_n^\alpha(\omega,\beta)$$
$$(ii) \frac{\frac{D_\omega^n f(z)^\alpha}{\alpha^n (z-\omega)^\alpha} - \beta}{1-\beta} \in P(\omega)$$
$$(iii) \frac{\frac{f(z)^\alpha}{(z-\omega)^\alpha} - \beta}{1-\beta} \in P_n(\omega)$$

Definition 2.2. *Let $p_\omega \in P(\omega)$. Let τ be any fixed real number such that $\tau-(n-1) > 0$ for $n \in N$. We define the $\omega, \tau - nth$ integral iteration of $p_\omega(z), z \in E$, and ω a fixed point in E as*

$$p_{\omega,\tau,n}(z) = \frac{\tau-(n-1)}{(z-\omega)^{\tau-(n-1)}} \int_\omega^z (t-\omega)^{\tau-n} p_{\omega,\tau,n-1}(t) dt, n \geq 1 \tag{9}$$

with $p_{\omega,\tau,0}(z) = p_\omega(z)$.

Since $p_{\omega,\tau,0}(z) \in P(\omega)$, then the transform $p_{\omega,\tau,n}(z)$ is analytic and $p_{\omega,\tau,n}(\omega) = 1$ and $p_{\omega,\tau,n}(z) \neq 0$ for $z \in E$ and ω is a fixed point in E. Let us denote the family of iterations above by $P_n^\tau(\omega)$. With $p_\omega(z)$ given by (2) we find that

$$p_{\omega,\tau,n}(z) = 1 + \sum_{k=1}^{\infty} B_{n,k}^\tau (z-\omega)^k \tag{10}$$

where

$$B_{n,k}^\tau = \frac{\tau(\tau-n)...(\tau-(n-1))}{(\tau+k)(\tau+k-1)...(\tau+k-(n-1))} B_k, k \geq 1$$

The B_k multipliers can be written factorially as

$$\frac{\tau(\tau-1)...(\tau-(n-1))}{(\tau+k)(\tau+k-1)...(\tau+k-(n-1))} = \frac{\tau!(\tau+k-n)!}{(\tau+k)!(\tau-n)!}, k \geq 1$$

where $(\tau)_n$ represents Pochammer symbol defined by

$$(\tau)_n = \frac{\Gamma(\tau+n)}{\Gamma(\tau)} = \begin{cases} 1 & if\ n=0 \\ \tau(\tau+1)...(\tau+(n-1)) & if\ n \geq 1 \end{cases}$$

Throughout this work the fraction $\frac{(\tau-(n-1))}{(\tau+k-(n-1))_n}$ the multiplier of B_k shall be denoted by $(\tau)_{\frac{n}{k}}$. thus we have

$$B^{\tau}_{n,k} = \frac{(\tau-(n-1))_n}{(\tau+k-(n-1))_n} B_k = (\tau)_{\frac{n}{k}} B_k \tag{11}$$

where

$$|B_k| \leq \frac{2}{(1+d)(1-d)^k}, k \geq 1$$

with $(\tau)_{\frac{0}{k}} = 1$.

By $p_{\omega,\tau,0}(z) = L_{\omega,0}(z) = \frac{1+z}{1-z}$ we easily see that the $\omega, \tau - nth$ integral iteration of the Mobius function is

$$L_{\omega,\tau,n}(z) = \frac{\tau-(n-1)}{(z-\omega)^{\tau-(n-1)}} \int_{\omega}^{z} (t-\omega)^{\tau-n} L_{\omega,\tau,n-1}(t)dt, n \geq 1 \tag{12}$$

The $L_{\omega,\tau,n}(z)$ will play a central role in the family $P^{\tau}_n(\omega)$ similar to the role of the Mobius function $L_0(z)$ in the family P. Now from (11) and the fact that $|B_k| \leq \frac{2}{(1+d)(1-d)^k}, k \geq 1$ we have a refinement analouge by extension of Caratheodory Lemma, that is, we have the following inequality

$$\left|B^{\tau}_{n,k}\right| \leq \frac{2}{(1+d)(1-d)^k}[\tau]_{\frac{n}{k}}, k \geq 1 \tag{13}$$

with equality if and only if $p_{\omega,\tau,n}(z) = L_{\omega,\tau,n}(z)$ given by (12).

From Definition 2.1 and 2.2 we note the following $P^{\tau}_1(\omega) = P_1(\omega)$ and $P^{\tau}_1(0) = P_1(0) = P_1$. We shall use the following results without proof to characterize $P^{\tau}_n(\omega)$

Theorem 2.1 *Let $\gamma \neq 1$ be a nonnegative real number, then for any fixed τ and each $n \geq 1$*

$$Re\ p_{\omega,\tau,n-1}(z) > \gamma \Rightarrow Rep_{\omega,\tau,n}(z) > \gamma, 0 \leq \gamma < 1, \tag{14}$$

and

$$Rep_{\omega,\tau,n-1}(z) > \gamma \Rightarrow Rep_{\omega,\tau,n}(z) < \gamma, \gamma > 1 \tag{15}$$

Corollary 2.1 $p^{\tau}_{\omega,n} \subset P(\omega), n \geq 1$

Theorem 2.2 $p^{\tau}_{\omega,n+1} \subset P^{\tau}_n(\omega), n \geq 1$

Theorem 2.3 *Let $p_{\omega,\tau,n} \in P_n^{\tau}(\omega)$. Then*

$$(i).\, |p_{\omega,\tau,n}| \leq 1 + 2\sum_{k=1}^{\infty} \frac{[\tau]_{\frac{n}{k}}}{(1+d)(1-d)^k}(r+d)^k, k \geq 1, |z| = r, |\omega| = d$$

$$(ii).Rep_{\omega,\tau,n} \geq 1 + 2\sum_{k=1}^{\infty} \frac{[\tau]_{\frac{n}{k}}}{(1+d)(1-d)^k}(-(r+d))^k, k \geq 1$$

The results are sharp.

Corollary 2.2 $p_{\omega,\tau,n} \in P_n^{\tau}(\omega)$ if and only if $p_{\omega,\tau,n}(z) \prec L_{\omega,\tau,n}(z)$.

Remark 2.3 If we choose $n = 0$ in the above Corollary we see that $p_\omega \in P(\omega)$ if and only if $p_\omega(z) \prec L_\omega(z)$. Also if we choose $n = 0$ and $\omega = 0$ in the same Corollary we see that $p_0 \in P(0) \Rightarrow p \in P$ if and only if $p(z) \prec L(z)$ which is well known.

Remark 2.4 For $z \in E$ and ω a fixed point in E, the following are equivalent

$$(i).p_\omega \prec L_\omega,$$
$$(ii).p_\omega \in P(\omega)$$
$$(iii).p_{\omega,\tau,n} \in P_n^{\tau}(\omega)$$
$$(iv).p_{\omega,\tau,n}(z) \prec L_{\omega,\tau,n}(z)$$

Theorem 2.4 $P_n^{\tau}(\omega)$ *is a convex set.*

Proof. Let $p_{\omega,\tau,n}, q_{\omega,\tau,n} \in P_n^{\tau}(\omega)$. Then for nonnegative real number σ_1 and σ_2 with $\sigma_1 + \sigma_2 = 1$, we have

$$\sigma_1 p_{\omega,\tau,n} + \sigma_2 q_{\omega,\tau,n} = \frac{\tau - (n-1)}{(z-\omega)^{\tau-(n-1)}} \int_{\omega}^{z} (t-\omega)^{\tau-n}(\sigma_1 p_{\omega,\tau,n-1} + \sigma_2 q_{\omega,\tau,n-1})(t)dt \quad (16)$$

since $\sigma_1 p_{\omega,\tau,0} + \sigma_2 q_{\omega,\tau,0} = \sigma_1 p_\omega(z) + \sigma_2 q_\omega(z) \in P(\omega)$ for $p_\omega, q_\omega \in P(\omega)$, then the result follows immediately.

3.Characterization of the class $B_{\omega,n}^{\tau}(\beta)$

In this section we discuss the main interest which includes inclusion, closure under certain integral transformation and coefficient inequalities. But before then we first proof the following lemma similar to Lemma A.

Lemma B *Let $f \in A(\omega)$ and $\alpha, \beta, D_\omega^n$ are as earlier defined. Then the following are equivalent:*

$$(i).f \in B_{\omega,n}^\tau(\beta)$$
$$(ii).\frac{\frac{L_{\omega,n}^\tau f(z)}{z-\omega} - \beta}{1-\beta} \in P(\omega)$$
$$(iii).\frac{\frac{f(z)}{(z-\omega)} - \beta}{1-\beta} \in P_n^\tau(\omega)$$

Proof: That $(i) \Leftrightarrow (ii)$ is clear from Definition 2.2. Now (ii) is true if and only if there exists $p_\omega \in P(\omega)$ such that

$$L_{\omega,n}^\tau f(z) = (z-\omega)[\beta + (1-\beta)p_\omega(z)] = (z-\omega) + (1-\beta)\sum_{k=1}^{\infty} B_k(z-\omega)^{k+1} \quad (17)$$

On the application of the operator $l_{\omega,n}^\tau$ on (17), we have (17) if and only if

$$f(z) = (z-\omega) + (1-\beta)\sum_{k=1}^{\infty} B_{n,k}^\tau(z-\omega)^{k+1} \quad (18)$$

if and only if

$$\frac{\frac{f(z)}{z-\omega} - \beta}{1-\beta} = 1 + \sum_{k=1}^{\infty} B_{n,k}^\tau(z-\omega)^k \quad (19)$$

The right hand side of (19) is a function in $P_n^\tau(\omega)$ and this proves the Lemma.

Theorem 3.1 *The following inclusion holds true for a fixed τ satisfying $\tau - (n-1) > 0$:*

$$B_{\omega,n+1}^\tau(\beta) \subset B_{\omega,n}^\tau, n \in N \quad (20)$$

where $0 \le \beta < 1$ and ω is a fixed point in E.

Proof: Let $f \in B_{\omega,n+1}^\tau(\beta)$, then by Lemma B $\frac{\frac{f(z)}{z-\omega}-\beta}{1-\beta} \in P_{n+1}^\tau(\omega)$. By Theorem 2.3, $\frac{\frac{f(z)}{z-\omega}-\beta}{1-\beta} \in P_n^\tau(\omega)$. That is again by Lemma B $f \in B_{\omega,n}^\tau(\beta)$.

Theorem 3.2 *The class $B_{\omega,1}^\tau(\beta)$ consists of univalent functions in E.*

Proof. Let $f \in B_{\omega,1}^\tau(\beta)$. Then by Lemma B, $\frac{\frac{f(z)}{z-\omega}-\beta}{1-\beta} \in P_1^\tau(\omega)$. Since τ is just any fixed integer satisfying $\tau-(n-1) > 0$, we have $\tau > 0$ for $n = 1$, and by Remark

2, it follows that $\frac{\frac{f(z)}{z-\omega}-\beta}{1-\beta} \in P_1^\tau(\omega)$. Thus by Lemma A, it implies that a function $f(z)$ belongs to the class $T_1^\tau(\omega,\beta) \equiv T_1^\alpha(\omega,\beta)$ which consists only of univalent functions in E.

Corollary 3.1 *For $n \geq 1, B_{\omega,n}^\tau(\beta)$ consists only of univalent functions in E.*

Theorem 3.3 *Let $f(z) \in B_{\omega,n}^\tau(\beta)$. Then we have sharp inequalities*

$$|a_k| \leq \frac{2(1-\beta)}{(1+d)(1-d)^{k-1}}[\tau]_{\frac{n}{k-1}}, k \geq 2$$

Equality is attained for

$$f(z) = (z-\omega) + 2(1-\beta)\sum_{k=2}^{\infty}\frac{[\tau]_{\frac{n}{k-1}}}{(1+d)(1-d)^{k-1}}(z-\omega)^k \tag{21}$$

Proof. The results follows from (19) and inequality (13).

Theorem 3.4 *The $B_{\omega,n}^\tau(\beta)$ is closed under the Bernard integral type [10].*

$$F_\omega(z) = \frac{c+1}{(z-\omega)^c}\int_\omega^z (t-\omega)^{c-1}f(t)dt, c+1 > 0 \tag{22}$$

Proof. From (22) we have

$$\frac{\frac{F_\omega(z)}{(z-\omega)}-\beta}{1-\beta} = \frac{\rho}{(z-\omega)^\rho}\int_\omega^z (t-\omega)^{\rho-1}\left(\frac{\frac{f(t)}{(t-\omega)}-\beta}{1-\beta}\right)dt \tag{23}$$

where $\rho = c+1$. Since $f \in B_{\omega,n}^\tau(\beta)$, taking $\rho = c+1 = \tau - n$ we write (23) as

$$\frac{\frac{F_\omega(z)}{(z-\omega)}-\beta}{1-\beta} = \frac{\tau-n}{(z-\omega)^{\tau-n}}\int_\omega^z (t-\omega)^{(\tau-n)-1}p_{\omega,\tau,n}(t)dt \tag{24}$$

which implies

$$\frac{\frac{F_\omega(z)}{(z-\omega)}-\beta}{1-\beta} \in P_{n+1}^\tau(\omega)$$

Thus by Theorem 2.2 we have

$$\frac{\frac{F_\omega(z)}{(z-\omega)}-\beta}{1-\beta} \in P_n^\tau(\omega)$$

Hence $F_\omega \in B^\tau_{\omega,n}(\beta)$.

Theorem 3.5 *Let $f \in B^\tau_{\omega,n}(\beta)$. Then*

$$(r+d)+2(1-\beta)\sum_{k=2}^{\infty}\frac{(-1)^{k-1}[\tau]_{\frac{n}{k-1}}}{(1+d)(1-d)^{k-1}}(r+d)^k \leq \tag{25}$$

$$|f(z)| \leq (r+d)+2(1-\beta)\sum_{k=2}^{\infty}\frac{[\tau]_{\frac{n}{k-1}}}{(1+d)(1-d)^{k-1}}(r+d)^k$$

Proof. If we take

$$p_{\omega,\tau,n}(z)=\frac{\frac{f(z)}{z-\omega}-\beta}{1-\beta}$$

in Theorem 2.3 the result follows. Upper bound equality will be attained for the function given in (21) while the equality in the lower bound is realized for the function

$$f(z)=(z-\omega)+2(1-\beta)\sum_{k=2}^{\infty}\frac{(-1)^{k-1}[\tau]_{\frac{n}{k-1}}}{(1+d)(1-d)^{k-1}}(z-\omega)^k$$

and this complete the proof.

Theorem 3.6 *$B^\tau_{\omega,n}(\beta)$ is a convex family of analytic and univalent functions.*

Proof. Let $f, g \in B^\tau_{\omega,n}(\beta)$. Then by Lemma B there exists $p_{\omega,\tau,n}, q_{\omega,\tau,n} \in P^\tau_n(\omega)$ such that

$$f(z)=(z-\omega)[\beta+(1-\beta)p_{\omega,\tau,n}(z)]$$

and

$$g(z)=(z-\omega)[\beta+(1-\beta)q_{\omega,\tau,n}(z)]$$

Therefore for any nonnegative real number σ_1 and σ_2 with $\sigma_1+\sigma_2=1$, we have

$$\begin{aligned}
h_\omega(z) &= \sigma_1 f(z)+\sigma_2 g(z)\\
&= (z-\omega)\sigma_1[\beta+(1-\beta)p_{\omega,\tau,n}(z)]+(z-\omega)\sigma_2[\beta+(1-\beta)q_{\omega,\tau,n}(z)]\\
&= (z-\omega)[(\sigma_1+\sigma_2)\beta+(1-\beta)(\sigma_1 p_{\omega,\tau,n}(z)+\sigma_2 q_{\omega,\tau,n}(z))]\\
&= (z-\omega)[\beta+(1-\beta)(\sigma_1 p_{\omega,\tau,n}+\sigma_2 q_{\omega,\tau,n})].
\end{aligned}$$

applying Theorem 2.4 and the proof is complete.

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