A NOTE ON PSEUDO-SYMMETRIC NUMERICAL SEMIGROUPS

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Abstract. In this study, we give some results on pseudo-symmetric numerical semigroups which generated by three elements, and we investigate the sutructures $\frac{S}{d}$ and $\frac{I}{d}$, for d a positive integer and I an ideal of S pseudo-symmetric numerical semigroup.

2000 Mathematics Subject Classification: 20M14.

Keywords and phrases: Numerical semigroup, Ideal, Pseudo, Symmetric, Gaps.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote, the sets of integers and non-negative integers, respectively. A subset S of $\mathbb{N}$ is called a numerical semigroup if it is closed and associative under addition and $0 \in S$. Furthermore, a subset $\{s_1, s_2, ...s_p\}$ of the set S is a generating set of S provided that

$$< s_1, s_2, ...s_p \;\; >= \{k_1 s_1 + k_2 s_2 + ... + k_n s_p : k_1, k_2, ..., k_p \in \mathbb{N}\}.$$

It was observed in [1] that the set $\mathbb{N} \backslash S$ is a finite set if and only if

$$g.c.d\{s_1, s_2, ...s_p\} = 1.$$

The Frobenius number of S, denoted by $g(S)$, is the largest integer not in S. That is, $g(S) = max\{x : x \in \mathbb{Z} \backslash S\}$. We define $n(S) = \sharp(\{0, 1, 2, ..., g(S)\} \cap S)$. It is also well-known that $S = \{0, s_1, s_2, ..., s_{n-1}, s_n = g(S) + 1, \rightarrow\}$, where "$\rightarrow$ " means that every integer greater than $g(S) + 1$ belongs to S and $n = n(S)$, $s_i < s_{i+1}$, for $i = 1, 2, ..., n$.([2]).

S is symmetric if for every $x \in \mathbb{Z} \backslash S$ the integer $g(S) - x$ belongs to S. Similarly, a numerical semigroup S is pseudo-symmetric if $g(S)$ is even and the only integer such that $x \in \mathbb{Z} \backslash S$ and $g(S) - x \notin S$ is $x = \frac{g(S)}{2}$ ([3]).

The elements of $\mathbb{N}\backslash S$, denoted by $H(S)$, are called the gaps of S. A gap x of a numerical semigroup S is said to be fundamental if $\{2x, 3x\} \subset S$. We denote by $FH(S)$ the set of all fundamental gaps of S ([5]).

A subset I of S is said to be an ideal if $I + S \subseteq I$. In other words, I is an ideal of S if and only if $x \in I$ and $s \in S$ implies $x + s \in I$. An ideal I of S is said to be generated by $A \subseteq S$ if $I = A + S$. We also say that the ideal I is finitely generated if there exists a finite set $A \subseteq S$ such that $I = A + S$. Finally, we say I is principal if it can be generated by a single element. That is, there exists $x_0 \in S$ such that $I = \{x_0\} + S = \{x_0 + s : s \in S\}$. In this case, we usually write $[x_0]$ instead of $\{x_0\} + S$. ([4]).

For S a numerical semigroup and d is positive integer, we define $\frac{S}{d} = \{x \in \mathbb{N} : dx \in S\}$ to be the quotient of S by d. The set $\frac{I}{d} = \{x \in S : dx \in I\}$ is called an ideal which quotient of I by $x \in S, x \neq 0$, where I be an ideal of S. The elements of S/I, denoted by $H(I)$, are called the gaps of I. ([3]).

In this paper, we assume that S is pseudo-symmetric numerical semigroup which generated by three elements s_1, s_2, s_3. We write $\frac{S}{d}$, when $d = \frac{g(S)}{2}$ and $d > \frac{g(S)}{2}$ for $d \in \mathbb{N}$ in section 2.

Section 3 consists of relations between $\frac{S}{d} = \{x \in \mathbb{N} : dx \in S\}$ and $\frac{I}{d} = \{x \in S : dx \in I\}$

2. Results

In this section, we give some results on $\frac{S}{d}$, where $S =< s_1, s_2, s_3 >$ is a pseudo-symmetric numerical semigroup.

Definition 2.1. *For S numerical semigroup and d is positive integer, we define $\frac{S}{d} = \{x \in \mathbb{N} : dx \in S\}$ to be the quotient of S by d.*

Note 2.2. *$\frac{S}{d} = \{x \in \mathbb{N} : dx \in S\}$ is a numerical semigroup which containing S, and if $d \in S$ then $\frac{S}{d} = \mathbb{N}$, where d is a positive integer.([3]). The following covering result follows from Definition 2.1.*

Corollary 2.3. *Let S be a numerical semigroup and d is a positive integer. Then $d \in FS(H)$ if and only if $\frac{S}{d} = \mathbb{N}\backslash\{1\}$.*([3]).

Theorem 2.4. *Let S be a pseudo-symmetric numerical semigroup and d is a positive integer. If $d > \frac{g(S)}{2}$ then $\frac{S}{d} = \mathbb{N}\backslash\{1\}$.*

Proof. If $d > \frac{g(S)}{2}$then $2d > g(S)$. Thus, we obtain that $3d > 2d > g(S)$ and $\{2d, 3d\} \subset S$ since $d \geq 1$ and $2d, 3d \in S$. That is $d \in FS(H)$, and we have that $\frac{S}{d} = \mathbb{N}\backslash\{1\}$ from Corollary 2.3.

Theorem 2.5. *Let S be a pseudo-symmetric numerical semigroup and d is a positive integer. If $d = \frac{g(S)}{2}$ then $\frac{S}{d} =< 3, 4, 5 >$.*

Proof. $x \in \frac{S}{d} \Longrightarrow dx \in S \Longrightarrow \frac{g(S)}{2}x \in S \Longrightarrow x = 0$ *or* $x > 2$. Because;
(i) If $x = 1$ then $\frac{g(S)}{2}1 \in S$. This is a contradiction.
(ii) If $x = 2$ then $\frac{g(S)}{2}2 = g(S) \in S$. This is a contradiction. Thus, we find that

$$x \in \{0, 3, 4, 5, \to ...\} =< 3, 4, 5 > .$$

Now, we suppose $a \in< 3, 4, 5 >$. Then, there exist $k_1, k_2, k_3 \in \mathbb{N}$ such that $a = 3k_1 + 4k_2 + 5k_3$. In this case,we write $da = 3dk_1 + 4dk_2 + 5dk_3 = 3\frac{g(S)}{2}k_1 + 4\frac{g(S)}{2}k_2 + 5\frac{g(S)}{2}k_3 = 3\frac{g(S)}{2}k_1 + 2g(S)k_2 + 5\frac{g(S)}{2}k_3 \in S$, since $2g(S) \in S$. Thus, we have $a \in \frac{S}{d}$.

Corollary 2.6. *Let S be a pseudo-symmetric numerical semigroup and d is a positive integer. Then, $\frac{S}{d}$ is symmetric or pseudo-symmetric.*

Example 2.7. Let be $S =< 7, 8, 25 >= \{0, 7, 8, 14, 15, 16, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32, 33, 35, \to ...\}$. The Frobenius number of S is $g(S) = 34$. The set of gaps and fundamental gaps of S, $H(S) = \{1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 17, 18, 19, 20, 26, 27, 34\}$ and $FH(S) = \{11, 12, 18, 19, 20, 26, 27, 34\}$, respectively. If $d = \frac{g(S}{2} = 17$ then $\frac{S}{17} = \{x \in \mathbb{N} : 17x \in S\} = \{0, 3, 4, 5, \to ...\} =< 3, 4, 5 >$ is pseudo-symmetric numerical semigroup. If $d = 20 > 17$ then $\frac{S}{20} = \{x \in \mathbb{N} : 20x \in S\} = \{0, 2, 3, 4, 5, \to ...\} =< 2, 3 >= \mathbb{N}\backslash\{1\}$ is symmetric numerical semigroup.If $d = 5$ then $\frac{S}{5} = \{x \in \mathbb{N} : 5x \in S\} = \{0, 3, 5, 6, 7, \to ...\} =< 3, 5, 7 >$ is pseudo-symmetric numerical semigroup. If $d = 2$ then $\frac{S}{2} = \{x \in \mathbb{N} : 2x \in S\} = \{0, 4, 7, 8, 11, 12, 14, 15, 16, 18, \to ...\} =< 4, 7 >$ is symmetric numerical semigroup.

3. The relations between $\frac{S}{d}$ and $\frac{I}{d}$

In this section, we will give some results related the relation between $\frac{S}{d}$ and $\frac{I}{d}$, where d is positive integer and I is a principal of S.

Definition 3.1. *Let I be an ideal of S. The set $\frac{I}{d} = \{s \in S : sd \in I\}$ is called an ideal which quotient of I by $d \in S, d \neq 0$.*

Theorem 3.2. *Let I be an ideal of S and $x \in S, x \neq 0$. Then the following conditions are satisfied:*
(1) *$\frac{I}{x}$ is a ideal of S.*
(2) *$I \subseteq \frac{I}{x} \subseteq \frac{I}{kx}$, for all $k \in \mathbb{N}, k > 0$.*

Proof. (1) $\forall a \in \frac{I}{x}, \forall s \in S, a+s \in \frac{I}{x}$: $a \in \frac{I}{x} \Longrightarrow ax \in I \Longrightarrow \forall s \in S, xa+sx = x(a+s) \in I$ since I is an ideal of S. (2)$a \in I \Longrightarrow ax \in I$ for $x \in S \Longrightarrow (ax)k \in I$ for all $k \in \mathbb{N}, k > 0 \Longrightarrow a \in \frac{I}{kx}$.

Theorem 3.3. *Let I be an ideal of S and $x \in S, x \neq 0$. Then the following conditions are satisfied:*
(1) *If $x \in I$ then $\frac{I}{x} = S \backslash \{0\}$.*
(2) $\frac{I}{x} \subseteq \frac{S}{x}$.

Proof. Let I be an ideal of S and $x \in S, x \neq 0$.
(1) Let $I = [s]$. Then,there exist $m \in S\backslash\{0\}$ and $s_0 \in S$ such that $mx = s + s_0$. In this case, $mx \in I = [s]$. That is, $m \in \frac{I}{x}$.
(2) $a \in \frac{I}{x} \Longrightarrow ax \in I, a \in S \Longrightarrow ax \in S, a \in \mathbb{N} \Longrightarrow a \in \frac{S}{x}$.

Example 3.4. Let be $S =< 4, 9, 11 >= \{0, 4, 8, 9, 11, 12, 13, 15, \rightarrow ...\}$. The Frobenius number of S is $g(S) = 14$. Then the principal ideal $I = [9]$ of S is given by: $I = [9] = 9 + S = \{9, 13, 17, 18, 20, 21, 22, 24, \longrightarrow ...\}$. Thus, we obtain that $\frac{I}{4} = \{8, 9, 11, 12, 13, 15, \longrightarrow ...\} \subset \frac{S}{4} = \mathbb{N}$ and $\frac{I}{4} \subseteq \frac{I}{20} = \{4, 8, 9, 11, 12, 13, 15, \longrightarrow ...\} = S\backslash\{0\}$.

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