SUPERORDINATION RESULTS IN THE COMPLEX PLANE

Camelia Mădălina Bălăeţi

Abstract. In this paper are given certain superordination results using the integral operator (Definition 1.). These results are related to some normalized holomorphic functions in the unit disc $U = \{z \in C : |z| < 1\}$.

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1. Introduction and preliminary results

Let $\mathcal{H}(U)$ be the space of holomorphic functions in the unit disk U of the complex plane $U = \{z \in C : |z| < 1\}$.

We consider the class $A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots\}$ with $A_1 = A$ and for $a \in C$, $n \in N^*$ we let

$$H[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots\}.$$

Denote with K the class of convex functions in U, defined by

$$K = \{f \in H(U) : f(0) = f'(0) - 1 = 0,\ \mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\}.$$

Since we use the terms of subordination and superordination, we review here those definitions.

If $f, g \in \mathcal{H}(U)$, then f is said to be *subordinate* to g, or g is said *superordinate* to f, if there is a function $w \in \mathcal{H}(U)$, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$.

In such a case we write $f \prec g$ or $f(z) \prec g(z)$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let Ω be a set in the complex plane C, and p be an analytic function in the unit disk with $\psi(\gamma, s, t; z) : C^3 \times U \to C$. In [1] S.S. Miller and P.T. Mocanu determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p''(z))\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of functions p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z)|\ z \in U\}.$$

This results have been first presented in [2].

Definition 1. [2]*Let $\varphi : C^2 \times U \to C$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z); z)$ are univalent in U and satisfy the (first-order) differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z); z) \tag{1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant.

Note that the best subordinant is unique up to a rotation of U.

For Ω a set in C, with φ and p as given in Definition 1., suppose (1) is replaced by

$$\Omega \subset \{\varphi(p(z), zp'(z); z)|\ z \in U\}. \tag{2}$$

Although this more general situation is a "differential containment", the condition in (2) will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extend to this generalization.

Before obtaining some of the main results we need to introduce a class of univalent functions defined on the unit disc that have some nice boundary properties.

Definition 2. [2] *We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.*

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

In order to prove the new results we shall use the following lemma:

Lemma 1. [2]*Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ and* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, n] \cap Q$ and $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U with*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

then

$$q(z) \prec p(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Definition 3. [3]*For $f \in A_n$ and $m \geq 0$, $m \in N$, the operator $I^m f$ is defined by*

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = \int_0^z f(t)t^{-1}dt$$

$$I^m f(z) = I[I^{m-1}f(z)],\ z \in U.$$

Remark 1. If we denote $l(z) = -\log(1-z)$, then

$$I^m f(z) = [\underbrace{(l * l * \ldots * l)}_{n-times} * f](z),\ f \in \mathcal{H}(U),\ f(0) = 0$$

By "$*$" we denote the Hadamard product or convolution (i.e. if $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$ then $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$).

Remark 2. $I^m f(z) = \int_0^z \int_0^{t_m} \ldots \int_0^{t_2} \frac{f(t_1)}{t_1 t_2 \ldots t_m} dt_1 dt_2 \ldots dt_m$.

Remark 3. $D^m I^m f(z) = I^m D^m f(z) = f(z)$, $f \in \mathcal{H}(U)$, $f(0) = 0$, where $D^m f(z)$ is the Sălăgean differential operator.

Remark 4. For $m = 0$ the integral operator is the Alexander operator.

2. Main results

Theorem 1.*Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2+z} \tag{3}$$

with $h(0) = 1$.

Let $f \in A_n$ and suppose that $[I^m f(z)]'$ is univalent and $[I^{m+1} f(z)]' \in [1, n] \cap Q$. If

$$h(z) \prec [I^m f(z)]',\ z \in U \tag{4}$$

then

$$q(z) \prec [I^{m+1} f(z)]',\ z \in U, \tag{5}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \Big(1 + t + \frac{t}{2+t}\Big) t^{\frac{1}{n}-1} dt, \tag{6}$$

$$q(z) = 1 + \frac{z}{n+1} + \frac{1}{n} M(z) \frac{1}{z^{\frac{1}{n}}}$$

and

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2+t} dt.$$

The function q is convex and it is the best subordinant.

Proof. Let $f \in A_n$. By using the properties of the integral operator $I^m f$ we have

$$I^m f(z) = z[I^{m+1} f(z)]', \ z \in U. \tag{7}$$

Differentiating (7), we obtain

$$[I^m f(z)]' = [I^{m+1} f(z)]' + z[I^{m+1} f(z)]'', \ z \in U. \tag{8}$$

If we let $p(z) = [I^{m+1} f(z)]'$, then (8) becomes

$$[I^m f(z)]' = p(z) + zp'(z), \ z \in U.$$

Then (4) becomes

$$h(z) \prec p(z) + zp'(z), \ z \in U.$$

By using Lemma 1., for $\gamma = 1$, we have

$$q(z) \prec p(z) = [I^{m+1} f(z)]', \ z \in U$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \Big(1 + t + \frac{t}{2+t}\Big) t^{\frac{1}{n}-1} dt = 1 + \frac{z}{n+1} + \frac{1}{n} M(z) \frac{1}{z^{\frac{1}{n}}}$$

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2+t} dt.$$

Moreover, the function q is convex and it is the best subordinant.

If $n = 1$, from Theorem 1. we obtain the next corollary.

Corollary 1. *Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2+z}$$

with $h(0) = 1$.

Let $f \in A$ and suppose that $[I^m f(z)]'$ is univalent and $[I^{m+1} f(z)]' \in \mathcal{H}[1,1] \cap Q$. If

$$h(z) \prec [I^m f(z)]', \ z \in U$$

then

$$q(z) \prec [I^{m+1}f(z)]', \; z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z \Big(1+t+\frac{t}{2+t}\Big)dt,$$

$$q(z) = 1 + \frac{z}{2} + M(z)\frac{1}{z}$$

and

$$M(z) = z - 2\ln(2+z) + \ln 2, \; z \in U.$$

The function q is convex and it is the best subordinant.

Theorem 2.*Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2+z}$$

with $h(0) = 1$. Let $f \in A_n$ and suppose that $[I^m f(z)]'$ is univalent and $\frac{I^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$.

If

$$h(z) \prec [I^m f(z)]', \; z \in U, \tag{9}$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \; z \in U, \tag{10}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \Big(1+t+\frac{t}{2+t}\Big)t^{\frac{1}{n}-1}dt = 1 + \frac{z}{n+1} + \frac{1}{n}M(z)\frac{1}{z^{\frac{1}{n}}}$$

and

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2+t}dt, \; z \in U.$$

The function q is convex and it is the best subordinant.

Proof. We let

$$p(z) = \frac{I^m f(z)}{z}, \; z \in U$$

and we obtain

$$I^m f(z) = zp(z), \; z \in U. \tag{11}$$

By differentiating (11) we obtain

$$[I^m f(z)]' = p(z) + zp'(z), \; z \in U$$

Then (9) becomes

$$h(z) \prec p(z) + zp'(z), \ z \in U.$$

By using Lemma 1. we have

$$q(z) \prec p(z) = \frac{I^m f(z)}{z}, \ z \in U,$$

where

$$q(z) = 1 + \frac{z}{n+1} + \frac{1}{n} M(z) \frac{1}{z^{\frac{1}{n}}}$$

with

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2+t} dt, \ z \in U.$$

Moreover, the function q is convex and it is the best subordinant.

If $f \in A$, then we have the next corollary.

Corollary 2. *Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2+z}$$

with $h(0) = 1$. Let $f \in A$ and suppose that $[I^m f(z)]'$ is univalent and $\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q$. If $h(z) \prec [I^m f(z)]'$, $z \in U$, then $q(z) \prec \frac{I^m f(z)}{z}$, $z \in U$, where

$$q(z) = \frac{1}{z} \int_0^z \Big(1 + t + \frac{t}{2+t}\Big) dt = 1 + \frac{z}{2} + M(z)\frac{1}{z}$$

and

$$M(z) = z - 2\ln(2+z) + 2\ln 2, \ z \in U.$$

The function q is convex and it is the best subordinant.

References

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Camelia Mădălina Bălăeţi
Department of Mathematics
University of Petroşani
20, Universităţii Str., 332006 Petroşani, Romania
email: *madalina@upet.ro*