UNIFORMLY STARLIKE AND CONVEX UNIVALENT FUNCTIONS BY USING CERTAIN INTEGRAL OPERATORS

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Abstract. An extension of k-uniformly starlike and convex functions are introduced by making use of an integral operator. Inclusion relations and coefficient bounds for these classes are determined and consequently, some known results are generalized.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are holomorphic (analytic) in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. For $\alpha \geq 1, 0 \leq \beta < 1, 0 < k < 1, \sigma > 0$, and for

$$I^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\frac{\log z}{t}\right)^{\sigma-1} f(t)dt = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n, \qquad (1)$$

we define the following three classes of functions.

i) The class $HP_\sigma(k, \alpha, \beta)$ consisting of functions $f \in \mathcal{H}$ satisfying

$$Re\left\{z\frac{(I^\sigma f(z))'}{I^\sigma f(z)}\right\} > k\left|z\frac{(I^\sigma f(z))'}{I^\sigma f(z)} - \alpha\right| + \beta, \quad z \in \mathcal{U}. \qquad (2)$$

ii) The class $KP_\sigma(k, \alpha, \beta)$ consisting of function $f \in \mathcal{H}$ such that $zf' \in HP_\sigma(k, \alpha, \beta)$. Therefore $f \in KP_\sigma(k, \alpha, \beta)$ if and only if

$$Re\left\{1 + \frac{z(I^\sigma f(z))''}{(I^\sigma f(z))'}\right\} > k\left|\frac{z(I^\sigma f(z))''}{(I^\sigma f(z))'} + 1 - \alpha\right| + \beta, \quad z \in \mathcal{U}. \qquad (3)$$

iii) The class $LP_\sigma(k,\alpha,\beta)$ consisting of functions $f \in \mathcal{H}$ such that

$$R\left\{\frac{I^\sigma f(z)}{I^{\sigma+1} f(z)}\right\} > k\left|\frac{I^\sigma f(z)}{I^{\sigma+1} f(z)} - \alpha\right| + \beta, \quad z \in U. \tag{4}$$

It is easy to show that a function $f \in \mathcal{H}$ belongs to the respective classes $HP_\sigma(k,\alpha,\beta)$, $KP(k,\alpha,\beta)$, and $LP(k,\alpha,\beta)$ if and only if the respective integral functions $\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}$, $1+\frac{z(I^\sigma f(z))''}{(I^\sigma f(z))'}$, and $\frac{I^\sigma f(z)}{I^{\sigma+1} f(z)}$ belong to D where

$$D = \left\{u+iv : \left(u - \frac{\beta-\alpha k^2}{1-k^2}\right)^2 - \frac{k^2}{1-k^2}v^2 > \frac{k^2(\alpha-\beta)^2}{(1-k^2)^2}, \quad u>0\right\} \tag{5}$$

is the hyperbolic domain in the right half plane with vertex at $\left(\frac{\beta+\alpha k}{1+k}, 0\right)$.

The above three classes include various new classes of analytic univalent functions as well as many well-known classes that have been studied earlier. For example, $HP_0(k,1,0)$ consists of k-uniformly starlike functions studied by Kanas and Wiśniowska [2,3,4]. In particular, $HP_0(1,1,0)$ is the class of parabolic starlike functions studied by Rønning [7]. The special case $KP_0(k,1,0)$ consists of k-uniformly convex functions which also was studied in [2,3,4]. In particular, $KP_0(1,1,0)$ is the class of uniformly convex functions studied in [7]. In this paper we study various inclusion relations for the above three classes $HP_\sigma(k,\alpha,\beta)$, $KP_\sigma(k,\alpha,\beta)$, and $LP_\sigma(k,\alpha,\beta)$. We then introduce some coefficient bounds for the functions in these classes. First we give an inclusion relation for the class

2.Inclusion Relations

To prove our main results, we shall need the following lemma which is due to Eenigenburg, Miller, Mocanu, and Reade [1].

Lemma 2.1. *Let $\beta, \gamma \in \mathrm{C}$ and h be an analytic function in $\mathcal{U}$ with $h(0)=1$ and $Re\{\beta h(z)+\gamma\} > 0$. If $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ is analytic in $\mathcal{U}$, then $p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z)$ implies $p(z) \prec h(z)$, where the symbol $\prec$ denotes the usual subordination.*

Theorem 2.2. *$HP_\sigma(k,\alpha,\beta) \subset HP_{\sigma+1}(k,\alpha,\beta)$.*

Proof. For $f \in \mathcal{H}$ suppose that $f \in HP_\sigma(k,\alpha,\beta)$. Then the function f needs to satisfy the required condition (2). For the operator I^σ acting on the function f we note that

$$z(I^\sigma f(z))' = 2I^\sigma f(z) - I^{\sigma+1} f(z). \tag{6}$$

Letting $p(z) = \frac{z(I^{\sigma+1}f(z))'}{I^{\sigma+1}f(z)}$ and differentiating with respect to z we obtain

$$p(z) + \frac{zp'(z)}{p(z)+1} = \frac{z(I^{\sigma}f(z))'}{I^{\sigma}f(z)}.$$

Then, by Lemma 2.1, we obtain $\left\{\frac{z(I^{\sigma+1}f(z))'}{I^{\sigma+1}f(z)}\right\}_{z\in U} \subset D$ since $\left\{\frac{z(I^{\sigma}f(z))'}{I^{\sigma}f(z)}\right\}_{z\in U} \subset D$, for D is a convex domain. This completes the proof.

In the next two theorems we examine the inclusion relations between the classes of functions defined by the conditions (2), (3), and (4).

Theorem 2.3. $KP_\sigma(k,\alpha,\beta) \subset HP_\sigma(k,\alpha,\beta)$.

Proof. Let $f \in KP_\sigma(k,\alpha,\beta)$. For $p(z) = \frac{z(I^{\sigma}f(z))'}{I^{\sigma}f(z)}$ a logarithmic differentiation yields $p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{z(I^{\sigma}f(z))''}{(I^{\sigma}f(z))'}$. Now the theorem follows from Lemma 2.1 and the fact that $\left\{1 + \frac{z(I^{\sigma}f(z))''}{(I^{\sigma}f(z)'}\right\}_{z\in U} \subset D$, for D is a convex domain.

Theorem 2.4. *If* $\frac{1}{2} \leq \beta < 1$ *then* $LP_\sigma(k,\alpha,\beta) \subset HP_{\sigma+1}(k, 2\alpha-1, 2\beta-1)$.

Proof. Let $f \in LP_\sigma(k,\alpha,\beta)$. Then by applying the fact (6) in (3) we can write

$$\frac{1}{2}Re\left\{\frac{z(I^{\sigma+1}f(z))'}{I^{\sigma+1}f(z)}\right\} + \frac{1}{2} > k\left|\frac{1}{2}\left(\frac{z(I^{\sigma+1}f(z))'}{I^{\sigma+1}f(z)}\right) + \frac{1}{2} - \alpha\right| + \beta.$$

With a simple manipulation we obtain

$$Re\left\{\frac{z(I^{\sigma+1}f(z))'}{I^{\sigma+1}f(z)}\right\} > k\left|\frac{z(I^{\sigma+1}f(z))'}{I^{\sigma+1}f(z)} - (2\alpha-1)\right| + 2\beta - 1.$$

Therefore, according to the condition (2), $f \in HP_{\sigma+1}(k, 2\alpha-1, 2\beta-1)$.

3.Coefficient Bounds

In this section we give coefficient bounds for function series expansion in the classes $HP_\sigma(k,\alpha,\beta)$, $KP_\sigma(k,\alpha,\beta)$ and $LP_\sigma(k,\alpha,\beta)$. For the Caratheodory class $\mathcal{P}$ of functions $p \in \mathcal{P}$ we define $\mathcal{P}(p_k)$, $k \geq 0$, by $p \prec p_k$ in $\mathcal{U}$, where the function p_k maps the unit disk conformally onto the region $\Omega_k = \{w \in \mathrm{C} : Re(w) > k|w-1|\}$ such that $1 \in \Omega_k$.

For $0 < k < 1$, Ω_k is a hyperbolic domain and the corresponding map has the form,

$$\begin{aligned} p_k(z) &= \frac{1}{1-k^2}\cos\left\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2}{1-k^2} \qquad (7) \\ &= 1 + \frac{1}{1-k^2}\sum_{n=1}^{\infty}\left[\sum_{j=1}^{2n} 2^j \begin{pmatrix} A \\ j \end{pmatrix}\begin{pmatrix} 2n-1 \\ 2n-j \end{pmatrix}\right] z^n, \end{aligned}$$

where $A = \frac{2}{\pi} cos^{-1}\ k$ and the branch of $\sqrt{z}$ is chosen such that $I_m\sqrt{z} \geq 0$.

The family $\mathcal{P}(p_k)$ and its extremal functions are studied in [3]. Note that the function p_k in (7) has non-negative coefficients. According to (2), a function f is in $HP_\sigma(k,\alpha,\beta)$ if and only if $\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} = p(z)$ is so that $p(0)=1$, and $p(\mathcal{U}) \subset D$ for D a convex domain. By using the properties of the hyperbolic domains for functions $f \in HP_\sigma(k,\alpha,\beta)$, we have $Re\left\{\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}\right\} > \frac{\alpha k+\beta}{1+k}$ and $\left|arg\ \frac{z(I^\sigma f(z))'}{I^\sigma f(z)}\right| < \tan^{-1}\frac{\sqrt{1-k^2}}{k}$. Now we are ready to state and prove the following lemma, which we shall need to prove our results in this section.

Lemma 3.1. *Let $\alpha \geq 1$, $0 \leq \beta < 1$, $\alpha+\beta \geq 2$, and $0 < k < \frac{1-\beta}{\alpha-1}$. Then the function*

$$Q(z) = \frac{\beta-\alpha k^2}{1-k^2} + \frac{\alpha-\beta}{1-k^2}\cos\left\{Ai\log\left(\frac{1+\sqrt{\psi(z)}}{1-\sqrt{\psi(z)}}\right)\right\}$$

is so that $Q(0)=1$, $Q'(0)>0$, and $Q(\mathcal{U}) \subset D$ where

$$\psi(z) = \begin{cases} \frac{(\sqrt[A]{N}+1)^2 z + (\sqrt[A]{N}-1)^2}{(\sqrt[A]{N}+1)^2 + (\sqrt[A]{N}-1)^2 z} & ;\ \alpha > 1 \\ z & ;\ \alpha = 1 \end{cases}$$

and

$$N = \frac{1-\beta+k^2(\alpha-1)+\sqrt{[(1-\underline{\ }) + k^2(\alpha-1)]^2-(\alpha-\beta)^2}}{\alpha-\beta}.$$

Proof. By making use of the properties of conformal mappings it is easy to see that

$$p(z) = \frac{\beta-\alpha k^2}{1-k^2} + \frac{\alpha-\beta}{1-k^2}\quad\cos\left\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\},\quad z \in \mathcal{U}$$

is analytic and univalent with $p(\mathcal{U}) \subset D$ and $p(0) = \alpha$. Let μ be the real number such that $\mu \in \mathcal{U}$ and $p(\mu) = 1$. So the function $Q(z) = (po\psi)(z)$ where $\psi(z) = \frac{z+\mu}{1+\mu z}$ is the Möbius transformation function which maps the open unit disk $\mathcal{U}$ onto itself and satisfies the conditions $Q(0)=1$, $Q'(0)>0$, and $Q(\mathcal{U}) \subseteq D$. For finding the value of μ, since $p(\mu)=1$, therefore we have

$$\left(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}\right)^{-A} + \left(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}\right)^{A} = \frac{2[1-\beta+k^2(\alpha-1)]}{\alpha-\beta}.$$

After an easy computation, without loss of generality, we can write

$$\left(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}\right)^{A} = \frac{1-\beta+k^2(\alpha-1)+\sqrt{[(1-\beta)+k^2(\alpha-1)]^2-(\alpha-\beta)^2}}{\alpha-\beta} = N.$$

It is easy to see that if $\alpha > 1$ then $0 < N < 1$ so $\mu = \left(\frac{\sqrt[A]{N}-1}{\sqrt[A]{N}+1}\right)^2$. Also if $\alpha = 1$ then $N = 1$ and so $\mu = 0$. Therefore we have

$$\psi(z) = \begin{cases} \frac{(\sqrt[A]{N}+1)^2 z + (\sqrt[A]{N}-1)^2}{(\sqrt[A]{N}+1)^2 + (\sqrt[A]{N}-1)^2 z} & ;\ \alpha > 1 \\ z & ;\ \alpha = 1. \end{cases}$$

So for $\alpha = 1$ we have $Q'(z) = \frac{(1-\beta)Ai}{(1-k^2)\sqrt{z}(1-\sqrt{z})} \sin\left\{Ai \log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}$ and $Q'(0) = \frac{2A^2(1-\beta)}{1-k^2} > 0$.

For $\alpha > 1$, after an easy computation, we have

$$Q'(0) = \frac{A(\alpha-\beta)(1-N^2)(1+\sqrt[A]{N^2})}{N(1-k^2)(1-\sqrt[A]{N^2})} > 0.$$

This completes the proof.

Theorem 3.2. *Let* $f \in HP_\sigma(k,1,\beta)$ *and* $\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} = 1 + \sum_{n=1}^{\infty} t_n z^n$ *then*

$$\sum_{n=1}^{\infty} |t_n|^2 \leq \left(\frac{1-\beta}{1-k^2}\right)^2 \left[3 + \frac{2}{\cos(A\pi)} + \frac{4}{\cos(A\pi/2)}\right].$$

Proof. According to Lemma 3.1, the function which maps $\mathcal{U}$ conformally onto the region D is given by

$$Q(z) = \frac{\beta-k^2}{1-k^2} + \frac{1-\beta}{1-k^2}\cos\left\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}.$$

Obviously we have

$$\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} - 1 \prec \frac{1-\beta}{1-k^2}\left[\cos\left\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - 1\right] = \sum_{n=1}^{\infty} T_n z^n, z \in \mathcal{U}.$$

Now by the well known results of Littlewood [6, p. 35] and Lang [5, p.200] we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} |t_n|^2 \leq \sum_{n=1}^{\infty} |T_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|Q(e^{i\theta}) - 1\right|^2 d\theta \\
&= \frac{(1-\beta)^2}{2\pi(1-k^2)^2}\int_0^{2\pi} \left|cosh\ \left(A\log\frac{1+\sqrt{e^{i\theta}}}{1-\sqrt{e^{i\theta}}}\right) - 1\right|^2 d\theta \\
&= \frac{(1-\beta)^2}{2\pi(1-k^2)^2}\int_0^{2\pi} \left|i^A(cot\ \frac{\theta}{4})^A + (-i)^A(\tan\frac{\theta}{4})^A - 1\right|^2 d\theta
\end{aligned}$$

$$\leq \frac{(1-\beta)^2}{2\pi(1-k^2)^2}\int_0^{2\pi}\left[(cot\ \ \frac{\theta}{4})^A+(\tan\frac{\theta}{4})^A+1\right]^2 d\theta$$
$$=\frac{3(1-\beta)^2}{(1-k^2)^2}+\frac{4(1-\beta)^2}{\pi(1-k^2)^2}\left[\int_0^\infty \frac{x^{2A}}{1+x^2}dx+2\int_0^\infty \frac{x^A}{1+x^2}dx\right]$$
$$=\frac{3(1-\beta)^2}{(1-k^2)^2}+\frac{4(1-\beta)^2}{\pi(1-k^2)^2}\left[\frac{-\pi e^{-2A\pi i}}{\sin(2A\pi)}\left(\frac{e^{A\pi i}-e^{3A\pi i}}{2i}\right)-\frac{2\pi e^{-A\pi i}}{\sin(A\pi)}\left(\frac{e^{A\frac{\pi}{2}\ i}-e^{A\frac{3\pi}{2}i}}{2i}\right)\right]$$
$$=\left(\frac{1-\beta}{1-k^2}\right)^2\left[3+\frac{2}{\cos(A\pi)}+\frac{4}{\cos(A\pi/2)}\right].$$

Theorem 3.3. *Let* $0<k<min\{1,\frac{1-\beta}{\alpha-1}\}$ *and* $f\in\mathcal{H}$. *If*

$$\sum_{n=2}^{\infty}[(k+1)|n-\alpha|+\alpha-\beta]\left(\frac{2}{n+1}\right)^{\sigma}|a_n|<1-\beta-k(\alpha-1) \tag{8}$$

then $f\in HP_\sigma(k,\alpha,\beta)$.

Proof. According to the condition (2), it is sufficient to show that

$$k\left|\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}-\alpha\right|-Re\left\{\frac{z(I^\sigma f(z)'}{I^\sigma f(z)}-\alpha\right\}<\alpha-\beta. \tag{9}$$

The left hand side in the inequality in (9) can be written as

$$k\left|\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}-\alpha\right|-Re\left\{\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}-\alpha\right\}$$
$$\leq(k+1)\left|\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}-\alpha\right|$$
$$\leq(k+1)\frac{\alpha-1+\sum_{n=2}^\infty|n-\alpha|\left(\frac{2}{n+1}\right)^\sigma|a_n|}{1-\sum\limits_{n=2}^\infty\left(\frac{2}{n+1}\right)^\sigma|a_n|}.$$

On the other hand, by (8), we have

$$(k+1)\frac{\alpha-1+\sum_{n=2}^\infty|n-\alpha|\left(\frac{2}{n+1}\right)^\sigma|a_n|}{1-\sum_{n=2}^\infty\left(\frac{2}{n+1}\right)^\sigma|a_n|}<\alpha-\beta.$$

Therefore the required condition (9) follows and so the proof is complete.

Similar coefficient bounds can be obtained for the classes $KP_\sigma(k,\alpha,\beta)$ and $LP_\sigma(k,\alpha,\beta)$ by replacing $\frac{z(I^\sigma f(z))'}{I^\sigma f(z)}=1+\sum_{n=1}^\infty t_nz^n$ with $1+\frac{z(I^\sigma f(z))''}{(I^\sigma f(z))'}=1+$

$\sum_{n=1}^{\infty} t_n z^n$ and $\frac{I^{\sigma} f(z)}{I^{\sigma+1} f(z)} = 1 + \sum_{n=1}^{\infty} t_n z^n$ respectively. For the sake of simplicity, we omit the trivial details.

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