BOUNDARY MATHEMATICAL PROBLEMS IN VISCOUS LIQUID THREE-DIMENSIONAL FLOWS

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Abstract. One presents the specific boundary conditions, which intervene in the problem of three-dimensional viscous liquid steady flow into the suction chamber of a pump station.

Thus, for the uniform flow of the viscous and heavy liquid in the entrance and going out sections of the domain, we must consider the two-dimensional special problems:

- for the fluid **entrance section** we must solve a mixed Dirichlet-Neumann problem for a Poisson equation with unknown constant, depending of the slope angle of the suction chamber bottom, which can be determinate by a successive calculus cycle of the computer program, satisfying a normalization condition, which for the given value of mean input velocity and entrance section dimensions ensure the desired flow-rate,

- for the fluid **going out pipe section** we must solve a Dirichlet problem for an other Poisson's equation, having the pressure drop gradient in the suction pipe as unknown constant, which can be determinate also by a successive calculus cycle satisfying a normalization condition, which establishes the pressure drop gradient on the pipe length for the same flow-rate.

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1. The practical importance of the problem

The theoretical research concerning the three-dimensional flow of the real liquid in a suction chamber of a pump [1], is justified by the flow effect with or without appearance of whirls, with or without air training or of the appearance of the cavitation destructive phenomenon at the liquid entry in a pump, as well as of the negative their influence on the hydraulic and energetic pump efficiency, the quick cavitational erosions, the machine vibrations followed by the fast wear, eventually by the pump suction less and by their pumping cease, extremely dangerous for the pumping station, for instance of nuclear electric power station.

These situation may come in the sight in special work conditions linked with the low level of the water in the river bed or in the pump suction chamber, or by a inefficient design of this installation for different cases of pump working [2]÷[4].

2. Partial differential equations of a viscous and heavy liquid steady flow

The three-dimensional **motion equations** [7] in Cartesian trihedron (fig.1) are:

$$U'_T + U'_X U + U'_Y V + U'_Z W + \frac{1}{\rho} P'_X = \upsilon \,(U''_{X^2} + U''_{Y^2} + U''_{Z^2}) + g \sin\alpha \qquad (1)$$

$$V'_T + V'_X U + V'_Y V + V'_Z W + \frac{1}{\rho} P'_Y = \upsilon \,(V''_{X^2} + V''_{Y^2} + V''_{Z^2}) \qquad (2)$$

$$W'_T + W'_X U + W'_Y V + W'_Z W + \frac{1}{\rho} P'_Z = \upsilon \,(W''_{X^2} + W''_{Y^2} + W''_{Z^2}) + g \cos\alpha, \qquad (3)$$

and the **mass conservation equation** for an incompressible fluid is

$$U'_X + V'_Y + W'_Z = 0. \qquad (4)$$

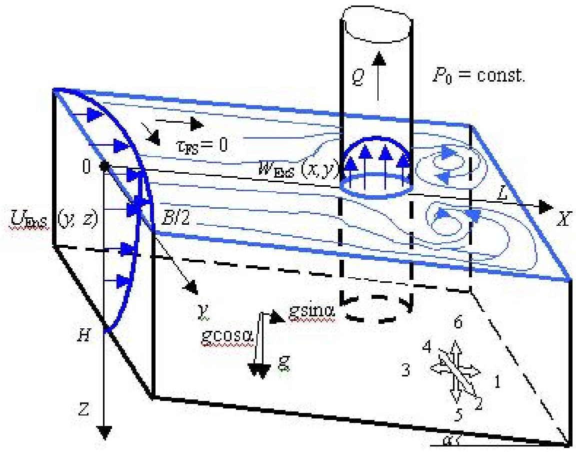

Figure 1: Scheme of the studied pump suction chamber and the character of the liquid flow

3. Dimensionless form of the equation system

As characteristic physical magnitudes of this flow we considered:

- the suction chamber width $B \cong H$, approximately equal with its height,
- the liquid mean velocity of the entrance in suction chamber $U_m = Q/BH$,
- the air atmospheric pressure on its free surface P_0,
- and the period T_0 of whirl appearance on the free surface.

With the new dimensionless variables:

$$x = X/B, \qquad y = Y/B, \; z = Z/H, \qquad t = T/T_0, \tag{5}$$

and functions:

$$u = U/U_m, \qquad v = V/U_m, \qquad w = W/U_m, \qquad p = P/P_0, \tag{6}$$

the partial differential equations (1) to (4) become in the dimensionless form:

$$\mathrm{Sh}\; u'_t + u'_x u + u'_y v + u'_z w + \mathrm{Eu}\; p'_x = \frac{1}{\mathrm{Re}} \triangle_{x,y,z}\, u + \frac{\sin\alpha}{\mathrm{Fr}}, \tag{1'}$$

$$\mathrm{Sh}\; v'_t + v'_x u + v'_y v + v'_z w + \mathrm{Eu}\; p'_y = \frac{1}{\mathrm{Re}} \triangle_{x,y,z}\, v, \tag{2'}$$

$$\text{Sh } w'_t + w'_x u + w'_y v + w'_z w + \text{Eu } p'_z = \frac{1}{\text{Re}} \triangle_{x,y,z} w + \frac{\cos\alpha}{\text{Fr}}, \qquad (3')$$

in which one puts into evidence the following criteria of hydrodynamic flow similarity, numbers of: Strouhal $\text{Sh} = B/T_0 U_m$, Euler $\text{Eu} = P_0/\rho U_m^2$, Reynolds $\text{Re} = BU_m/v$, Froude $\text{Fr} = U_m^2/gB$, the mass conservation equation being an invariant

$$u'_x + v'_y + w'_z = 0. \qquad (4')$$

4. The solving numerical method

The numerical integration of partial differential equation system [6] were performed by iterative calculus of unknown functions u, v, w and p, given by the algebraic relations associated to the partial differential equations, by introducing their expressions deduced from finite Taylor's series developments in a cubical grid, having the same step $\delta x = \delta y = \delta z = \chi$ [5]

$$f'_{x,y,z} = \frac{f_{1,2,3} - f_{3,4,6}}{2\chi} \text{ and } f''_{x^2,y^2,z^2} = \frac{f_{1,2,5} - 2f_0 + f_{3,4,6}}{\chi^2} \qquad (7)$$

5. Boundary conditions for three-dimensional steady flow

- **on the solid walls**, due to the molecular adhesion condition we considered

$$u_{SW} = v_{SW} = w_{SW} = 0 \qquad (8)$$

and consequently, from (3') for $\alpha \approx 0$, the hydrostatic pressure distribution

$$p_{SW}(z) = 1 + \frac{\gamma H}{P_0} z = 1 + Ar \cdot z, \qquad (9)$$

in which I introduced the **hydrostatic similarity number** $Ar = gH/P_0$, dedicated to the Archimedean lift discoverer,

- **on the free surface**, considered plane due to the liquid important weight and small flow velocities, we have the conditions

$$p(x,y,0)_{FS} = 1 \qquad and \qquad w_{FS}(x,y,0) = 0, \tag{10}$$

excepting a null set of points constituted by the whirl centres on the free surface, in whose neighbourhood the fluid has a descent motion.

Neglecting the liquid friction with the air, we shall cancel the both shearing stress components:

$$\nu_{zx}|_{SL} = u'_z + w'_x = u'_z|_{SL} = 0, \ \rightarrow \ u_6 = u_5$$

$$\tau_{zy}|_{SL} = v'_z + w'_y = v'_z|_{SL} = 0, \ \rightarrow \ v_6 = v_5 \tag{11}$$

the value w_6 being calculated from the mass conservation equation, written in finite differences and that is not unstable in this local appliance

$$\frac{u_1 - u_3}{2x} + \frac{v_2 - v_4}{2x} + \frac{w_5 - w_6}{2x} = 0, \ \rightarrow w_6|_{FS} = w_5 + u_1 - u_3 + v_2 - v_4, \tag{12}$$

- **in the entrance section**, we considered the uniformly and steady flow at the normal depth of the current parallel with the channel bottom slope $i = \sin a$, which leads us to the condition $v|_{EnS} = w|_{EnS} \equiv 0$ and to the hydrostatic pressure repartition (9), the distribution of $u(y,z)$ velocity component being obtained from the first motion equation, which in these boundary conditions become a **Poisson** type equation with the constant *a priori* unknown,written also in finite differences

$$\triangle_{x,z} u + \frac{\text{Re}}{\text{Fr}} \sin\alpha = 0, \ \rightarrow u_0 = \frac{1}{4}\sum_{i=1}^{4} u_i + \frac{a^2}{4}\frac{\text{Re}}{\text{Fr}} \sin\alpha, \tag{13}$$

whose numerical solving is possible by an iterative cycle on the computer, until to the obtaining of slope value i, necessary to the given Re and Fr numbers corresponding to the flow velocity U_m, when the heavy force component $g\sin\alpha$ is balanced by the interior friction forces, acting by the liquid adhesion to the channel bottom and its solid walls.

The boundary conditions for the Poisson equation (13) for solving in the frame of plane mixed problem **Dirichlet-Neumann**: $U_{SW} = 0$ on the solid walls and $\Im_{zx}|_{FS} = \mu U'_z = 0$ on the free surface, lead as to solve a **mixed problem Dirichlet-Neumann**, whose normalization condition

$$\int_0^B \int_0^H U_i(y,z)dydz = Q = BHU_m, \tag{14}$$

or in the iterative numerical calculus for the dimensionless case [5]

$$\chi^2 \sum_{k=2}^{k_M-1} \sum_{j=2}^{j_N-1} u_{k_M,j} + \frac{\chi^2}{2} \sum_{j=2}^{j_N-1} u_{k_M,j} + \frac{\chi^2}{2}\frac{1}{4} \sum_{j=2}^{j_N-1} u_{k=2,j} + 2\frac{\chi^2}{2}\frac{1}{4} \sum_{k=2}^{k_M-1} u_{k,j=2} = q = 1, \tag{14'}$$

represents the flow-rate which brought by this velocity repartition in the entrance section, for the given values of U_m, therefore Re and Fr numbers, as well as the considered grid step χ.

For the starting of the calculus it is necessary to introduce only a single velocity value in a point as a seed, for instance $u_S = 2$ in the point $j = 20$ and $k = 40$ in the middle of the free surface, in the rest of the domain the initial arbitrary values being equal with zero, or, by admittance of a paraboloidal repartition a the initial given arbitrary values concerning the velocity distribution in the domain, in the shape

$$u_{va}(y,z) = 8\left(1-z^2\right)\cdot\left(\frac{1}{2}-y\right)\cdot\left(\frac{1}{2}+y\right) = 8\left(1-z^2\right)\cdot\left(\frac{1}{4}-y^2\right),$$

$$u_{va}(j,k) = 4u_S \cdot a^2\,(j-1)\cdot(k-1)\cdot[2(k-1)a]\,[2(j-1)a] \tag{15}$$

because we have

$$y = (j-1)a - \frac{1}{2} \qquad \textit{and} \qquad y = 1-(k-1)a \tag{16}$$

- **in the exit section**, we shall considerate that the streamlines are parallel with the vertical pipeline walls, $u_{ExS} = v_{ExS} \equiv 0$, the velocity component $w(x,y)$ distribution in the uniformly and steady flow verifying also a Poisson's type equation with an a priori unknown constant, but this time due to the ignorance of the pressure drop on the vertical pipeline p'_z, deduced from the third motion equation (3') in approximation $\cos a \approx 1$, using and the mass conservation equation (4')

$$\text{Re Eu } p'_z = \frac{\text{Re}}{\text{Fr}}\cos\alpha + \triangle_{x,y}w, \;\rightarrow\; w_0 = \frac{1}{4}\left[\sum_{i=1}^{4} w_i + \chi^2\text{Re}\left(\frac{1}{\text{Fr}} - \text{Eu}p'_z\right)\right]. \tag{17}$$

The boundary condition to solve the Poisson equation (17) in the frame of **Dirichlet problem**, is given by the liquid adhesion $w_{ExS} \equiv 0$ on the interior solid wall of suction pipeline.

The Poisson type equation solving, in which one does not known *a priori* the pressure drop p'_z along the pipe, can be made by numerical way, also by an iterative cycle on computer, yielding up to the arbitrary values $p'_z \gtrsim 0.1$, which exceed a few the pressure distribution deduced from the hydrostatic repartition (9), until the velocity integration w_{Ex} in the exit section (fig. 1) shall equalize the unitary value of dimensionless flow-rate, calculated in the case of non-symmetrical or symmetrical supposed velocity repartitions, using the forms [5]:

$$q_{nonsym} \approx \chi^2 \cdot \left(\sum_{i=17}^{20}\sum_{j=7}^{10} w_{ij} + \frac{1}{8}\sum_{i=17.2}\sum_{j=8}^{9} w_{ij} + \frac{1}{8}\sum_{i=18}^{19}\sum_{j=7.1} w_{ij}\right) = 1,$$

$$q_{sym} \approx \chi^2 \cdot [9w_{17.8} + 4(w_{18.8} + w_{17.7})] = 1 \tag{18}$$

6. Conclusions on the used boundary specific problems in the three-dimensional case

6.1. The Dirichlet problem proposed by German mathematician Peter Gustav Lejeune Dirichlet (1805-1859) is generally specific to the equations of elliptic type, corresponding to the repartition of scalar physic magnitudes given by harmonic functions, for example of the temperature distribution in a homogeneous medium, without convection ($U = V \equiv 0$) and without interior heat sources in virtue of Laplace's equation

$$\rho c(T'_t + T'_X \cdot U + T'_Y \cdot V) = \lambda\,(T''_{X^2} + T''_{Y^2}) + q \;\rightarrow \triangle_{X,Y}T = T''_{X^2} + T''_{Y^2} = 0, \tag{19}$$

having the mean formula in finite differences

$$\frac{T_1 - 2T_0 + T_3}{\chi^2} + \frac{T_2 - 2T_0 + T_4}{\chi^2} = 0 \ \rightarrow \ T_0 = \frac{1}{4}\sum_{i=1}^{4} T_i, \tag{20}$$

6.2. For the Neumann problem formulated by the German physician Carl Gottfried Neumann (1832-1925), the specific boundary condition of this problem consists in to give the **values of the function derivative f'_n on the normal to the boundary** C of the domain D.

To specify the unknown function values in the domain, it is necessary in this case to fulfil the so-called normalization condition, giving the flux through the frontier

$$\phi = \int_C f'_n dl \tag{21}$$

In the three-dimensional case of **mixed boundary problems**, by combination of these two classical problems, to solve the Poisson equations with unknown constants, intervene the well-known normalization conditions, even for the Dirichlet problem only.

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