RESTRICTION OF STABLE BUNDLES ON A JACOBIAN OF GENUS 2 TO AN EMBEDDED CURVE

Cristian Anghel

This paper is dedicated to the memory of
Professor Gheorghe Galbură

Abstract. The aim of this note is to describe the restriction map from the moduli space of stable rank 2 bundle with small c_2 on a jacobian X of dimension 2, to the moduli space of stable rank 2 bundles on the corresponding genus 2 curve C embedded in X.

2000 *Mathematics Subject Classification*: 14D20, 14H60.

1. Introduction

Let C a smooth curve of genus 2 and X his jacobian wich is a smooth projective algebraic surface. We denote by $M_{(2,\ C,\ i)}$ for $i = 1$ *or* 2 the moduli space of rank 2 bundle on X with $c_1 = C$ and $c_2 = i$. Also we denote by $M_{(2,\ K)}$ the moduli space of rank 2 bundle on C with determinant K i.e. the canonical class of C. Obviously, for any $E \in M_{(2,\ C,\ i)}$ the restriction $E_{|C}$ is a rank 2 bundle on C with determinant K.

The natural questions wich appear are the followings: is $E_{|C}$ a stable (or at least semi-stable) bundle on C and if yes, what is the induced map $M_{(2,\ C,\ i)} \longrightarrow M_{(2,\ K)}$? As we shall see, the answer depend on i: for $i = 1$, the restriction is semi-stable, but for $i = 2$ and E generic in $M_{(2,\ C,\ 2)}$ the restriction is stable. Also, in the second case we can describe for some non-generic bundles E what is the restriction $E_{|C}$.

2. Previously known results

For X the jacobian of a genus 2 curve C, we denote by $F_0 = \mathcal{O}(C) \otimes \mathcal{J}_0$, where $\mathcal{J}_0$ is the sheaf of ideals of the origin of X. Also, using F_0 we can construct a unique extension $0 \longrightarrow \mathcal{O}_X \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow 0$ wich has $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$. The first result we need is the following, proved in [2]:

Theorem 2.1. *For any rank 2 bundle E on X with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$ there are uniques $x,\ y \in X$ such that $E \simeq {T_x}^* F_{-1} \otimes P_y$, where ${T_x}^*$ is the pull-back by the x-translation and P_y is the line bundle on X wich correspond to y by the canonical isomorphism $X \longrightarrow \widehat{X}$ defined by the principal polarisation C. As consequence the moduli space is isomorphic with $X \times X$.*

It is very easy to verify that the condition for E the have $det(E) = \mathcal{O}_X(C)$ is that $x = -2y$; so we have the following:

Remark 2.2 The moduli space of rank 2 bundles on X with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$ is isomorphic with X.

For the moduli space on C we need the following theorem proved in [3]:

Theorem 2.3 *Let F a semi-stable rank 2 bundle on C with determinant equal with the canonical class of C, and x_0 a Weierstrass point of C. Let $D_F = \{\xi \in Pic^1(C) \ \mid H^0(\xi \otimes F \otimes \mathcal{O}(-x_0)) \neq 0\}$. With these notations, D_F is a divisor of the linear system $\mid 2C \mid$ on $Pic^1(C)$ and the map $F \longrightarrow D_F$ is an isomorphism between the moduli space of rank two bundles with canonical determinant and $\mathbf{P}^3$.*

For the case $c_2 = 2$ we need the following result proved in [1] and [4]:

Theorem 2.4. *$M_{(2,\ C,\ 2)}$ is isomorphic with $X \times Hilb^3(X)$, and for any $E \in M_{(2,\ C,\ 2)}$ there exist an unique exact sequence of the form:*

$$0 \longrightarrow {T_x}^* \mathcal{O}_X(-C) \longrightarrow H \longrightarrow E \longrightarrow 0$$

where H is an homogenous rank 3 bundle on X.

By [2] a generic homogenous rank 3 bundle has the form $P_a \oplus P_b \oplus P_c$ with $a \neq b \neq c$ and it is clear that the condition for E the have $det(E) = \mathcal{O}_X(C)$ is that $x = -a - b - c$; so we have the following:

Remark 2.5. The moduli space of rank 2 bundles on X with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 2$ is birational with $Sym^3(X)$.

3. The restriction theorems

Using the previous notations we have the followings:

Theorem 3.1 *For generic $y \in X$ the restriction $E_{|C}$ of $E \simeq T_{-2y}{}^{*}F_{-1} \otimes P_y$ is semi-stable but not stable. The rational restriction map $X - - \to \mathbf{P}^3$ is the quotient by the natural involution of X and the image is the Kummer surface.*

Theorem 3.2 *For generic $E \in Hilb^3(X)$ the restriction $E_{|C}$ is stable. The restriction $E_{|C}$, viewed in $\mathbf{P}^3 = \mid 2C \mid$ is the unique divisor of $\mid 2C \mid$ wich contain the 3 points a, b, c of the corresponding H. Also, the fiber over a point $C' \in \mid 2C \mid$ is birational with $Hilb^3(C')$.*

The main idea in the proof of the previous theorems is to obtain an explicit description of $D_{E_{|C}}$ for generic E in the corresponding moduli space. In the first case for generic $y \in X$ and $E \simeq T_{-2y}{}^{*}F_{-1} \otimes P_y$ we obtain that $D_{E_{|C}}$ is the union of the two translate of C by y and $-y$. For $c_2 = 2$ and generic E, $D_{E_{|C}}$ is the hyperplane wich pass by the 3 points wich determine the homogenous bundle H associated with E by 2.3 above. The full details will appear elsewere.

References

[1] S. Mukai, *Fourier functor and its applications to the moduli of bundles on an abelian variety*, Algebraic Geometry, Sendai, 1985.

[2] S. Mukai, *Duality between $D(X)$ and $D(\widehat{X})$ with applications to Picard sheaves*, Nagoya Math. J. 81 1981 153-173.

[3] R. Narasimhan, *Moduli of vector bundles on a compact riemann surface*, Ann. of Math. 1969 14-51.

[4] H. Umemura, *Moduli spaces of stables vector bundles over abelian surfaces*, Nagoya Math. J. 71 1980 47-60.

Author:

Cristian Anghel
Department of Mathematics
Institute of Mathematics of the Romanian Academy
Calea Griviţei nr. 21 Bucureşti Romania
email:*Cristian.Anghel@imar.ro*