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THE CLASSIFICATION OF CURVES ON THE DUAL UNIT LORENTZ SPHERE

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Abstract. In this paper, curves on the dual unit Lorentz sphere were investigated and made classifications of these. And also some examples related to these were given.

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1. Introduction

Dual numbers were introduced by William Kingdon Clifford (1845-1879) as a tool for his geometrical investigations. After him, E. Study used dual numbers and dual vectors in his research on line geometry and kinematics [1]. He devoted special attention to the representation of directed lines by dual unit vectors. Recently, several researchers have been using dual quantities in their investigations concerning spatial mechanisms and kinematics. There are numerous papers on kinematics and spatial mechanisms in which dual quantities have been used [2], [3], [4].

The set of directed lines in the Euclidean 3-space $\mathbb{R}^3$ is one-to-one correspondence with the points of dual unit sphere at dual space $\mathbb{D}^3$ of triples of dual numbers. A differentiable curve on dual unit sphere S^2 of $\mathbb{D}^3$ corresponds to a ruled surface in space of lines $\mathbb{R}^3$, [1].

If we take the space of Lorentzian lines $\mathbb{L}^3$ with signature $(+,+,-)$ instead of the space of lines $\mathbb{R}^3$, E. Study's mapping can be stated as follows:

The dual time-like and space-like unit vectors of dual hyperbolic and Lorentzian unit spheres H_0^2 and S_1^2 at dual Lorentzian space are one-to-one correspondence with the directed time-like and space-like lines of the space of Lorentzian lines, respectively, [5]. According to E.Study's mapping, the geometry of some curves on unit sphere in dual Lorentz space gives the geometry of space-like or time-like ruled surfaces in $\mathbb{L}^3$. Therefore, to study the theory of curves on the unit sphere in dual Lorentz space is more practical than that of ruled surfaces in $\mathbb{L}^3$.

In this study, using E.Study's mapping, the classification of curves (or these corresponding ruled surfaces) on the dual unit sphere in $\mathbb{L}^3$-Lorentz space was defined and some examples related to them were given. In fact, the classification of ruled surfaces in $\mathbb{L}^3$ had been investigated by several researches, [6], [7], [8], [9] But, the originality of this paper lies in the classification of the curves on the dual unit Lorentz sphere.

2. Preliminaries

Let $\mathbb{L}^3$ be a three-dimensional Lorentz space, that is, the real vector space $\mathbb{R}^3$ provided with the inner product

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{\mathbb{L}} = u_1v_1 + u_2v_2 - u_3v_3$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. A vector $\overrightarrow{u} = (u_1, u_2, u_3)$ of $\mathbb{L}^3$ is said to be space-like if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{\mathbb{L}} > 0$ or $\overrightarrow{u} = 0$, time-like if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{\mathbb{L}} < 0$ and light-like or null if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{\mathbb{L}} = 0$ and $\overrightarrow{u} \neq 0$. The norm of the vector $\overrightarrow{u} \in \mathbb{L}^3$ is defined by $\|\overrightarrow{u}\|_{\mathbb{L}} = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle|}$. We also consider the time orientation as follows: Let $\overrightarrow{e} = (0, 0, 1)$. A time-like vector $\overrightarrow{u}$ is future-pointing (resp. past-pointing) if and only if $\langle \overrightarrow{u}, \overrightarrow{e} \rangle_{\mathbb{L}} < 0$ (resp. $\langle \overrightarrow{u}, \overrightarrow{e} \rangle_{\mathbb{L}} > 0$). And also a vector $\overrightarrow{u} = (u_1, u_2, u_3)$ is time-like and future-pointing if and only if $u_1^2 + u_2^2 - u_3^2 < 0$ and $u_3 > 0$, in other words, if and only if $\sqrt{u_1^2 + u_2^2} < u_3$. The cross product of two vectors $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{L}^3$ is given by

$$\overrightarrow{u} \wedge_{\mathbb{L}} \overrightarrow{v} = (u_2v_3 - v_2u_3,\ v_1u_3 - u_1v_3,\ u_2v_1 - u_1v_2).$$

Moreover, for the vectors $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in \mathbb{L}^3$ the equalities,

$$< \overrightarrow{u} \wedge_{\mathbb{L}} \overrightarrow{v}, \overrightarrow{w} >= \det(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}) \ ,$$

$$(\overrightarrow{u} \wedge_{\mathbb{L}} \overrightarrow{v}) \wedge_{\mathbb{L}} w = -\langle \overrightarrow{u}, \overrightarrow{v} \rangle \overrightarrow{w} + \langle \overrightarrow{v}, \overrightarrow{w} \rangle \overrightarrow{u}$$

are satisfied, where det denotes the usual determinant in $\mathbb{R}^3$. The hyperbolic and Lorentzian unit spheres are

$$H_0^2 = \{\overrightarrow{a} \in \mathbb{L}^3 \mid \langle \overrightarrow{a}, \overrightarrow{a} \rangle_{\mathbb{L}} = -1\}$$

and

$$S_1^2 = \{\overrightarrow{a} \in \mathbb{L}^3 \mid \langle \overrightarrow{a}, \overrightarrow{a} \rangle_{\mathbb{L}} = 1\}$$

respectively. There are two components of H_0^2 passing through $(0, 0, 1)$ and $(0, 0, -1)$ a future pointing hyperbolic sphere and past pointing hyperbolic unit sphere, and they are denoted by H_0^{2+} and H_0^{2-}, respectively, [10].

Let a regular curve $\overrightarrow{\alpha} : I \to \mathbb{L}^3,\ I \subset \mathbb{R}$, and be $\dot{\overrightarrow{\alpha}}(s)$ the tangent vector of $\overrightarrow{\alpha}(s)$, for all $s \in I$. If

i) $\left\langle \dot{\overrightarrow{\alpha}}(s), \dot{\overrightarrow{\alpha}}(s) \right\rangle > 0$ then $\overrightarrow{\alpha}$ is space-like curve,

ii) $\left\langle \dot{\overrightarrow{\alpha}}(s), \dot{\overrightarrow{\alpha}}(s) \right\rangle < 0$ then $\overrightarrow{\alpha}$ is time-like curve,

iii) $\left\langle \dot{\overrightarrow{\alpha}}(s), \dot{\overrightarrow{\alpha}}(s) \right\rangle = 0,\ \dot{\overrightarrow{\alpha}} \neq \overrightarrow{0}$, then $\overrightarrow{\alpha}$ is null curve, [11].

3. Dual Lorentz Space

We know that a dual number has the form $A = a + \varepsilon a^*$ where a and a^* are real numbers and ε stands for the dual unit which is subjected to the rules, [2].

$$\varepsilon \neq 0, \qquad 0\varepsilon = \varepsilon 0 = 0, \qquad 1\varepsilon = \varepsilon 1 = \varepsilon, \qquad \varepsilon^2 = 0 .$$

The order of two dual numbers $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ is defined as, [6]:

$$\begin{aligned} a &> b \Longrightarrow A > B;\ a < b \Longrightarrow A < B;\ \ a = b \text{ and } a^* > b^* \Longrightarrow A > B; \\ a &= b \text{ and } a^* < b^* \Longrightarrow A < B;\ \ a = a^* \text{ and } b = b^* \Leftrightarrow A = B. \end{aligned}$$

The set of all dual numbers is a ring and denoted by $\mathbb{D}$. The set of triples of dual numbers

$$\mathbb{D}^3 = \{\overrightarrow{A} = (A_1,\ A_2, A_3) \mid A_1,\ A_2,\ A_3 \in \mathbb{D}\}$$

is a module over the ring $\mathbb{D}$ which is called $\mathbb{D}$-Module or dual space. The elements of $\mathbb{D}^3$ are called as dual vectors. A dual vector can be written in the form $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$, where $\overrightarrow{a},\ \overrightarrow{a}^* \in \mathbb{R}^3$.

The Lorentzian inner product of two dual vectors $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ and $\overrightarrow{B} = \overrightarrow{b} + \varepsilon \overrightarrow{b}^*$ is defined as

$$\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle_{\mathbb{L}} = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle_{\mathbb{L}} + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b}^* \right\rangle_{\mathbb{L}} + \left\langle \overrightarrow{a}^*, \overrightarrow{b} \right\rangle_{\mathbb{L}} \right).$$

The norm of a dual vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$, $\overrightarrow{a} \neq \vec{0}$, is defined by

$$\left\| \overrightarrow{A} \right\|_{\mathbb{L}} = \| \overrightarrow{a} \|_{\mathbb{L}} + \varepsilon \frac{\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle_{\mathbb{L}}}{\| \overrightarrow{a} \|_{\mathbb{L}}}.$$

The dual vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ is dual unit vector if and only if $\| \overrightarrow{a} \|_{\mathbb{L}} = 1$ and $\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle_{\mathbb{L}} = 0$, i.e. $\left\| \overrightarrow{A} \right\|_{\mathbb{L}} = 1 \Leftrightarrow \| \overrightarrow{a} \|_{\mathbb{L}} = 1$ and $\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle_{\mathbb{L}} = 0$.

A dual vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ is said to be time-like if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle_{\mathbb{L}} < 0$, space-like if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle_{\mathbb{L}} > 0$ or $\overrightarrow{a} = \vec{0}$, and light-like (or null) if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle_{\mathbb{L}} = 0$ and $\overrightarrow{a} \neq \vec{0}$. The set of all dual vectors $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ such that $\langle \overrightarrow{a}, \overrightarrow{a} \rangle_{\mathbb{L}} = 0$ is called the dual light-like (or dual null) cone.

The time orientation can be considered as follows:
A dual time-like vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ is future pointing (resp. past pointing) if and only if $\overrightarrow{a}$ is future pointing (resp. past pointing). Therefore, we call it dual Lorentzian space the set of all dual time-like, dual space-like and dual light-like (or null) vectors provided by the structure ($\mathbb{D}^3$, $\langle , \rangle_{\mathbb{L}}$) and this space is denoted by $\mathbb{D}^3_1$. Then we define the dual hyperbolic and Lorentzian unit spheres in $\mathbb{D}^3_1$. They are defined by

$$H_0^2 = \left\{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in \mathbb{D}^3_1 \mid \left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle_{\mathbb{L}} = -1, \quad \overrightarrow{a},\ \overrightarrow{a}^* \in \mathbb{L}^3 \right\}$$

and

$$S_1^2 = \left\{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in \mathbb{D}^3_1 \mid \left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle_{\mathbb{L}} = 1, \quad \overrightarrow{a},\ \overrightarrow{a}^* \in \mathbb{L}^3 \right\}$$

respectively. There are two components of sphere H_0^2. We call the components of H_0^2 passing through $(0,0,1)$ and $(0,0,-1)$ future pointing dual hyperbolic unit sphere and past pointing dual hyperbolic unit sphere and denoted by H_0^{2+} and H_0^{2-}, respectively, [5]. With respect to this definition, we can rewrite:

$$\begin{aligned} H_0^{2+} &= \{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in H_0^2 \mid \overrightarrow{a} \text{ is a future pointing vector} \}, \\ H_0^{2-} &= \{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in H_0^2 \mid \overrightarrow{a} \text{ is a past pointing vector} \}. \end{aligned}$$

As in the case of the space $\mathbb{L}^3$, we define dual Lorentzian cross product of dual vectors $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ and $\overrightarrow{B} = \overrightarrow{b} + \varepsilon \overrightarrow{b}^*$ by

$$\overrightarrow{A} \wedge \overrightarrow{B} = \overrightarrow{a} \wedge_{\mathbb{L}} \overrightarrow{b} + \varepsilon \left(\overrightarrow{a} \wedge_{\mathbb{L}} \overrightarrow{b}^* + \overrightarrow{a}^* \wedge_{\mathbb{L}} \overrightarrow{b} \right)$$

Let a regular dual unit spherical curve be $\overrightarrow{\Psi} : I \subset \mathbb{R} \to \mathbb{D}^3_1$,

$\overrightarrow{\Psi}(s) = \vec{\psi}(s) + \varepsilon \vec{\psi}^*(s)$, for all $s \in I$. If

i) $\left\langle \dot{\vec{\psi}}(s), \dot{\vec{\psi}}(s) \right\rangle > 0$ then $\overrightarrow{\Psi}$ is said to be space-like dual curve,

ii) $\left\langle \dot{\vec{\psi}}(s), \dot{\vec{\psi}}(s) \right\rangle < 0$ then $\overrightarrow{\Psi}$ is said to be time-like dual curve,

iii) $< \dot{\vec{\psi}}(s), \dot{\vec{\psi}}(s) >= 0$, $\dot{\vec{\psi}} \neq 0$ and $\dot{\vec{\psi}}^* \neq 0$, then $\overrightarrow{\Psi}$ is said to be dual null curve.

Theorem 1. *There is one to one correspondence between directed space-like (or time-like) lines of* $\mathbb{R}^3_1$ *and ordered pair of vectors* $(\overrightarrow{a}, \overrightarrow{a}_0)$ *such that* $\langle \overrightarrow{a}, \overrightarrow{a}_0 \rangle_{\mathbb{L}} = 1$ *(or* $\langle \overrightarrow{a}, \overrightarrow{a}_0 \rangle_{\mathbb{L}} = -1$*) and* $\langle \overrightarrow{a}, \overrightarrow{a}_0 \rangle_{\mathbb{L}} = 0$*, [5].*

4. Dual Lorentzian Spherical Curves

A time-like ruled surface is defined as a surface generated by the motion of straight time-like line. Similarly, a space-like ruled surface is defined as a surface generated by the motion of a straight space-like line in $\mathbb{L}^3$. Using E. Study's mapping for the elements of Lorentzian spaces $\mathbb{D}_1^3$ and $\mathbb{L}^3$, space-like (resp. time-like) ruled surfaces are represented by a dual space-like (resp. time-like) unit vector $\overrightarrow{X}(t) = \overrightarrow{x}(t) + \varepsilon \overrightarrow{x}^*(t)$, $t \in I \subset \mathbb{R}$. Here the vector $\overrightarrow{x}$ is direction vector of the line and the vector $\overrightarrow{x}^*$ is usually called the moment of the line (with respect to origin O). According to E. Study's mapping, there exists a one-to-one correspondence between the set of all directed lines in $\mathbb{L}^3$ -space and the set of all dual unit vectors in $\mathbb{D}_1^3$ -space. So, a differentiable curve on dual unit sphere in $\mathbb{D}_1^3$ corresponds to a ruled surface in space of lines $\mathbb{L}^3$ and this correspondence is one-to-one (figure 1). Moreover, the drall of the orbit surface of X-dual curve is

$$\Delta_X = \frac{\langle dx, dx^* \rangle}{\langle dx, dx \rangle}$$

And also, a developable trajectory ruled surface is characterized by

$$\Delta_X = 0, \ [4]. \tag{1}$$

Therefore the terms dual Lorentzian unit spherical curve, time-like and space-like ruled surfaces are synonymous in this study.

Theorem 2. *The ruled surface corresponding to differentiable dual curve $\overrightarrow{X}(t) = \overrightarrow{x}(t) + \epsilon \overrightarrow{x}^*(t)$, $t \in I \subset \mathbb{R}$, on the dual unit Lorentz sphere can be given by equation*

$$\overrightarrow{X}(t, \lambda) = \overrightarrow{x}(t) \wedge_{\mathbb{L}} \overrightarrow{x}^*(t) + \lambda \overrightarrow{x}(t).$$

Proof. The proof is made similarly as in $\mathbb{R}^3$. Namely, let a subset $M \subset \mathbb{R}^3$ be a regular surface. i.e. provided that for each point $p \in M$ there exist a neighborhood V of p in $\mathbb{R}^3$ and a map $\overrightarrow{X} : U \to \mathbb{R}^3$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap M$ such that:

(i) $\overrightarrow{X}$ is differentiable,

(ii) $\overrightarrow{X} : U \to V \cap M$ is a homeomorphism. This means that $\overrightarrow{X}$ has a continuous inverse $\overrightarrow{X}^{-1} : V \cap M \to U$ such that $\overrightarrow{X}^{-1}$ is the restriction to $V \cap M$ of a continuous map $\overrightarrow{F} : W \to \mathbb{R}^2$, where W is an open subset of $\mathbb{R}^3$ that contains $V \cap M$,

(iii) each map $\overrightarrow{X} : U \to M$ is a regular patch.

Therefore, a ruled surface M in $\mathbb{R}^3$ is a regular surface that has a parametrization $\overrightarrow{X} : U \to M$ of the form

$$\Omega : \overrightarrow{X}(t, v) = \overrightarrow{a}(t) + v \overrightarrow{x}(t) \tag{2}$$

where $\overrightarrow{a}$ and $\overrightarrow{x}$ are curves in $\mathbb{R}^3$ with $\dot{a}$ never $\vec{0}$. The curve $\overrightarrow{a}$ is called the directrix or base curve of the ruled surface, and $\overrightarrow{x}$ is called the director curve. The rulings of the ruled surface are the straight lines $v \to \overrightarrow{a}(t) + v\overrightarrow{x}(t)$ (Figure 1), [12].

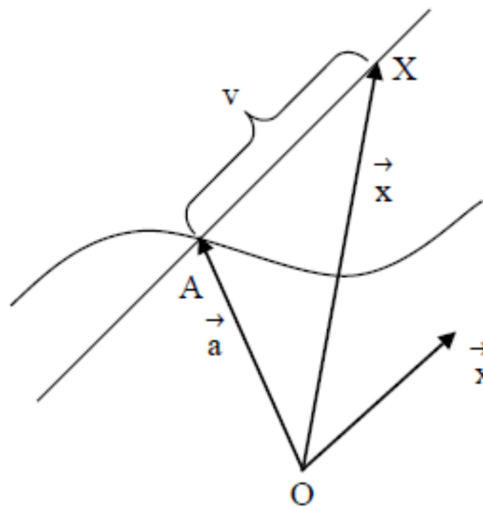

Figure 1:

Since $\overrightarrow{x}^*(t) = \overrightarrow{a}(t) \wedge_{\mathbb{L}} \overrightarrow{x}(t)$, we may solve $\overrightarrow{a}(t)$ as

$$\overrightarrow{a}(t) = \overrightarrow{x}(t) \wedge_{\mathbb{L}} \overrightarrow{x}^*(t) + \mu\overrightarrow{x}(t)$$

here μ is a real scalar. Hence Eq. (2) becomes

$$\overrightarrow{X}(t,\lambda) = \overrightarrow{x}(t) \wedge_{\mathbb{L}} \overrightarrow{x}^*(t) + \lambda\overrightarrow{x}(t), \ (v + \mu = \lambda).$$

If we assume $\overrightarrow{X}(t) = \overrightarrow{x}(t) + \varepsilon\overrightarrow{x}^*(t), \ t \in I \subset \mathbb{R}$ a dual curve on unit sphere in $\mathbb{D}_1^3$, the following assumptions can be made. First of all, we consider that $\overrightarrow{X}$ space-like curve or time-like curve. In this case, we can give eight different kinds of ruled surface corresponding to $\overrightarrow{X}(t) = \overrightarrow{x}(t) + \varepsilon\overrightarrow{x}^*(t), \ t \in I \subset \mathbb{R}$, as follows:
If $\overrightarrow{X}(t)$ is space-like dual curve or time-like dual curve, then the ruled surface Ω corresponding to unit dual curve $\overrightarrow{X}(t)$ is said to be of type Ω_+ or type Ω_-, respectively. Also, the ruled surface of type Ω_+ can be divided into three types. In this case, if

i) $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^* \right\rangle > 0$ then Ω is said to be of type Ω_+^1,

ii) $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^* \right\rangle = 0$ then Ω is said to be of type Ω_+^2,

iii) $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^{*} \right\rangle < 0$ then Ω is said to be of type Ω_{+}^{3}.

For the ruled surface of type Ω_{-}, if

i) $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^{*} \right\rangle > 0$ then Ω is said to be of type Ω_{-}^{1},

ii) $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^{*} \right\rangle = 0$ then Ω is said to be of type Ω_{-}^{2}.

iii) $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^{*} \right\rangle < 0$ then Ω is said to be of type Ω_{-}^{3}

In other words, if $\left\langle \dot{\vec{x}}, \dot{\vec{x}} \right\rangle = 0$ and $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^{*} \right\rangle > 0$ or $\left\langle \dot{\vec{x}}, \dot{\vec{x}}^{*} \right\rangle < 0$ then Ω is said to be of type Ω^{4}. Finally, if $\overrightarrow{X}(t)$ is light-like (or null) curve then the ruled surface Ω is called a null scroll.

Note that the ruled surfaces Ω_{+}^{2} and Ω_{-}^{2} become developable surfaces.

By using Theorem 2 and above definitions, we can give the following examples.

5. Some Examples

Example 5.1 The dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = \left(s \sin s, s \cos s, \sqrt{s^2+1}\right) + \varepsilon \left(\sqrt{s^2+1} \sin s, \sqrt{s^2+1} \cos s, s\right)$$

is a ruled surface of type Ω_{+} in $\mathbb{L}^3$. For this surface, if $s > 0$ then Ω is a ruled surface of type Ω_{+}^{1} (Figure 2) and if $s < 0$ then Ω is a ruled surface of type Ω_{+}^{3} (Figure 3).

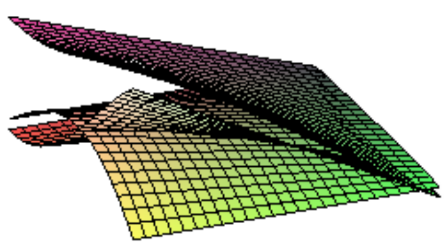

Figure 2:

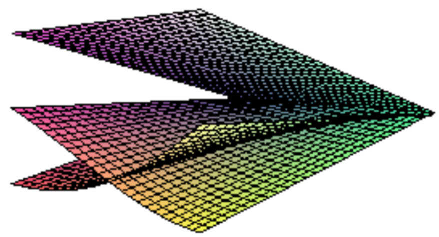

Figure 3:

Example 5.2 The unit dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = \left(\sqrt{s^2+1}, s, \sqrt{2}s\right) + \varepsilon\left(\sqrt{s^2+1}, s - \frac{1}{s}, \sqrt{2}s\right),\ s \neq 0$$

is a ruled surface of type Ω_-^1 in $\mathbb{L}^3$ (Figure 4).

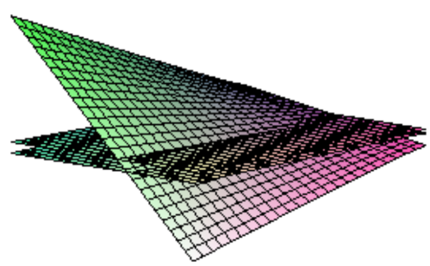

Figure 4:

Example 5.3 The unit dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = (s, 1, s) + \varepsilon(-s, 0, -s)$$

is a null scroll (Figure 5).

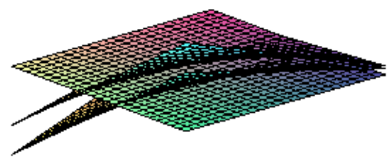

Figure 5:

Example 5.4 The unit dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{Y}(s) = \left(s, \sqrt{2s^2+1}, \sqrt{3}s\right) + \varepsilon\left(\sqrt{3}s, 0, s\right),$$

is a ruled surface of type Ω_-^3 (Figure 6).

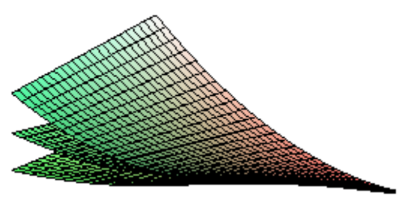

Figure 6:

Example 5.5 For a smooth function $\phi(s)$, the dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = (\sin s, \cos s, 0) + \varepsilon(0, 0, \phi(s))$$

is a ruled surface of type Ω_+^2 (Figure 7).

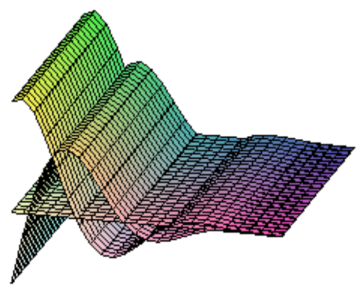

Figure 7:

Example 5.6 For a smooth function $\varphi(s)$, the unit dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = (\sinh s, 0, \cosh s) + \varepsilon(0, \varphi(s), 0)$$

is a ruled surface of type Ω_+^2 in $\mathbb{L}^3$ (Figure 8).

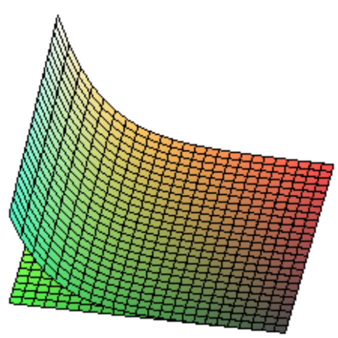

Figure 8:

Example 5.7 For a smooth function $\gamma(s)$, the unit dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = (0, \cosh s, \sinh s) + \varepsilon(\gamma(s), 0, 0)$$

is a ruled surface of type Ω_-^2 in $\mathbb{L}^3$ (Figure 9).

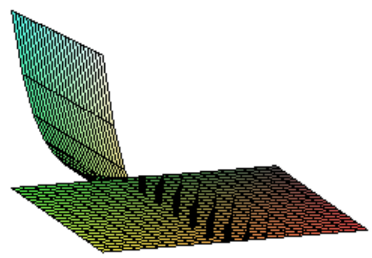

Figure 9:

Example 5.8 The unit dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = (s, 1, s) + \varepsilon(0, s^2, s)$$

is a ruled surface of type Ω^4 (Figure 10).

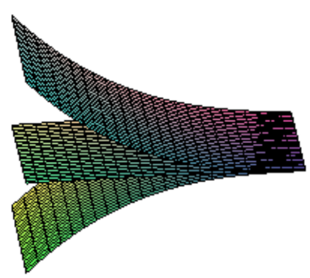

Figure 10:

Example 5.9 The unit dual spherical curve in $\mathbb{D}_1^3$ defined by

$$\overrightarrow{X}(s) = (s, 1, s) + \varepsilon \left(s, s^2, 0\right)$$

is a ruled surface type of Ω^4 (Figure 11).

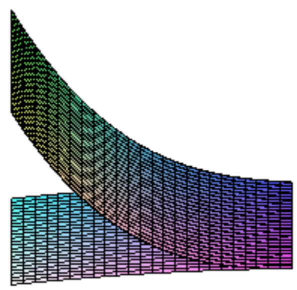

Figure 11:

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