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COMMON BEST PROXIMITY POINT AND BEST PROXIMITY POINT FOR CYCLIC CONTRACTIONS AND SEMI CYCLIC CONTRACTIONS ON PROBABILISTIC MENGER SPACES

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Abstract. In this paper we introduce the concepts of cyclic contraction, semi cyclic contraction pair, reverse cyclic contraction and property UC in probabilistic Menger spaces. We present some common best proximity point and best proximity point results for cyclic contractions and semi cyclic contraction pairs in probabilistic Menger spaces and as a result, we prove probabilistic version of classical Banach contraction principle with a new method.

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1. Introduction and Preliminaries

Fixed point theory is a method for finding a solution to nonlinear equation $Tx = x$ for mapping $T : A \to A$, where A is a subset of a metric space, a normed linear space, a topological vector space. When T is a nonself mapping, the equation $Tx = x$ has no solution. In this case, we try to detect an element x that is close proximity to Tx. In fact, best approximation theorems and best proximity point theorems are applied for finding this element. In 1969, best proximity point theory was introduced by Fan [6]. After it became a field of active research and it was studied by many scholars, including Prolla [21], Reich [23] and Sehgal and Singh [29, 30]. In 2010, Sadiq Basha obtained best proximity point theorem for contractions [25]. Best proximity point theorems for relatively nonexpansive mappings and contractive mappings have been studied in [1, 24]. Kirk et al. [11] introduced the concept of the cyclic contraction mapping in metric space and they also established fixed point results for such mappings. In 2010, Pacurar and Rus [19] obtained some results about fixed point theorems for cyclic φ-contractions. Also Karpagam and Agrawal [10] proved some best proximity point theorems for cyclic orbital Meir–Keeler contraction mappings. (see also [5, 12, 17, 20]). Gabeleh and Abkar [7], proved semi cyclic contractive pairs of mappings in Banach spaces have best proximity points. Some results of best proximity points for semi cyclic φ-contraction pair of mappings are obtained in [37]. For more results in this area see [2, 31]. The notation of UC property was introduced by Suzuki et al. in [36]. They extend the Eldred and Veeramani theorem to metric spaces with the property UC.

Probabilistic metric space (abbreviated, PM space) that is one of the generalization of metric space, introduced and studied by Karl Menger [13]. Schweizer and Sklar studied the properties of spaces introduced by Menger, they gave some results about topology, convergence of sequences and completeness of these spaces [26, 27]. Afterward, PM space was developed in various directions. Sehgal and Bharucha-Reid [28], were given the first result on fixed point theory in PM spaces. After that many scholars studied various types of contractions and related fixed point theorems in PM spaces (for example, see [9, 18, 22]). Su and Zhang [35], proved some theorems about existence of best proximity points in PM spaces. Lately Shayanpour et al. achieved some best proximity point theorems for proximal contraction and proximal nonexpansive mappings in probabilistic Banach spaces [34]. For further existence results, we refer to [32, 33].

A mapping $F : [-\infty, \infty] \to [0, 1]$ is called a distribution function if F is a nondecreasing and left continuous function and $F(-\infty) = 0$ and $F(\infty) = 1$. Let Γ^+ be the set of all the distribution functions such that $F(0) = 0$. The set of all $F \in \Gamma^+$ for which $\lim_{t\to\infty} F(t) = 1$ will be denoted by D^+. With usual pointwise ordering of functions, the spaces Γ^+ and D^+ are partially ordered and $\epsilon_0 = \chi_{(0,\infty)}$ is a maximal element of them.

Let X be a nonempty set and $F : X \times X \to \Gamma^+$ $(F(x, y) = F_{x,y})$ be a mapping such that

(PM1) $F_{x,y} = \epsilon_0$, iff $x = y$,

(PM2) $F_{x,y} = F_{y,x}$,

(PM3) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$,

for every $x, y, z \in X$ and $t, s \geq 0$. Then the pair (X, F) is called a probabilistic metric space.

For definitions of triangular norm, probabilistic Menger space, complete probabilistic Menger space, etc. and known results one can see [26, 8].

Throughout this paper, (X, F, Δ) is a probabilistic Menger space such that $\mathrm{Ran}F \subseteq D^+$ and $\Delta_m(a, b) = \min\{a, b\}$ show the minimum t-norm.

To prove our theorems in this paper we need the following simple Lemma. [32]

Let (X, F) be a PM space. If there exists $q \in (0,1)$ such that for all $t > 0$, $F_{x,y}(qt) \geq F_{z,w}(t) \geq F_{x,y}(t)$ where $x, y, z, w \in X$, then $x = y$ and $z = w$.

[38] If (X, F, Δ) is a complete probabilistic Menger space, then (X^2, M, Δ) is also a complete probabilistic Menger space, where for every $(x, y), (u, v) \in X^2$ and $t \geq 0$

$$M_{(x,y),(u,v)}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\}.$$

[3, 2.5.3] If (X, F, Δ) is a probabilistic Menger space with continuous t-norm Δ, then the probabilistic distance function F is a low semi continuous function of points, that is, for any fixed point $t > 0$, if $x_n \to x$ and $y_n \to y$, then

$$\liminf_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Definition 1. *Let (X, F, Δ) be a probabilistic Menger space and $T : X \to X$ be a mapping. The mapping T is continuous at a point $x \in X$ if for every sequence (x_n) in X, which converges to x, the sequence (Tx_n) in X converges to Tx.*

Definition 2. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). A self mapping T on $A \cup B$ is said to be cyclic if $T(A)$ is a subset of B and $T(B)$ is a subset of A.*

Definition 3. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). A cyclic self mapping T on $A \cup B$ is said to be cyclic contraction if there exists $0 < a < 1$ such that*

$$F_{Tx,Ty}(t) \geq \Delta(F_{x,y}(\frac{t}{a}), F_{A,B}(t)), \tag{1}$$

for all $x \in A, y \in B$ and all $t > 0$, where $F_{A,B}(t) = \sup_{x\in A, y\in B} F_{x,y}(t)$, which is called the probabilistic distance of A and B.

Let $X = [0,1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for any $x, y \in X$ and $t > 0$, then with a simple calculation one can see that (X, F, Δ_m) is a probabilistic Menger space. Let $A = [0, \frac{1}{2}]$, $B = [\frac{1}{3}, 1]$ and define

$$Tx = \begin{cases} \frac{1}{3}, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} - \frac{1}{3}x, & \frac{1}{2} < x \leq 1. \end{cases}$$

It is clear that T is a cyclic mapping. If $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{3}, \frac{1}{2}]$, then $Tx = Ty = \frac{1}{3}$

and $F_{Tx,Ty}(t) = 1$ for any $t > 0$, hence (1) holds. If $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$, then

$$F_{Tx,Ty}(t) = \frac{t}{t + |Tx - Ty|} = \frac{t}{t + |\frac{1}{3} - \frac{1}{2} + \frac{1}{3}y|} \geq \frac{t}{t + \frac{1}{3}|y - x|}$$
$$= F_{x,y}\left(\frac{t}{\frac{1}{3}}\right)$$
$$\geq \min\{F_{x,y}(\frac{t}{\frac{1}{3}}), F_{A,B}(t)\},$$

for any $t > 0$, hence T is a cyclic contraction.

Definition 4. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). A cyclic self mapping T on $A \cup B$ is said to be reverse cyclic contraction if there exists $0 < a < 1$ such that*

$$F_{x,y}(t) \geq \Delta(F_{Tx,Ty}(\frac{t}{a}), F_{A,B}(t)),$$

for all $x \in A, y \in B$ and all $t > 0$.

Let $X = \mathbb{R}$, for every $x, y \in X$ and $t > 0$, define $F_{x,y}(t) = \frac{t}{t+|x-y|}$, then with a simple calculation one can see that (X, F, Δ_m) is a probabilistic Menger space. Let $A = [0, \infty)$, $B = (-\infty, 0]$ and define $T : A \cup B \to A \cup B$ as

$$Tx = \begin{cases} -x^3 - 2x, & x \geq 0, \\ -2x, & x \leq 0. \end{cases}$$

For every $x \in A$, $y \in B$ and $t > 0$ we have

$$F_{x,y}(t) = \frac{t}{t + |x - y|} \geq \frac{t}{t + \frac{1}{2}|Tx - Ty|}$$
$$= F_{Tx,Ty}(\frac{t}{\frac{1}{2}}) \geq \min\{F_{Tx,Ty}(\frac{t}{\frac{1}{2}}), F_{A,B}(t)\},$$

therefore T is reverse cyclic contraction.

Definition 5. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). The pair (A, B) satisfies the property UC if for all sequences (x_n) and (x'_n) in A and for any sequence (y_n) in B, $\lim_{n\to\infty} F_{x_n,x'_n}(t) = 1$ whenever $\lim_{n\to\infty} F_{x_n,y_n}(t) = F_{A,B}(t)$ and $\lim_{n\to\infty} F_{x'_n,y_n}(t) = F_{A,B}(t)$, for all $t > 0$.*

Definition 6. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). Let S, T be two cyclic self mappings on $A \cup B$. The cyclic mapping S is said to be a T-cyclic contraction if there exists $0 < a < 1$ such that*

$$F_{Sx,Sy}(t) \geq \Delta(F_{Tx,Ty}(\frac{t}{a}), F_{A,B}(t)),$$

for all $x \in A, y \in B$ and all $t > 0$.

Let $X = \mathbb{R}^2$ and $F_{(x,y),(u,v)}(t) = \frac{t}{t+|x-u|+|y-v|}$ for any $t > 0$ and $(x, y), (u, v) \in X$, then it is easy to see that (X, F, Δ_m) is a probabilistic Menger space. Define $A = \{(x, y) : x \leq 0, y \in \mathbb{R}\}$, $B = \{(x, y) : x \geq 1, y \in \mathbb{R}\}$ and

$$T(x,y) = \begin{cases} (3, 1 + x^4 - y^4), & (x,y) \in A, \\ (0, 1 + x^4 - y^3), & (x,y) \in B, \end{cases} \qquad S(x,y) = \begin{cases} (1, \frac{x^4-y^4}{4}), & (x,y) \in A, \\ (0, \frac{x^4-y^3}{4}), & (x,y) \in B. \end{cases}$$

Clearly $F_{A,B}(t) = \frac{t}{t+1}$, for any $t > 0$ and T, S are cyclic self mappings. For any $(x, y) \in A$ and $(u, v) \in B$ we have

$$\begin{aligned} F_{S(x,y),S(u,v)}(t) &= \frac{t}{t + 1 + |\frac{x^4-y^4}{4} - \frac{u^4-v^3}{4}|} \\ &= \frac{t}{t + \frac{1}{4}(3 + |x^4 - y^4 - u^4 + v^3|) + \frac{1}{4} \times 1} \\ &= \frac{4t}{4t + 3 + |x^4 - y^4 - u^4 + v^3| + 1} \\ &\geq \min\left\{ \frac{2t}{2t + 3 + |x^4 - y^4 - u^4 + v^3|}, \frac{2t}{2t+1} \right\} \\ &\geq \min\left\{ \frac{2t}{2t + 3 + |x^4 - y^4 - u^4 + v^3|}, \frac{t}{t+1} \right\} \\ &= \min\left\{ F_{T(x,y),T(u,v)}\left(\frac{t}{\frac{1}{2}}\right), F_{A,B}(t) \right\}, \end{aligned}$$

for every $t > 0$, so S is a T-cyclic contraction.

Definition 7. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ) and S, T be two self mappings on $A \cup B$. The pair (S, T) is semi cyclic contraction if $S(A) \subseteq B$, $T(B) \subseteq A$ and there exists $0 < a < 1$ such that*

$$F_{Sx,Ty}(t) \geq \Delta(F_{x,y}(\frac{t}{a}), F_{A,B}(t)), \tag{2}$$

for all $x \in A, y \in B$ and all $t > 0$.

Let $X = [0,1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$, for every $x, y \in X$ and $t > 0$. Then it is easy to see that (X, F, Δ_m) is a probabilistic Menger space. Let $A = [0, \frac{1}{2}]$ and $B = [\frac{1}{3}, 1]$. Define $S, T : A \cup B \to A \cup B$ by

$$S(x) = \begin{cases} \frac{1}{2} - \frac{1}{3}x, & x \in A, \\ 1, & x \in B, \end{cases} \qquad T(x) = \begin{cases} 0, & x \in A, \\ \frac{1}{2} - \frac{1}{3}x, & x \in B. \end{cases}$$

Clearly $S(A) \subseteq B$, $T(B) \subseteq A$, $S(B) \nsubseteq A$ and $T(A) \nsubseteq B$ so neither S nor T is cyclic. Also for every $x \in A, y \in B$ and $t > 0$ we have

$$F_{Sx,Ty}(t) = \frac{t}{t + \frac{1}{3}|x-y|} = F_{x,y}(\frac{t}{\frac{1}{3}}) \geq \min\{F_{x,y}(\frac{t}{\frac{1}{3}}), F_{A,B}(t)\}.$$

Hence the pair (S, T) is semi cyclic contraction. When $S = T$, a semi cyclic contraction pair is a cyclic contraction. Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ_m) and $T : A \cup B \to A \cup B$ be reverse cyclic contraction, Then T is a T^3-cyclic contraction.

Proof. Since T is reverse cyclic contraction, then for some $0 < a < 1$, for all $x \in A, y \in B$ and all $t > 0$ we have

$$\begin{aligned} F_{Tx,Ty}(t) &\geq \min\left\{F_{T^2x,T^2y}(\frac{t}{a}), F_{A,B}(t)\right\} \\ &\geq \min\left\{\min\left\{F_{T^3x,T^3y}\left(\frac{t}{a^2}\right), F_{A,B}\left(\frac{t}{a}\right)\right\}, F_{A,B}(t)\right\} \\ &= \min\left\{F_{T^3x,T^3y}\left(\frac{t}{a^2}\right), \min\left\{F_{A,B}\left(\frac{t}{a}\right), F_{A,B}(t)\right\}\right\} \\ &\geq \min\left\{F_{T^3x,T^3y}\left(\frac{t}{a^2}\right), F_{A,B}(t)\right\}. \end{aligned}$$

Therefore T is a T^3-cyclic contraction.

Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ_m) and $T : A \cup B \to A \cup B$ be a cyclic contraction. Then T^3 is a T-cyclic contraction.

Proof. Since T is cyclic contraction, then for some $0 < a < 1$, for all $x \in A, y \in B$

and all $t > 0$ we have

$$\begin{aligned} F_{T^3x,T^3y}(t) &\geq \min\left\{F_{T^2x,T^2y}(\frac{t}{a}), F_{A,B}(t)\right\} \\ &=\geq \min\left\{\min\left\{F_{Tx,Ty}\left(\frac{t}{a^2}\right), F_{A,B}\left(\frac{t}{a}\right)\right\}, F_{A,B}(t)\right\} \\ &= \min\left\{F_{Tx,Ty}\left(\frac{t}{a^2}\right), \min\left\{F_{A,B}\left(\frac{t}{a}\right), F_{A,B}(t)\right\}\right\} \\ &\geq \min\left\{F_{Tx,Ty}\left(\frac{t}{a^2}\right), F_{A,B}(t)\right\}, \end{aligned}$$

therefore T^3 is a T-cyclic contraction.

Definition 8. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). A cyclic self mapping T on $A \cup B$ is relatively continuous at a point $x \in A$ if for any $\epsilon \in (0, 1)$ and $t > 0$, there exists a $\delta \in (0, 1)$ such that*

$$F_{A,B}(t) < F_{x,y}(t) + \delta \ \Rightarrow \ F_{A,B}(t) < F_{Tx,Ty}(t) + \epsilon,$$

for any $y \in B$.

In the same way, we can define relatively continuous at a point in B. A cyclic mapping T is relatively continuous if it is relatively continuous at each point of its domain. Let $X = \mathbb{R}$, $A = B = [1, +\infty)$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for any $x, y \in X$ and $t > 0$. Then it is easy to see that (X, F, Δ_m) is a probabilistic Menger space and for any $t > 0$, $F_{A,B}(t) = 1$. Now define $T : [1, +\infty) \to [1, +\infty)$ by $Tx = 1 + \ln x$. If $\epsilon > 0$, $x \in [1, +\infty)$, $t > 0$, $\delta \leq \epsilon$ and for each $y \in B$ that

$$1 = F_{A,B}(t) < F_{x,y}(t) + \delta = \frac{t}{t + |x - y|} + \delta,$$

then

$$F_{Tx,Ty}(t) + \epsilon = \frac{t}{t + |\ln x - \ln y|} + \epsilon \geq \frac{t}{t + |x - y|} + \delta > 1.$$

So T is a relatively continuous. Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ_m) and $x \in A$, then a cyclic self mapping T on $A \cup B$ is relatively continuous at x iff $F_{Tx,Ty_n}(t) \to F_{A,B}(t)$, for any sequence $(y_n) \subseteq B$ and any $t > 0$ such that $F_{x,y_n}(t) \to F_{A,B}(t)$.

Proof. Let T be relatively continuous at x and (y_n) be a sequence in B and $t_0 > 0$ such that $F_{x,y_n}(t_0) \to F_{A,B}(t)$. We show that $F_{Tx,Ty_n}(t) \to F_{A,B}(t)$. Suppose $\epsilon > 0$,

by assumptions there exists a $\delta \in (0,1)$ such that for any $y \in B$ if $F_{A,B}(t) < F_{x,y}(t) + \delta$ then $F_{A,B}(t) < F_{Tx,Ty}(t) + \epsilon$. Since $F_{x,y_n}(t) \to F_{A,B}(t)$, so there exists a $N \in \mathbb{N}$ such that for any $n \geq N$, $F_{A,B}(t) < F_{x,y_n}(t) + \delta$. Therefore $F_{A,B}(t) < F_{Tx,Ty_n}(t) + \epsilon$ for any $n \geq N$, so $F_{Tx,Ty_n}(t) \to F_{A,B}(t)$.
Now suppose that for any sequence $(y_n) \subseteq B$ and any $t > 0$ if $F_{x,y_n}(t) \to F_{A,B}(t)$, then $F_{Tx,Ty_n}(t) \to F_{A,B}(t)$. We show that T is relatively continuous at x. Arguing by contradiction, we assume that T is not relatively continuous at x. Then there exist ϵ and t, such that for any $n \in \mathbb{N}$, there exists a $y_n \in B$, that $F_{A,B}(t) < F_{x,y_n}(t) + \frac{1}{n}$ and $F_{A,B}(t) \geq F_{Tx,Ty_n}(t) + \epsilon$. It is clear that $\lim\limits_{n\to\infty} F_{x,y_n}(t) = F_{A,B}(t)$ and $\lim\limits_{n\to\infty} F_{Tx,Ty_n}(t) \neq F_{A,B}(t)$, which is a contradiction. Therefore T is relatively continuous at x.

Let $X = \mathbb{R}$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$, then it is easy to see that (X, F, Δ_m) is a complete probabilistic Menger space, so by Lemma 1, $(\mathbb{R}^2, M, \Delta_m)$ is also a complete probabilistic Menger space, where

$$M_{(x,y),(u,v)}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\},$$

for every $(x,y),(u,v) \in \mathbb{R}^2$ and $t > 0$. Define $A = \{(x,1) : 0 \leq x \leq 1\}$, $B = \{(2,y) : -1 \leq y \leq 0\}$ and $T : A \cup B \to A \cup B$ by

$$T(x,1) = \begin{cases} (2, x-1), & x \in [0,1] \cap \mathbb{Q}, \\ (2,0), & x \in [0,1] \cap \mathbb{Q}^c, \end{cases} \qquad T(2,y) = (1+y, 1).$$

Clearly

$$M_{A,B}(t) = \sup_{\substack{0 \leq x \leq 1 \\ -1 \leq y \leq 0}} \min\{\frac{t}{t+|x-2|}, \frac{t}{t+|1-y|}\} = \frac{t}{t+1}.$$

It is easy to see that T is not continuous, while by using Lemma 1, we can show that T is relatively continuous. Let $(x,1) \in A$ and $(2, y_n) \in B$ such that

$$\lim_{n\to\infty} M_{(x,1),(2,y_n)}(t) = \min\{\frac{t}{t+|x-2|}, \lim_{n\to\infty} \frac{t}{t+|1-y_n|}\} = \frac{t}{t+1} = M_{A,B}(t),$$

if $x \in [0,1] \cap \mathbb{Q}$, then we have

$$M_{T(x,1),T(2,y_n)}(t) = \min\{\frac{t}{t+|1-y_n|}, \frac{t}{t+|x-2|}\} = \frac{t}{t+1} = M_{A,B}(t),$$

if $x \in [0,1] \cap \mathbb{Q}^c$, then we have

$$M_{T(x,1),T(2,y_n)}(t) = \min\{\frac{t}{t+|1-y_n|}, \frac{t}{t+1}\} = \frac{t}{t+1} = M_{A,B}(t),$$

therefore T is relatively continuous.

Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). If T and S are two self mappings on $A \cup B$ such that T is relatively continuous mapping and S is a T-cyclic contraction, then S is relatively continuous.

Proof. Let $x \in A$, we show that if (y_n) is a sequence in B and $t > 0$ such that $F_{x,y_n}(t) \to F_{A,B}(t)$, then $F_{Sx,Sy_n}(t) \to F_{A,B}(t)$. Since S is a T-cyclic contraction, so there exists $0 < a < 1$ such that

$$F_{A,B}(t) = \lim_{n\to\infty} F_{x,y_n}(t) = \liminf_{n\to\infty} F_{x,y_n}(t) \leq \liminf_{n\to\infty} F_{x,y_n}\left(\frac{t}{a}\right).$$

By hypothesis, we have

$$F_{Sx,Sy_n}(t) \geq \min\left\{F_{Tx,Ty_n}(\frac{t}{a}), F_{A,B}(t)\right\}$$

Now taking lim inf as $n \to \infty$, we get

$$\begin{aligned}\liminf_{n\to\infty} F_{Sx,Sy_n}(t) &\geq \min\left\{\liminf_{n\to\infty} F_{Tx,Ty_n}\left(\frac{t}{a}\right), F_{A,B}\left(t\right)\right\}\\ &\geq \min\left\{F_{A,B}(t), F_{A,B}\left(t\right)\right\}\\ &= F_{A,B}(t).\end{aligned}$$

Hence

$$F_{A,B}(t) \geq \limsup_{n\to\infty} F_{Sx,Sy_n}(t) \geq \liminf_{n\to\infty} F_{Sx,Sy_n}(t) \geq F_{A,B}(t),$$

Therefore $\lim_{n\to\infty} F_{Sx,Sy_n}(t) = F_{A,B}(t)$.

Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). If $T : A \cup B \to A \cup B$ is a cyclic contraction, then T is relatively continuous.

Definition 9. *Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ). An element $x \in A$ is called a common best proximity point of the mappings $S, T : A \to B$, if*

$$F_{x,Sx}(t) = F_{A,B}(t) = F_{x,Tx}(t),$$

for all $t > 0$.

Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ_m) such that the pairs (A, B) and (B, A) satisfy the property UC. Let S, T be cyclic self mappings on $A \cup B$ and S be a T-cyclic contraction. If there exist $x \in A$ and $y \in B$ such that x, y are common best proximity points of T and S, then $F_{x,y}(t) = F_{A,B}(t)$ for all $t > 0$.

Proof. Let $x \in A$ and $y \in B$ be common best proximity points of T and S, so $F_{y,Sy}(t) = F_{x,Sx}(t) = F_{A,B}(t) = F_{x,Tx}(t) = F_{y,Ty}(t)$, for all $t > 0$. Since the pairs (A, B) and (B, A) satisfy the property UC, we have $F_{Tx,Sx}(t) = \epsilon_0(t) = F_{Ty,Sy}(t)$ so $Tx = Sx$ and $Ty = Sy$. By hypothesis, we have

$$\begin{aligned} F_{Sx,Sy}(t) &\geq \min\left\{F_{Tx,Ty}\left(\frac{t}{a}\right), F_{A,B}(t)\right\} \\ &= \min\left\{F_{Sx,Sy}\left(\frac{t}{a}\right), F_{A,B}(t)\right\} \\ &\geq \min\left\{F_{Sx,Sy}\left(\frac{t}{a^2}\right), F_{A,B}(t)\right\}. \end{aligned}$$

Now by induction on n we can show that

$$F_{Sx,Sy}(t) \geq \min\left\{F_{Sx,Sy}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}. \tag{3}$$

Taking limit as $n \to \infty$ from (3), so $F_{Sx,Sy}(t) = F_{A,B}(t)$. Since $F_{x,Sx}(t) = F_{A,B}(t)$, $F_{Sx,Sy}(t) = F_{A,B}(t)$ and (A, B) satisfies the property UC, we have $Sy = x$. By similar way we can show that $Sx = y$, therefore $F_{x,y}(t) = F_{A,B}(t)$ for all $t > 0$.

Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ) and the pairs (A, B) and (B, A) satisfy the property UC. If $(x_n) \subseteq A$ and $(y_n) \subseteq B$ are sequences such that one of the following conditions holds

$$\lim_{n\to\infty} \inf_{m\geq n} F_{x_m,y_n}(t) = F_{A,B}(t), \quad or \quad \lim_{m\to\infty} \inf_{n\geq m} F_{x_m,y_n}(t) = F_{A,B}(t), \quad \forall t > 0,$$

then (x_n) is Cauchy sequence.

Proof. Arguing by contradiction, we assume that (x_n) is not Cauchy sequence, then there exist $\epsilon > 0$, $\lambda \in (0, 1)$ and subsequence (m_k) and (l_k) such that $l_k < m_k$ and $F_{x_{l_k},x_{m_k}}(\epsilon) \leq 1 - \lambda$.

Now if $\lim\limits_{m\to\infty} \inf\limits_{n\geq m} F_{x_m,y_n}(\epsilon) = F_{A,B}(\epsilon)$, then $F_{A,B}(\epsilon) = \lim\limits_{m\to\infty} \inf\limits_{n\geq m} F_{x_m,y_n}(\epsilon) \leq \lim\limits_{k\to\infty} F_{x_{m_k},y_{m_k}}(\epsilon) \leq F_{A,B}(\epsilon)$ so $\lim\limits_{k\to\infty} F_{x_{m_k},y_{m_k}}(\epsilon) = F_{A,B}(\epsilon)$ and by the same way we can show that $\lim\limits_{k\to\infty} F_{x_{l_k},y_{m_k}}(\epsilon) = F_{A,B}(\epsilon)$, and if $\lim\limits_{n\to\infty} \inf\limits_{m\geq n} F_{x_m,y_n}(\epsilon) = F_{A,B}(\epsilon)$, then $\lim_{k\to\infty} F_{x_{m_k},y_{l_k}}(\epsilon) = F_{A,B}(\epsilon)$ and $\lim_{k\to\infty} F_{x_{l_k},y_{l_k}}(\epsilon) = F_{A,B}(\epsilon)$. In both cases, since (A, B) has the property UC, we have $\lim_{k\to\infty} F_{x_{l_k},x_{m_k}}(\epsilon) = 1$, which this contradicts with $F_{x_{l_k},x_{m_k}}(\epsilon) \leq 1 - \lambda$. Therefore (x_n) is Cauchy sequence.

In this paper we give some resultes about common best proximity points in probabilistic Menger spaces (X, F, Δ_m). We show that if A, B are two nonempty closed subsets of X and T, S are two cyclic self mappings on $A \cup B$ such that S is a T-cyclic contraction and T is continuous, then under certain conditions T and S have unique common best proximity points in A and B. We also prove that if A, B are two nonempty closed subsets of X and T, S are two cyclic self mappings on $A \cup B$ such that S is a T-cyclic contraction and T is relatively continuous, then under certain conditions T and S have unique common best proximity points in A and B. Next we show that if A, B are two nonempty closed subsets of X and cyclic self mapping T on $A \cup B$ is reverse cyclic contraction, then under certain conditions T has unique best proximity points in A and B. Also, we prove that every contraction on complete probabilistic Menger space (X, F, Δ_m) has a unique fixed point. Finally, we prove that if A, B are two nonempty closed subsets of X and (S, T) is semi cyclic contraction pair, then under certain conditions S and T have a unique best proximity point.

2. Main results

Now we first bring the following theorem about existence and uniqueness of common best proximity points for cyclic contractions in probabilistic Menger spaces.

Theorem 1. *Let (X, F, Δ_m) be a complete probabilistic Menger space, A and B be two nonempty closed subsets of X such that pairs (A, B) and (B, A) satisfy the property UC. Let S, T be two cyclic self mappings on $A \cup B$ such that satisfying the following conditions:*

(i) $S(A) \subseteq T(A)$ and $S(B) \subseteq T(B)$;

(ii) S is a T-cyclic contraction;

(iii) S and T commute;

(iv) T is continuous.

Then there exist unique x in A and y in B such x and y are common best proximity points of T and S. Furthermore for any $t > 0$, $F_{x,y}(t) = F_{A,B}(t)$.

Proof. Let $x_0 \in A$, since $S(A) \subseteq T(A)$, there exists $x_1 \in A$ such that $Sx_0 = Tx_1$. Again there exists $x_2 \in A$ such that $Sx_1 = Tx_2$. By following this process one can find the sequence (x_n) in A such that $Sx_n = Tx_{n+1}$. Since S is a T-cyclic contraction, we have

$$F_{Sx_n,SSx_n}(t) \geq \min\left\{F_{Sx_{n-1},SSx_{n-1}}\left(\frac{t}{a}\right), F_{A,B}(t)\right\}. \tag{4}$$

Now by induction on n, we show that

$$F_{Sx_n,SSx_n}(t) \geq \min\left\{F_{Sx_0,SSx_0}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}. \tag{5}$$

By (4), the relation (5) holds for $n = 1$. Assume that the relation (5) holds for some $n = k$, that is

$$F_{Sx_k,SSx_k}(t) \geq \min\left\{F_{Sx_0,SSx_0}\left(\frac{t}{a^k}\right), F_{A,B}(t)\right\}.$$

So we have

$$\begin{aligned} F_{Sx_{k+1},SSx_{k+1}}(t) &\geq \min\left\{F_{Sx_k,SSx_k}\left(\frac{t}{a}\right), F_{A,B}(t)\right\} \\ &\geq \min\left\{\min\left\{F_{Sx_0,SSx_0}\left(\frac{t}{a^{k+1}}\right), F_{A,B}\left(\frac{t}{a}\right)\right\}, F_{A,B}(t)\right\} \\ &= \min\left\{F_{Sx_0,SSx_0}\left(\frac{t}{a^{k+1}}\right), F_{A,B}(t)\right\}. \end{aligned}$$

Thus (5) holds for $n = k + 1$. Now taking $\liminf$ as $n \to \infty$ from (5), we get

$$\liminf_{n\to\infty} F_{Sx_n,SSx_n}(t) = F_{A,B}(t),$$

so $\lim_{n\to\infty} F_{Sx_n,SSx_n}(t) = F_{A,B}(t)$. Similarly, we can show that $\lim_{n\to\infty} F_{Sx_{n+1},SSx_n}(t) = F_{A,B}(t)$. Since (A, B) satisfy the property UC, then $\lim_{n\to\infty} F_{Sx_{n+1},Sx_n}(t) = 1$. Now for any $t_0 > 0$, we show that

$$\lim_{m\to\infty} \inf_{n\geq m} F_{Sx_m,SSx_n}(t_0) = F_{A,B}(t_0).$$

For this purpose we show that for each $0 < \epsilon < F_{A,B}(t_0)$, there exists m_0 such that for any $m \geq m_0$,

$$\inf_{n\geq m} F_{Sx_m,SSx_n}(t_0) \geq F_{A,B}(t_0) - \epsilon.$$

Let $0 < \epsilon < F_{A,B}(t_0)$, we choose λ such that $a < \lambda < 1$. Since for any $t > 0$, $\lim_{n\to\infty} F_{Sx_n,SSx_n}(t) = F_{A,B}(t)$ and $\lim_{n\to\infty} F_{Sx_{n+1},Sx_n}(t) = 1$, then there exists m_0 such that for any $m \geq m_0$,

$$F_{Sx_m,SSx_m}(t_0) \geq F_{A,B}(t_0) - \epsilon, \tag{6}$$

and

$$F_{Sx_{m+1},Sx_m}((1-\lambda)t_0) \geq F_{A,B}(t_0) - \epsilon.$$

Now by induction we show that for any $m \geq m_0$ and any $n \geq m$,

$$F_{Sx_m,SSx_n}(t_0) \geq F_{A,B}(t_0) - \epsilon. \tag{7}$$

In the other words, we show that

$$\inf_{n\geq m} F_{Sx_m,SSx_n}(t_0) \geq F_{A,B}(t_0) - \epsilon.$$

By (6), it is clear that (7) holds for $n = m \geq m_0$. Assume that (7) holds for $n = k \geq m$, that is

$$F_{Sx_m,SSx_k}(t_0) \geq F_{A,B}(t_0) - \epsilon.$$

Now for $n = k + 1$, we have

$$\begin{aligned}
F_{Sx_m,SSx_{k+1}}(t_0) &\geq \Delta_m \left\{F_{Sx_m,Sx_{m+1}}((1-\lambda)t_0), F_{Sx_{m+1},SSx_{k+1}}(\lambda t_0)\right\} \\
&= \min \left\{F_{Sx_m,Sx_{m+1}}((1-\lambda)t_0), F_{Sx_{m+1},SSx_{k+1}}(\lambda t_0)\right\} \\
&\geq \min\{F_{A,B}(t_0) - \epsilon, \min\{F_{Tx_{m+1},TSx_{k+1}}(\frac{\lambda t_0}{a}), F_{A,B}(\lambda t_0)\}\} \\
&\geq \min \{F_{A,B}(t_0) - \epsilon, \min \{F_{Sx_m,SSx_k}(t_0), F_{A,B}(\lambda t_0)\}\} \\
&\geq \min \{F_{A,B}(t_0) - \epsilon, \min \{F_{A,B}(t_0) - \epsilon, F_{A,B}(\lambda t_0)\}\} \\
&\geq \min \{F_{A,B}(t_0) - \epsilon, \min \{F_{A,B}(t_0) - \epsilon, F_{A,B}(\lambda t_0) - \epsilon\}\} \\
&= \min \{F_{A,B}(t_0) - \epsilon, F_{A,B}(\lambda t_0) - \epsilon\} \\
&= F_{A,B}(\lambda t_0) - \epsilon.
\end{aligned}$$

Now letting $\lambda \to 1$, we obtain $F_{Sx_m,SSx_{k+1}}(t_0) \geq F_{A,B}(t_0) - \epsilon$. So (7) holds for $n = k+1$, therefore for all $t > 0$, $\lim_{m\to\infty} \inf_{n\geq m} F_{Sx_m,SSx_n}(t) = F_{A,B}(t)$, and by Lemma 1, (Sx_n) is Cauchy sequence. Since X is complete and B is closed set, then there exists $y \in B$ such that $Sx_n \to y$. Since $Sx_n = Tx_{n+1}$, so $Tx_n \to y$ and by continuity T we have

$$TSx_n \to Ty \quad \textit{and} \quad TTx_n \to Ty.$$

Since S and T commute, so $STx_n \to Ty$. On the other hand, we have

$$F_{STx_n,Sx_n}(t) \geq \min\left\{F_{TTx_n,Tx_n}(\frac{t}{a}), F_{A,B}(t)\right\}.$$

Now taking $\liminf$ as $n \to \infty$, by Proposition 1, we get

$$\begin{aligned} F_{Ty,y}(t) &\geq \min\left\{F_{Ty,y}\left(\frac{t}{a}\right), F_{A,B}(t)\right\} \\ &\geq \min\left\{F_{Ty,y}\left(\frac{t}{a^2}\right), F_{A,B}(t)\right\} \\ &\vdots \\ &\geq \min\left\{F_{Ty,y}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}. \end{aligned}$$

Now taking limit as $n \to \infty$, we obtain

$$F_{Ty,y}(t) \geq \min\{1, F_{A,B}(t)\} = F_{A,B}(t),$$

therefore $F_{Ty,y}(t) = F_{A,B}(t)$. Also

$$\begin{aligned} F_{Sy,Sx_n}(t) &\geq \min\left\{F_{Ty,Tx_n}(\frac{t}{a}), F_{A,B}(t)\right\} \\ &\geq \min\{F_{Ty,Tx_n}(t), F_{A,B}(t)\}, \end{aligned}$$

now taking $\liminf$ as $n \to \infty$, by Proposition 1, we have $F_{Sy,y}(t) = F_{A,B}(t)$. Since (A, B) satisfy the property UC, then $Sy = Ty$. Let $x = Sy = Ty$, hence $Tx = TSy = STy = Sx$. Since S is a T-cyclic contraction and $F_{Ty,Tx}(t) = F_{Ty,STy}(t) = F_{x,Sx}(t)$, we get

$$F_{x,Sx}(t) = F_{Sy,Sx}(t) \geq \min\left\{F_{x,Sx}\left(\frac{t}{a}\right), F_{A,B}(t)\right\}$$

By induction on n, we have

$$F_{x,Sx}(t) \geq \min\left\{F_{x,Sx}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}. \tag{8}$$

Taking limit as $n \to \infty$ from (8), we get $F_{x,Sx}(t) \geq F_{A,B}(t)$, and hence

$$F_{A,B}(t) = F_{x,Sx}(t) = F_{Ty,STy}(t) = F_{Ty,TSy}(t) = F_{Ty,Tx}(t) = F_{x,Tx}(t),$$

for all $t > 0$, therefore $x \in A$ and $y \in B$ are common best proximity points of T and S. Now suppose that there exists $y^* \in B$ such that $F_{y^*,Ty^*}(t) = F_{A,B}(t) = F_{y^*,Sy^*}(t)$, for all $t > 0$. Since (A, B) satisfy the property UC, so $Ty^* = Sy^*$. On the other hand since $F_{x,Tx}(t) = F_{A,B}(t) = F_{x,Sx}(t)$ and (B, A) satisfy the property UC, so $Sx = Tx$. By induction we can show that

$$F_{Sy^*,Sx}(t) \geq \min\left\{F_{Sy^*,Sx}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}, \tag{9}$$

Taking limit as $n \to \infty$ from (9), we get $F_{Sy^*,Sx}(t) \geq F_{A,B}(t)$, hence for all $t > 0$, $F_{Sy^*,Sx}(t) = F_{A,B}(t)$. Since (B, A) satisfy the property UC and $F_{y^*,Sy^*}(t) = F_{A,B}(t)$, then $Sx = y^*$. Because $F_{y,Ty}(t) = F_{A,B}(t) = F_{y,Sy}(t)$ for all $t > 0$, in similar way we can show that $Sx = y$. Therefore $y = y^*$. Similarly, we can show that $x \in A$ is a unique common best proximity point of T and S.

Let $X = \mathbb{R}^2$ and $F_{(x,y),(u,v)}(t) = \frac{1}{2}(\epsilon_0(t)+\epsilon_0(t-d((x,y),(u,v))))$, where $d((x,y),(u,v)) = |x-u|+|y-v|$, for all $(x,y),(u,v) \in X$ and $t > 0$, then $(\mathbb{R}^2, F, \Delta_m)$ is a complete probabilistic Menger space. Let $A = \{(2,x) : x \in [-\pi, 0]\}$, $B = \{(0,x) : x \in [0,\pi]\}$ and $S, T : A \cup B \to A \cup B$ be defined as

$$T(x,y) = \begin{cases} (0, \frac{1}{3}|y\cos y|), & (x,y) \in A, \\ \\ (2, -\frac{1}{3}|y\cos y|), & (x,y) \in B, \end{cases} \qquad S(x,y) = \begin{cases} (0,0), & (x,y) \in A, \\ \\ (2,0), & (x,y) \in B. \end{cases}$$

It is clear that the pairs (A, B) and (B, A) satisfy the property UC and

$$F_{A,B}(t) = \sup_{\substack{-\pi \leq x \leq 0 \\ 0 \leq y \leq \pi}} \frac{1}{2}(\epsilon_0(t) + \epsilon_0(t - (2 + |x-y|))) = \frac{1}{2}(\epsilon_0(t) + \epsilon_0(t-2)).$$

Also T and S are cyclic mappings and $ST = TS$. Now let $(2,x) \in A$ and $(0,y) \in B$, so we have

$$\begin{aligned} F_{S(2,x),S(0,y)}(t) &= F_{(0,0),(2,0)}(t) \\ &= \frac{1}{2}(\epsilon_0(t) + \epsilon_0(t-2)) \\ &= F_{A,B}(t) \\ &\geq \min\left\{F_{T(2,x),T(0,y)}\left(\frac{t}{a}\right), F_{A,B}(t)\right\}, \end{aligned}$$

for every $0 < a < 1$. Hence S is a T-cyclic contraction. Let $(2, x_n)$ be a sequence in A such that $(2, x_n) \to (2, x) \in A$, then

$$\lim_{n\to\infty} F_{(2,x_n),(2,x)}(t) = \frac{1}{2}(\epsilon_0(t) + \epsilon_0(t - |x_n - x|))) = 1,$$

therefore $x_n \to x$. Now we have

$$F_{T(2,x_n),T(2,x)}(t) = F_{(0,\frac{1}{3}|x_n\cos x_n|),(0,\frac{1}{3}|x_n\cos x_n|)}(t) \geq \frac{1}{2}(\epsilon_0(t) + \epsilon_0(t - |x_n - x|)),$$

hence $\lim\limits_{n\to\infty} F_{T(2,x_n),T(2,x)}(t) = 1$, therefore T is a continuous mapping on A. In similar way we can show that T is a continuous mapping on B. Therefore all the assumptions of Theorem 1 are satisfied and $(2,0)$ in A and $(0,0)$ in B are unique common best proximity points of S and T.

Theorem 2. *Let (X, F, Δ_m) be a complete probabilistic Menger space, A and B be two nonempty closed subsets of X such that pairs (A, B) and (B, A) satisfy the property UC. Let T and S be two cyclic self mappings on $A \cup B$ such that satisfying the following conditions:*

(i) $S(A) \subseteq T(A)$ *and* $S(B) \subseteq T(B)$;

(ii) S *is a* T*-cyclic contraction;*

(iii) S *and* T *commute;*

(iv) T *is relatively continuous.*

Then there exist unique x in A and y in B such that x and y are common best proximity points of T and S. Furthermore for any $t > 0$, $F_{x,y}(t) = F_{A,B}(t)$.

Proof. Let $(x_n) \subseteq A$ be the sequence obtained in Theorem 1, such that $Sx_n = Tx_{n+1}$ and for some $y \in B$, we have $Sx_n \to y$ and $Tx_n \to y$. Similarly, we can obtain a sequence $(y_n) \subseteq B$ such that $Sy_n = Ty_{n+1}$ and for some $x \in A$, $Sy_n \to x$ and $Ty_n \to x$. Since S is a T-cyclic contraction, we have

$$\begin{aligned} F_{Sx_n,Sy_n}(t) &\geq \min\left\{F_{Tx_n,Ty_n}\left(\frac{t}{a}\right), F_{A,B}\left(t\right)\right\} \\ &\geq \min\left\{F_{Sx_{n-1},Sy_{n-1}}\left(\frac{t}{a}\right), F_{A,B}(t)\right\}. \end{aligned}$$

With a process similar to the proof of Theorem 1, we can show that

$$F_{Sx_n,Sy_n}(t) \geq \min\left\{F_{Sx_0,Sy_0}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}. \tag{10}$$

Taking $\liminf$ as $n \to \infty$ from (10), we get $\liminf_{n\to\infty} F_{Sx_n,Sy_n}(t) \geq F_{A,B}(t)$, so by Proposition 1 for all $t > 0$,

$$F_{x,y}(t) = \liminf_{n\to\infty} F_{Sx_n,Sy_n}(t) = F_{A,B}(t).$$

On the other hand $Tx_n \to y$ and $F_{x,y}(t) = F_{A,B}(t)$, so $\liminf_{n\to\infty} F_{Tx_n,x}(t) = F_{x,y}(t) = F_{A,B}(t)$ and since

$$F_{A,B}(t) \geq \limsup_{n\to\infty} F_{Tx_n,x}(t) \geq \liminf_{n\to\infty} F_{Tx_n,x}(t) = F_{A,B}(t),$$

we have $\lim_{n\to\infty} F_{Tx_n,x}(t) = F_{A,B}(t)$ for all $t > 0$, similarly we can show that $\lim_{n\to\infty} F_{Sx_n,x}(t) = F_{A,B}(t)$ for all $t > 0$. Now by Proposition 1, S is relatively continuous so by Lemma

1, $F_{STx_n,Sx}(t) \to F_{A,B}(t)$. Since $F_{Sx_n,x}(t) \to F_{A,B}(t)$ and T is relatively continuous hence $F_{TSx_n,Tx}(t) \to F_{A,B}(t)$. Since S and T commute and (B, A) satisfy the property UC, we conclude that $Sx = Tx$. By induction on n we can show that

$$F_{Sx,Sx_n}(t) \geq \min\left\{F_{Sx,Sx_0}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}. \tag{11}$$

Taking lim inf as $n \to \infty$ from (11), we get $F_{Sx,x}(t) = F_{A,B}(t)$ for all $t > 0$. Since $Sx = Tx$, therefore $x \in A$ is common best proximity point of S and T. Similarly we can show that $y \in B$ is common best proximity point of S and T. The rest of the proof is similar to the proof of Theorem 1.

Let $X = \mathbb{R}$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. It is easy to see that $(\mathbb{R}, F, \Delta_m)$ is a complete probabilistic Menger space, so by Lemma 1, $(\mathbb{R}^2, M, \Delta_m)$ is also a complete probabilistic Menger space, where

$$M_{(x,y),(u,v)}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\},$$

for every $(x, y), (u, v) \in \mathbb{R}^2$ and $t > 0$. Let $A = \{(x, 1) : 0 \leq x \leq 1\}$ and $B = \{(2, y) : -1 \leq y \leq 0\}$, then

$$M_{A,B}(t) = \sup_{\substack{0\leq x\leq 1\\ -1\leq y\leq 0}} \min\{\frac{t}{t+|x-2|}, \frac{t}{t+|1-y|}\} = \frac{t}{t+1},$$

and the pairs (A, B) and (B, A) satisfy the property UC. Let $S, T : A \cup B \to A \cup B$ be defined as

$$T(x,1) = \begin{cases} (2, x-1), & x \in [0,1] \cap \mathbb{Q}, \\ (2,0), & x \in [0,1] \cap \mathbb{Q}^c, \end{cases} \qquad T(2,y) = (1+y, 1),$$

$$S(x,y) = \begin{cases} (2,0), & (x,y) \in A, \\ (1,1), & (x,y) \in B. \end{cases}$$

For every $(x, 1) \in A$, $(2, y) \in B$ and all $0 < a < 1$, we have

$$\begin{aligned} M_{S(x,1),S(2,y)}(t) = M_{(2,0),(1,1)}(t) &= \min\{F_{2,1}(t), F_{0,1}(t)\} \\ &= \min\left\{\frac{t}{t+1}, \frac{t}{t+1}\right\} \\ &= \frac{t}{t+1} = M_{A,B}(t) \\ &\geq \min\left\{M_{T(x,1),T(2,y)}(\frac{t}{a}), M_{A,B}(t)\right\}. \end{aligned}$$

Hence S is a T-cyclic contraction, S and T commute and T is relatively continuous. Hence by Theorem 2, T and S have unique common best proximity points $(1,1) \in A$ and $(2,0) \in B$. Let (X, F, Δ_m) be a complete probabilistic Menger space, A and B be two nonempty closed subsets of X such that pairs (A, B) and (B, A) satisfy the property UC. If $T : A \cup B \to A \cup B$ is cyclic contraction, then there exist unique points $x \in A$ and $y \in B$ such that x, y are best proximity points of T. Furthermore for any $t > 0$, $F_{x,y}(t) = F_{A,B}(t)$.

Proof. Since T is cyclic contraction, then by Corollary 1, T is relatively continuous and by Proposition 1, T^3 is T-cyclic contraction. Since T is cyclic mapping, then $T^3(A) \subseteq T(A) \subseteq B$ and $T^3(B) \subseteq T(B) \subseteq A$. Also T^3 and T commute therefore by Theorem 2, T^3 and T have unique common best proximity points $x \in A$ and $y \in B$ and $F_{x,y}(t) = F_{A,B}(t)$, for all $t > 0$, so the result follows.

In Corollary 2, if $A = X = B$, then we immediately achive the probabilistic version of classical Banach contraction principle. Every contraction on complete probabilistic Menger space (X, F, Δ_m) has a unique fixed point.

Theorem 3. *Let (X, F, Δ_m) be a complete probabilistic Menger space, A and B be two nonempty closed subsets of X such that pairs (A, B) and (B, A) satisfy the property UC. Let T and S be two cyclic self mappings on $A \cup B$ such that satisfying the following conditions:*

(i) $S(A) \subseteq T(A)$ and $S(B) \subseteq T(B)$;

(ii) S is a T-cyclic contraction;

(iii) S and T commute;

(iv) T is bijective.

Then there exist unique x in A and y in B such that x and y are common best proximity points of T and S. Furthermore for any $t > 0$, $F_{x,y}(t) = F_{A,B}(t)$.

Proof. Let $(x_n) \subseteq A$ be the sequence obtained in Theorem 1, so there exists a $y \in B$, such that $Sx_n \to y$ and $Tx_n \to y$. Similarly, we can obtain a sequence $(y_n) \subseteq B$ such that $Sy_n = Ty_{n+1}$ and for some $x \in A$, $Sy_n \to x$ and $Ty_n \to x$. By the same argument as in the proof of Theorem 2, we can show that $F_{x,y}(t) = F_{A,B}(t)$. Since S is a T-cyclic contraction, so we get

$$F_{ST^{-1}x, Sx_n}(t) \geq \min \left\{ F_{x, Tx_n}\left(\frac{t}{a}\right), F_{A,B}(t) \right\}. \tag{12}$$

Taking lim inf as $n \to \infty$ from (12), we have

$$\begin{aligned} F_{ST^{-1}x,y}(t) &= \liminf_{n\to\infty} F_{ST^{-1}x,Sx_n}(t) \\ &\geq \min\left\{\liminf_{n\to\infty} F_{x,Tx_n}\left(\frac{t}{a}\right), F_{A,B}\left(t\right)\right\} \\ &\geq \min\left\{F_{x,y}(t), F_{A,B}(t)\right\} \\ &= \min\left\{F_{A,B}(t), F_{A,B}(t)\right\} \\ &= F_{A,B}(t). \end{aligned}$$

Therefore $F_{ST^{-1}x,y}(t) = F_{A,B}(t)$ for all $t > 0$. Since $F_{ST^{-1}x,y}(t) = F_{A,B}(t) = F_{x,y}(t)$ and (A, B) satisfy the property UC, so $ST^{-1}x = x$ and since S and T commute, hence $Tx = TST^{-1}x = Sx$. By induction we can show that

$$F_{Sx,Sy_n}(t) \geq \min\left\{F_{Sx,Sy_0}\left(\frac{t}{a^n}\right), F_{A,B}(t)\right\}. \tag{13}$$

Taking lim inf as $n \to \infty$ from (13), we have $F_{Sx,x}(t) = F_{A,B}(t)$ for all $t > 0$. Since $Sx = Tx$, therefore $x \in A$ is common best proximity point of S and T. Similarly we can show that $y \in B$ is common best proximity point of S and T. The rest of the proof is similar to the proof of Theorem 1.

Let (X, F, Δ_m) be a complete probabilistic Menger space, A and B be two nonempty closed subsets of X such that pairs (A, B) and (B, A) satisfy the property UC. If $T : A \cup B \to A \cup B$ is a bijective reverse cyclic contraction, then there exist unique points $x \in A$ and $y \in B$ such that x, y are best proximity points of T. Furthermore for any $t > 0$, $F_{x,y}(t) = F_{A,B}(t)$.

Proof. Since T is reverse cyclic contraction, then by Proposition 1, T is a T^3-cyclic contraction. By hypothesis T^3 is bijective, also, T and T^3 commute. Therefore by Theorem 3, the desired result is achieved.

In Theorem 1, if $A = X = B$, then we immediately achieve the following result. Let (X, F, Δ_m) be a complete probabilistic Menger space. If $S, T : X \to X$ are two mappings such that satisfying the following conditions:

(i) $S(X) \subseteq T(X)$;

(ii) T is continuous;

(iii) S and T commute;

(iv) there exists $a \in (0,1)$ such that for any $x, y \in X$ and any $t > 0$,

$$F_{Sx,Sy}(t) \geq F_{Tx,Ty}(\frac{t}{a}).$$

Then S and T have unique common fixed point in X. Consider $X = \mathbb{R}^2$ and define $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for all $t > 0$, where d is Euclidean metric on $\mathbb{R}^2$. It is easy to see that (X, F, Δ_m) is a complete probabilistic Menger space. Consider self mappings S and T on X as follows:

$$S(x,y) = (7x, \frac{y}{3} + 4), \qquad T(x,y) = (11x, \frac{y}{2} + 3),$$

then

$$\begin{aligned} d(T(x,y), T(u,v)) &= \sqrt{(11x - 11u)^2 + (\frac{y}{2} + 3 - \frac{v}{2} - 3)^2} \\ &= \sqrt{121(x-u)^2 + \frac{1}{4}(y-v)^2} \\ &\geq \sqrt{110.25(x-u)^2 + \frac{1}{4}(y-v)^2} \\ &= \sqrt{\frac{9}{4}49(x-u)^2 + \frac{9}{4}\frac{1}{9}(y-v)^2} \\ &= \frac{3}{2}\sqrt{(7x-7u)^2 + (\frac{y}{3} - \frac{v}{3})^2} \\ &= \frac{3}{2}d(S(x,y), S(u,v)). \end{aligned}$$

Therefore

$$\begin{aligned} F_{S(x,y),S(u,v)}(t) &= \frac{t}{t + d(S(x,y), S(u,v))} \\ &\geq \frac{t}{t + \frac{2}{3}d(T(x,y), T(u,v))} \\ &= F_{T(x,y),T(u,v)}\left(\frac{t}{\frac{2}{3}}\right). \end{aligned}$$

It is clear that $TS = ST$, T is continuous and $S(X) \subseteq T(X)$. Hence all of the assumptions of Corollary 2 are satisfied, and $(0, 6)$ is a unique common fixed point of S and T. Let A and B be two nonempty subsets of a probabilistic Menger space (X, F, Δ_m), S and T be two self mappings on $A \cup B$ and $x_0 \in A$. If (S, T) is a semi cyclic contraction pair, $y_n = Sx_n$ and $x_{n+1} = Ty_n$, $(n = 0, 1, 2, \ldots)$, then $F_{x_n,Sx_n}(t) \to F_{A,B}(t)$ and $F_{y_n,Ty_n}(t) \to F_{A,B}(t)$, for all $t > 0$.

Proof. Since (S,T) is semi cyclic contraction pair, there exists $0<a<1$ such that for all $t>0$,

$$\begin{aligned} F_{x_n,Sx_n}(t) = F_{Ty_{n-1},Sx_n}(t) &\geq \min\{F_{y_{n-1},x_n}\left(\frac{t}{a}\right), F_{A,B}(t)\} \\ &= \min\{F_{Sx_{n-1},Ty_{n-1}}\left(\frac{t}{a}\right), F_{A,B}(t)\} \\ &\geq \min\left\{\min\left\{F_{y_{n-1},x_{n-1}}\left(\frac{t}{a^2}\right), F_{A,B}\left(\frac{t}{a}\right)\right\}, F_{A,B}\left(t\right)\right\} \\ &\geq \min\left\{F_{x_{n-1},Sx_{n-1}}\left(\frac{t}{a^2}\right), F_{A,B}(t)\right\}. \end{aligned}$$

By induction on n, we can show that

$$F_{x_n,Sx_n}(t) \geq \min\left\{F_{x_0,y_0}\left(\frac{t}{a^{2n}}\right), F_{A,B}(t)\right\}.$$

Now taking lim inf as $n\to\infty$, we conclude that $\liminf\limits_{n\to\infty} F_{x_n,Sx_n}(t) = F_{A,B}(t)$ for all $t>0$, hence $\lim\limits_{n\to\infty} F_{x_n,Sx_n}(t) = F_{A,B}(t)$. Similarly we can show that $\lim\limits_{n\to\infty} F_{y_n,Ty_n}(t) = F_{A,B}(t)$ for all $t>0$.

Let A and B be two nonempty subsets of a probabilistic Menger space (X,F,Δ_m), S and T be two self mappings on $A\cup B$ and $x_0\in A$. Let (S,T) be a semi cyclic contraction pair, $y_n = Sx_n$ and $x_{n+1} = Ty_n$, $(n=0,1,2,\ldots)$. If (x_n) and (y_n) have convergent subsequences (x_{n_k}) and (y_{n_k}) respectively such that for some $x\in A$ and $y\in B$, $x_{n_k}\to x$ and $y_{n_k}\to y$. Then for all $t>0$

$$F_{x,Sx}(t) = F_{A,B}(t) = F_{y,Ty}(t).$$

Proof. By Proposition 2, $F_{x_{n_k},y_{n_k}}(t)\to F_{A,B}(t)$ and by Proposition 1, $\liminf\limits_{k\to\infty} F_{x_{n_k},y_{n_k}}(t) = F_{x,y}(t)$, so $F_{x,y}(t) = F_{A,B}(t)$. Also by Proposition 2 for all $t>0$, we have $F_{y_{n_k},Ty_{n_k}}(t)\to F_{A,B}(t)$. Let $0<\epsilon<1$, then we have

$$F_{y,Ty_{n_k}}(t) \geq \Delta_m\left\{F_{y,y_{n_k}}((1-\epsilon)t), F_{y_{n_k},Ty_{n_k}}(\epsilon t)\right\}.$$

Now taking lim inf as $k\to\infty$, we get

$$\liminf_{k\to\infty} F_{y,Ty_{n_k}}(t) \geq \min\{1, F_{A,B}(\epsilon t)\} = F_{A,B}(\epsilon t),$$

If $\epsilon \to 1$, then $\lim_{k\to\infty} F_{y,Ty_{n_k}}(t) = F_{A,B}(t)$ for all $t > 0$. On the other hand

$$F_{Ty,y_{n_k}}(t) = F_{Ty,Sx_{n_k}}(t) \geq \min\left\{F_{y,x_{n_k}}(\frac{t}{a}), F_{A,B}(t)\right\}.$$

Now taking lim inf as $k \to \infty$, then we conclude that

$$F_{Ty,y}(t) \geq \min\left\{F_{y,x}(\frac{t}{a}), F_{A,B}(t)\right\}, \tag{14}$$

now, we have

$$\begin{aligned} F_{Ty,y}(t) &\geq \min\left\{F_{y,x}(\frac{t}{a}), F_{A,B}(t)\right\} \\ &= \min\left\{F_{A,B}(\frac{t}{a}), F_{A,B}(t)\right\} \\ &\geq \min\left\{F_{A,B}(t), F_{A,B}(t)\right\} \\ &= F_{A,B}(t), \end{aligned}$$

therefore $F_{Ty,y}(t) = F_{A,B}(t)$, for all $t > 0$. Similarly, we can show that $F_{Sx,x}(t) = F_{A,B}(t)$, for all $t > 0$, so the result follows.

Theorem 4. *Let (X, F, Δ_m) be a complete probabilistic Menger space, S and T be two self mappings on $A \cup B$ and A, B be two nonempty closed subsets of X such that pairs (A, B) and (B, A) satisfy the property UC. If (S, T) is a semi cyclic contraction pair, then S and T have a unique best proximity point.*

Proof. Let $x_0 \in A$, $y_n = Sx_n$ and $x_{n+1} = Ty_n$, then by Proposition 2, $F_{y_n,x_{n+1}}(t) = F_{y_n,Ty_n}(t) \to F_{A,B}(t)$ for all $t > 0$. Since (S, T) is a semi cyclic contraction pair, we have

$$\begin{aligned} F_{y_{n+1},x_{n+1}}(t) = F_{Sx_{n+1},Ty_n}(t) &\geq \min\left\{F_{x_{n+1},y_n}(\frac{t}{a}), F_{A,B}(t)\right\} \\ &\geq \min\left\{F_{x_{n+1},y_n}(t), F_{A,B}(t)\right\}, \end{aligned}$$

let $n \to \infty$, so we have $\lim_{n\to\infty} F_{y_{n+1},x_{n+1}}(t) = F_{A,B}(t)$ for all $t > 0$. Now since (B, A) has the property UC, then $F_{y_n,y_{n+1}}(t) \to 1$ for all $t > 0$. By the same argument as in the proof of Theorem 1, for any $t > 0$ we can show that

$$\lim_{m\to\infty} \inf_{n\geq m} F_{y_m,Ty_n}(t) = F_{A,B}(t).$$

By Lemma 1, (y_n) is a Cauchy sequence so there exists $y \in B$ such that $y_n \to y$. Similarly we can show that (x_n) is a Cauchy sequence and for some $x \in A$, $x_n \to x$. Now by Proposition 2, $F_{y,Ty}(t) = F_{A,B}(t) = F_{x,Sx}(t)$, for all $t > 0$.

Theorem 5. *Let (X, F, Δ_m) be a complete probabilistic Menger space, A and B be two nonempty closed subsets of X such that pairs (A, B) and (B, A) satisfy the property UC and $A \cap B \neq \emptyset$. Let S and T be two self mappings on $A \cup B$, if (S, T) is semi cyclic contraction pair, then S and T have a unique common fixed point in $A \cap B$.*

Proof. By the hypothesis, we have $F_{A,B}(t) = 1$, so $F_{Sx,Ty}(t) \geq F_{x,y}(\frac{t}{a})$. Let $x_0 \in A$, $y_n = Sx_n$ and $x_{n+1} = Ty_n$, $(n = 0, 1, 2, \ldots)$ and define $(z_n)_{n \geq 1}$ in $A \cup B$ as follows:

$$z_n = \begin{cases} Ty_k, & n = 2k, \\ Sx_k & n = 2k-1. \end{cases}$$

Now we show that (z_n) is a Cauchy sequence. If $n = 2k$, then

$$\begin{aligned} F_{z_{n+1},z_n}(t) = F_{Sx_{k+1},Ty_k}(t) &\geq F_{x_{k+1},y_k}(\frac{t}{a}) = F_{Ty_k,Sx_k}(\frac{t}{a}) \\ &\geq F_{y_k,x_k}(\frac{t}{a^2}) = F_{Sx_k,Ty_{k-1}}(\frac{t}{a^2}) \\ &\geq F_{x_k,y_{k-1}}(\frac{t}{a^3}) = F_{Ty_{k-1},Sx_{k-1}}(\frac{t}{a^3}) \\ &\geq F_{y_{k-1},x_{k-1}}(\frac{t}{a^4}) \geq \cdots \geq F_{y_1,x_1}(\frac{t}{a^{2k}}). \end{aligned}$$

If $n = 2k - 1$, then

$$\begin{aligned} F_{z_{n+1},z_n}(t) = F_{Ty_{k+1},Sx_k}(t) &\geq F_{y_{k+1},x_k}(\frac{t}{a}) = F_{Sx_{k+1},Ty_{k-1}}(\frac{t}{a}) \\ &\geq F_{x_{k+1},y_{k-1}}(\frac{t}{a^2}) = F_{Ty_k,Sx_{k-1}}(\frac{t}{a^2}) \\ &\geq F_{y_k,x_{k-1}}(\frac{t}{a^3}) = F_{Sx_k,Ty_{k-2}}(\frac{t}{a^3}) \\ &\geq F_{x_k,y_{k-2}}(\frac{t}{a^4}) = F_{Ty_{k-1},Sx_{k-2}}(\frac{t}{a^4}) \\ &\geq F_{y_{k-1},x_{k-2}}(\frac{t}{a^5}) \geq \cdots \geq F_{y_1,x_1}(\frac{t}{a^{2k-1}}). \end{aligned}$$

Taking $\liminf$ as $n \to \infty$, then $F_{z_{n+1},z_n}(t) \to 1$ for any $t > 0$. For every $p \in \mathbb{N}$ we have

$$F_{z_{n+p},z_n}(t) \geq \Delta_m^p \left(F_{z_n,z_{n+1}}(\frac{t}{p}), F_{z_{n+1},z_{n+2}}(\frac{t}{p}), \cdots, F_{z_{m-1},z_m}(\frac{t}{p}) \right)$$

taking $\liminf$ as $n \to \infty$, then we have $F_{z_{n+p},z_n}(t) \to 1$ for any $t > 0$, therefore (z_n) is Cauchy sequence and there exists $z \in A \cup B$ such that $z_n \to z$. Clearly, $z \in A \cap B$.

On the other hand for all $t > 0$,

$$F_{z_{2k-1},Tz}(t) = F_{Sx_k,Tz}(t) \geq F_{x_k,z}(\frac{t}{a}) = F_{Ty_{k-1},z}(\frac{t}{a}) = F_{z_{2(k-1)},z}(\frac{t}{a}),$$

taking lim inf as $k \to \infty$, then we have $F_{z,Tz}(t) = \lim_{k\to\infty} F_{z_{2k-1},Tz}(t) \geq 1$, therefore $F_{z,Tz}(t) = 1$ for all $t > 0$ and this implies that $z = Tz$. Similarly we can show that $z = Sz$. If $Tw = w = Sw$ for some $w \in A \cap B$, then we have

$$F_{w,z}(t) = F_{Sw,Tz}(t) \geq F_{w,z}(\frac{t}{a}),$$

so by Lemma 1, we have $w = z$.

Let $X = \mathbb{R}^2$ and $F_{(x,y),(u,v)}(t) = \frac{t}{t+d((x,y),(u,v))}$, where

$$d((x,y),(u,v)) = \begin{cases} |x| + |u| + |y - v|, & y \neq v, \\ |x - u|, & y = v, \end{cases}$$

for all $(x, y), (u, v) \in X$ and $t > 0$, then it is easy to see that $(\mathbb{R}^2, F, \Delta_m)$ is a complete probabilistic Menger space. Let $A = \{(x, 0) : 1 \leq x \leq 3\}$, $B = \{(x, \frac{1}{2}) : 0 \leq x \leq 2\}$ and $S, T : A \cup B \to A \cup B$ be defined as

$$S(x,y) = \begin{cases} (0, \frac{1}{2}), & (x,y) \in A, \\ \\ (x,y), & (x,y) \in B, \end{cases} \qquad T(x,y) = \begin{cases} (x,y), & (x,y) \in A, \\ \\ (1,0), & (x,y) \in B. \end{cases}$$

It is clear that the pairs (A, B) and (B, A) satisfy the property UC and

$$F_{A,B}(t) = \sup_{\substack{1\leq x\leq 3 \\ 0\leq y\leq 2}} \frac{t}{t + |x| + |y| + \frac{1}{2}} = \frac{t}{t + \frac{3}{2}}.$$

Obviously $S(A) \subseteq B$, $T(B) \subseteq A$, $S(B) \not\subseteq A$ and $T(A) \not\subseteq B$, so S and T are not cyclic mappings. Now if $(x, 0) \in A$ and $(y, \frac{1}{2}) \in B$, then we have

$$\begin{aligned} F_{S(x,0),T(y,\frac{1}{2})}(t) &= \frac{t}{t + \frac{3}{2}} = F_{A,B}(t) \\ &\geq \min\left\{F_{(x,0),(y,\frac{1}{2})}\left(\frac{t}{a}\right), F_{A,B}(t)\right\}, \end{aligned}$$

for all $0 < a < 1$. Hence (S, T) is a semi cyclic contraction pair. Therefore all the assumptions of Theorem 4 are satisfied and $(1, 0)$ in A and $(0, \frac{1}{2})$ in B are best proximity points of S and T respectively.

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