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RESULTS CONCERNING EXTENDED GENERALIZED BESSEL-MAITLAND FUNCTION

S.W. Khan

Abstract. Through this paper, we display the extended generalized Bessel-Maitland function (EGBMF) and obtain some integral representations of it. The extended fractional derivative [6] of the generalized Bessel-Maitland function gives the extended generalized Bessel-Maitland function. Furthermore we display interesting relationships of this function with Laguerre polynomials and Whittaker functions. At long last the Mellin transform of this function is evaluated in terms of generalized Wright hypergeometric function and Euler transform is also evaluated.

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1. Introduction

The theory of Bessel function is intimately connected with the theory of certain types of differential equations. A detailed account of application of Bessel function is represented in Watson [1]. At the latest innumerable authors have done work on Bessel function ([4],[5],[7]-[10]). The Bessel-Maitland function is a generalization of Bessel function, introduced by Edward Maitland Wright [11]. It is given by

$$J_{\nu}^{\mu}(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\mu n+\nu+1)n!}. \tag{1}$$

The applications of Bessel-Maitland function are found in the field of applied sciences, engineering, biological, chemical and Physical sciences [1]. The generalized Bessel-Maitland function investigated and studied in [7] and is defined as

$$J_{\nu,q}^{\mu,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n+\nu+1)n!}, \tag{2}$$

where $\mu, \nu, \gamma \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0$ and $q \in (0,1) \cup \mathbb{N}$.
In this paper, we extend the generalized Bessel-Maitland $J_{\nu,q}^{\mu,\gamma}(z)$ in the following way

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+qn, c-\gamma)(c)_{qn}(-z)^n}{B(\gamma, c-\gamma)\Gamma(\mu n+\nu+1)n!}, \quad (p>0, q \in \mathbb{N}, \Re(c) > \Re(\gamma) > 0), \tag{3}$$

which will be known as extended generalized Bessel-Maitland function (EGBMF). Using the fact $\frac{(\gamma)_{qn}}{(c)_{qn}} = \frac{B(\gamma+qn, c-\gamma)}{B(\gamma, c-\gamma)}$, and where

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(1-t)}} dt, \qquad (\Re(p)>0, \Re(x)>0, \Re(y)>0). \tag{4}$$

For $p=0$, we get beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{5}$$

The Wright generalized hypergeometric function is defined as [2]

$$ {}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{array}{ccccc} (a_1,\alpha_1), & (a_2,\alpha_2), & \cdots, & (a_p,\alpha_p); & \\ & & & & z \\ (b_1,\beta_1), & (b_2,\beta_2), & \cdots, & (b_q,\beta_q); & \end{array} \right], \tag{6}$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i, \alpha_i k) z^k}{\prod_{j=1}^{q} \Gamma(b_j, \beta_j k) k!},$$

where the coefficients $\alpha_i (i = 1, 2, \cdots, p)$ and $\beta_j (j = 1, 2, \cdots, q)$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0.$$

The Mellin tansform [3] of the function $f(z)$ is defined as

$$M[f(z);s] = \int_0^{\infty} z^{s-1} f(z) dz = f^*(s), \quad \Re(s) > 0. \tag{7}$$

The inverse Mellin transform

$$f(z) = M^{-1}[f^*(s);x] = \frac{1}{2\pi i} \int f^*(s) x^{-s} ds.$$

The Euler(Beta) [3] transform of the function $f(z)$ is defined as

$$B[f(z):a,b] = \int_0^1 z^{a-1}(1-z)^{b-1} f(z) dz. \tag{8}$$

2. Characterization of EGBMF as integral

In this segment we start with the Theorem and then obtain some specific results.

Theorem 1. *The extended generalized Bessel-Maitland function can have the following integral representation*

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{1}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} e^{\frac{-p}{t(1-t)}} J_{\nu,q}^{\mu,c}(t^q z) dt, \tag{9}$$

where $p > 0, q \in \mathbb{N}, \Re(c) > \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\nu) > -1.$

Proof. Put equation (4) in equation (3), we get

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \sum_{n=0}^{\infty} \int_0^1 t^{\gamma+qn-1}(1-t)^{c-\gamma-1} e^{\frac{-p}{t(1-t)}} dt \frac{(c)_{qn}(-z)^n}{B(\gamma, c-\gamma)\Gamma(\mu n+\nu+1)n!}.$$

Reciprocate the order of summation and integration, that is sured under the assumptions of theorem, we get

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{1}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} e^{\frac{-p}{t(1-t)}} \sum_{n=0}^{\infty} \frac{(c)_{qn}(-t^q z)^n}{\Gamma(\mu n+\nu+1)n!} dt,$$

using (2), we get our result.

Corollary 2. *Consider* $t = \frac{u}{1+u}$ *in Theorem 2.1, we get*

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{1}{B(\gamma, c-\gamma)} \int_0^{\infty} \frac{u^{\gamma-1}}{(1+u)^c} e^{\frac{-p(1+u)^2}{u}} J_{\nu,q}^{\mu,c}\left(\frac{u^q z}{(1+u)^q}\right) du. \tag{10}$$

Corollary 3. *Consider* $t = \sin^2\theta$ *in Theorem 2.1, we get*

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{2}{B(\gamma, c-\gamma)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\gamma-1}(\cos\theta)^{2c-2\gamma-1} e^{\frac{-p}{\sin^2\theta\cos^2\theta}} J_{\nu,q}^{\mu,c}(z\sin^{2q}\theta) d\theta. \tag{11}$$

Corollary 4. *Recurrence relation for the extended generalized Bessel-Maitland function*

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = (\nu+1) J_{\nu+1,q}^{\mu,\gamma;c}(z;p) + \mu z \frac{d}{dz} J_{\nu+1,q}^{\mu,\gamma;c}(z;p), \tag{12}$$

where $p > 0, q \in \mathbb{N}, \Re(c) > \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\nu) > -1.$

Proof. Starting with right hand side of (12) and using (3), we obtain

$$(\nu+1)J^{\mu,\gamma;c}_{\nu+1,q}(z;p)+\mu z\frac{d}{dz}J^{\mu,\gamma;c}_{\nu+1,q}(z;p)=(\nu+1)\sum_{n=0}^{\infty}\frac{B_p(\gamma+qn,c-\gamma)(c)_{qn}(-z)^n}{B(\gamma,c-\gamma)\Gamma(\mu n+\nu+2)n!}$$

$$+\mu n\frac{d}{dz}\sum_{n=0}^{\infty}\frac{B_p(\gamma+qn,c-\gamma)(c)_{qn}(-z)^n}{B(\gamma,c-\gamma)\Gamma(\mu n+\nu+2)n!},$$

$$=\sum_{n=0}^{\infty}\frac{B_p(\gamma+qn,c-\gamma)(c)_{qn}(-z)^n(\mu n+\nu+1)}{B(\gamma,c-\gamma)\Gamma(\mu n+\nu+2)n!},$$

$$=J^{\mu,\gamma;c}_{\nu,q}(z;p).$$

3. Properties of the EGBMF allied with derivative

The classical Riemann-Liouville fractional derivative of order μ is determined through

$$D^{\mu}_{x}\{f(x)\}=\frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}dt,\qquad \Re(\mu)<0. \tag{13}$$

The extended Riemann-Liouville fractional derivative specified by Özarslan and Özergin [6] is specified as

$$D^{\mu,p}_{x}\{f(x)\}=\frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}e^{\frac{-px^2}{t(x-t)}}dt,\qquad \Re(\mu)<0,\Re(p)>0. \tag{14}$$

Theorem 5. *Let $p>0, q\in\mathbb{N}, \Re(\delta)>\Re(\lambda)>0, \Re(\mu)>0, \Re(\gamma)>0, \Re(\nu)>-1$, then*

$$D^{\lambda-c,p}_{z}\left(z^{\lambda-1}J^{\mu,c}_{\nu,q}(z^q)\right)=\frac{z^{c-1}}{\Gamma(c-\lambda)}B(\lambda,c-\lambda)J^{\mu,\lambda;c}_{\nu,q}(z^q;p). \tag{15}$$

Proof. Replace μ by $\lambda-c$ in the definition of the extended fractional derivative operators, we get

$$D^{\lambda-c,p}_{z}\left(z^{\lambda-1}J^{\mu,c}_{\nu,q}(z^q)\right)=\frac{1}{\Gamma(c-\lambda)}\int_0^z t^{\lambda-1}(z-t)^{c-\lambda-1}J^{\mu,c}_{\nu,q}(t^q)e^{\frac{-pz^2}{t(z-t)}}\ dt,$$

$$=\frac{z^{c-\lambda-1}}{\Gamma(c-\lambda)}\int_0^z t^{\lambda-1}\left(1-\frac{t}{z}\right)^{c-\lambda-1}J^{\mu,c}_{\nu,q}(t^q)e^{\frac{-pz^2}{t(z-t)}}\ dt.$$

Lay $u=\frac{t}{z}$ in above equation we acquire

$$=\frac{z^{c-1}}{\Gamma(c-\lambda)}\int_0^1 u^{\lambda-1}(1-u)^{c-\lambda-1}J_{\nu,q}^{\mu,c}(u^q z^q)e^{\frac{-p}{u(1-u)}}\,du.$$

Using (9), we obtain result.

Theorem 6. *The derivative formula of the extended generalized Bessel-Maitland function is*

$$\frac{d^n}{dz^n}J_{\nu,q}^{\mu,\gamma;c}(z;p)=(c)_q(c+q)_q\cdots(c+(n-1)q)_q J_{\nu+n\mu}^{\mu,\nu+nq;c+nq}(z;p). \tag{16}$$

Proof. Take on derivative w.r.t 'z' in (3), we get

$$\frac{d}{dz}J_{\nu,q}^{\mu,\gamma;c}(z;p)=(c)_q J_{\nu+\mu}^{\mu,\nu+q;c+q}(z;p). \tag{17}$$

Thereafter applying the derivative w.r.t 'z' in (17), we get

$$\frac{d^2}{dz^2}J_{\nu,q}^{\mu,\gamma;c}(z;p)=(c)_q(c+q)_q J_{\nu+2\mu}^{\mu,\nu+2q;c+2q}(z;p).$$

Repeating the derivation w.r.t 'z', we get our result

Theorem 7. *Under mentioned derivative formula of the extended generalized Bessel-Maitland function is real*

$$\frac{d^n}{dz^n}[z^\nu J_{\nu,q}^{\mu,\gamma;c}(\lambda z^\mu;p)]=z^{\nu-n}J_{\nu-n,q}^{\mu,\gamma;c}(\lambda z^\mu;p) \tag{18}$$

Proof. Replace z by λz^μ in (3) and take its product with z^ν, then taking its derivative w.r.t 'z' n times we get our result.

4. Affinity of the EGBMF with Laguerre polynomial and Whittaker function

Through this segment we render the portrayal of the extended generalized Bessel-Maitland function in terms of Laguerre polynomials and Whittaker's function.

Theorem 8. *For the extended generalized Bessel-Maitland function, we get*

$$e^{2p}J_{\nu,q}^{\mu,\gamma;c}(z;p)=\frac{(-1)^k}{B(\gamma,c-\gamma)}\sum_{m,n,k=0}^{\infty}\frac{L_n(p)L_m(p)(\gamma)_{qk}z^k}{\Gamma(\mu k+\nu+1)k!}B(\gamma+m+qk+1,c+n-\gamma+1). \tag{19}$$

Proof. By using the generating function of Laguerre polynomials [2]

$$e^{\frac{-pt}{(1-t)}} = \sum_{n=0}^{\infty} L_n(p) t^n (1-t), \tag{20}$$

we get,

$$e^{\frac{-p}{t(1-t)}} = e^{-2p} \sum_{m,n=0}^{\infty} L_n(p) L_m(p) t^{m+1} (1-t)^{n+1}, \qquad 0 < t < 1. \tag{21}$$

Put (21) in (9)

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{1}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1} (1-t)^{c-\gamma-1} e^{-2p} \sum_{m,n=0}^{\infty} L_n(p) L_m(p) t^{m+1} (1-t)^{n+1} J_{\nu,q}^{\mu,c}(t^q z) dt.$$

Applying (2), we get

$$= \frac{(-1)^k e^{-2p}}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1} (1-t)^{c-\gamma-1} \sum_{m,n,k=0}^{\infty} \frac{L_n(p) L_m(p) t^{m+qk+1} (1-t)^{n+1} (\gamma)_{qk} z^k}{\Gamma(\mu k + \nu + 1) k!}.$$

Reciprocate the order of integration and summation

$$= \frac{(-1)^k e^{-2p}}{B(\gamma, c-\gamma)} \sum_{m,n,k=0}^{\infty} \frac{L_n(p) L_m(p) (\gamma)_{qk} z^k}{\Gamma(\mu k + \nu + 1) k!} \int_0^1 t^{\gamma+m+qk} (1-t)^{c+n-\gamma} dt.$$

Using the definition of Beta function (5)

$$= \frac{(-1)^k e^{-2p}}{B(\gamma, c-\gamma)} \sum_{m,n,k=0}^{\infty} \frac{L_n(p) L_m(p) (\gamma)_{qk} z^k}{\Gamma(\mu k + \nu + 1) k!} B(\gamma + m + qk + 1, c + n - \gamma + 1).$$

Multiplying on both sides by e^{2p}, we get the required result.

Theorem 9. *For the extended generalized Bessel-Maitland function, we have*

$$e^{\frac{3p}{2}} J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{(-1)^k \Gamma(c-\gamma+1)}{B(\gamma, c-\gamma)} \sum_{m,k=0}^{\infty} \frac{L_m(p) (c)_{qk} z^k}{\Gamma(\mu k + \nu + 1) k!} p^{\frac{m+\gamma+qk-1}{2}} W_{\frac{\gamma-2c-qk-m-1}{2}, \frac{m+\gamma+qk}{2}}(p). \tag{22}$$

Proof. By using (20), we get

$$e^{\frac{-p}{t(1-t)}} = e^{-p} e^{\frac{-p}{t}} \sum_{m=0}^{\infty} L_m(p) t^m (1-t). \tag{23}$$

Put (23) in (9), we get

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{1}{B(\gamma,c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} e^{-p} e^{\frac{-p}{t}} (1-t) \sum_{m=0}^{\infty} L_m(p) t^m J_{\nu,q}^{\mu,c}(t^q z) dt.$$

Applying (2)

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{e^{-p}}{B(\gamma,c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma} e^{\frac{-p}{t}} \sum_{m=0}^{\infty} L_m(p) t^m \sum_{k=0}^{\infty} \frac{(c)_{qk}(-t^q z)^k}{\Gamma(\mu k+\nu+1)k!} dt.$$

Reciprocate the order of integration and summation

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{(-1)^k e^{-p}}{B(\gamma,c-\gamma)} \sum_{m,k=0}^{\infty} \frac{L_m(p)(c)_{qk} z^k}{\Gamma(\mu k+\nu+1)k!} \int_0^1 t^{m+qk+\gamma-1}(1-t)^{c-\gamma} e^{\frac{-p}{t}} dt.$$

Now using

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1} e^{\frac{-p}{t}} dt = \Gamma(\nu) p^{\frac{\mu-1}{2}} e^{\frac{-p}{2}} W_{\frac{1-\mu-2\nu}{2},\frac{\mu}{2}}(p), \qquad [\Re(\nu)>0, \Re(p)>0].$$

We get our result.

5. Integral transforms of EGBMF

Theorem 10. *Let* $\mu,\nu,\gamma,c,s \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(c) > \Re(\gamma) > 0, \Re(s) > 0$ *and* $q \in \mathbb{N}$*. The Mellin transform of extended generalized Bessel-Maitland function is specified through*

$$M\left(J_{\nu,q}^{\mu,\gamma;c}(z;p);s\right) = \frac{\Gamma(s)\Gamma(s+c-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)} {}_2\Psi_2 \left[\begin{matrix} (c,q), & (\gamma+s,q); & \\ & & -z \\ (\nu+1,\mu), & (c+2s,q); & \end{matrix} \right]. \quad (24)$$

Proof. Using Mellin transform (7) on extended generalized Bessel-Maitland function

$$M\left(J_{\nu,q}^{\mu,\gamma;c}(z;p);s\right) = \int_0^{\infty} p^{s-1} J_{\nu,q}^{\mu,\gamma;c}(z;p) dp.$$

Now using (9)

$$= \frac{1}{B(\gamma,c-\gamma)} \int_0^{\infty} p^{s-1} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} e^{\frac{-p}{t(1-t)}} J_{\nu,q}^{\mu,c}(t^q z) dt dp.$$

Reciprocate the order of integrals in overhead equation, which is admittable so far as the conditions in the statement of the Theorem, we get

$$= \frac{1}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} J_{\nu,q}^{\mu,c}(t^q z) \int_0^\infty p^{s-1} e^{\frac{-p}{t(1-t)}} dpdt.$$

Now exerting $u = \frac{p}{t(1-t)}$ and using the fact that $\Gamma(s) = \int_0^\infty u^{s-1}e^{-u}du$, we get

$$= \frac{\Gamma(s)}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma+s-1}(1-t)^{s+c-\gamma-1} J_{\nu,q}^{\mu,c}(t^q z)dt.$$

Using (2)

$$= \frac{\Gamma(s)}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma+s-1}(1-t)^{s+c-\gamma-1} \sum_{n=0}^{\infty} \frac{(c)_{qn}(t^q z)^n}{\Gamma(\mu n+\nu+1)n!} dt.$$

Reciprocate the order of summation and integration, we get

$$= \frac{\Gamma(s)}{B(\gamma, c-\gamma)} \sum_{n=0}^{\infty} \frac{(c)_{qn}(z)^n}{\Gamma(\mu n+\nu+1)n!} \int_0^1 t^{\gamma+s+qn-1}(1-t)^{s+c-\gamma-1}dt,$$

using the Beta function (5), we get

$$= \frac{\Gamma(s)}{B(\gamma, c-\gamma)} \sum_{n=0}^{\infty} \frac{(c)_{qn}(z)^n}{\Gamma(\mu n+\nu+1)n!} \frac{\Gamma(\gamma+s+qn)\Gamma(s+c-\gamma)}{\Gamma(2s+c+qn)}.$$

Considering $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$, $B(\gamma, c-\gamma) = \frac{\Gamma(\gamma)\Gamma(c-\gamma)}{\Gamma(c)}$, we get

$$= \frac{\Gamma(s)\Gamma(s+c-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(c+qn)\Gamma(\gamma+s+qn)(z)^n}{\Gamma(\mu n+\nu+1)\Gamma(c+2s+qn)n!}.$$

Using (6), we get our result.

Corollary 11. *Put $s = 1$ in Theorem 5.1, we get*

$$\int_0^\infty J_{\nu,q}^{\mu,\gamma;c}(z;p)dp = \frac{\Gamma(c-\gamma+1)}{\Gamma(c-\gamma)\Gamma(\gamma)} \, {}_2\Psi_2 \left[\begin{matrix} (c,q), & (\gamma+1,q); \\ (\nu+1,\mu), & (c+2,q)\ ; \end{matrix} \; z \right]. \tag{25}$$

Corollary 12. *Taking the inverse Mellin transform on both sides of (24), we get the important complex integral*

$$J_{\nu,q}^{\mu,\gamma;c}(z;p) = \frac{1}{2\pi i\Gamma(\gamma)\Gamma(c-\gamma)} \int_{\lambda-\iota\infty}^{\lambda+\iota\infty} \Gamma(s)\Gamma(s+c-\gamma) {}_2\Psi_2 \left[\begin{matrix} (c,q), & (\gamma+s,q); \\ (\nu+1,\mu), & (c+2s,q); \end{matrix} \; z \right] p^{-s}ds. \tag{26}$$

Theorem 13. *Let $\mu, \nu, \gamma, c, s \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(c) > \Re(\gamma) > 0, \Re(s) > 0$ and $q \in \mathbb{N}$. The Euler transform of extended generalized Bessel-Maitland function is specified through*

$$B\left\{J_{\nu,q}^{\mu,\gamma;c}(z^\mu;p) : \nu+1, 1\right\} = J_{\nu+1,q}^{\mu,\gamma;c}(1;p). \tag{27}$$

Proof. By definition of Euler transform (8) and (3), we get

$$B\left\{J_{\nu,q}^{\mu,\gamma;c}(z^\mu;p) : \nu+1, 1\right\} = \int_0^1 z^{\nu+1-1}(1-z)^{1-1}\sum_{n=0}^{\infty}\frac{B_p(\gamma+qn, c-\gamma)(c)_{qn}(-z^\mu)^n}{B(\gamma, c-\gamma)\Gamma(\mu n+\nu+1)n!}dz.$$

Reciprocating the summation and integral which is gauranteed under convergence condition, we get

$$= \sum_{n=0}^{\infty}\frac{(-1)^n B_p(\gamma+qn, c-\gamma)(c)_{qn}}{B(\gamma, c-\gamma)\Gamma(\mu n+\nu+1)n!}\int_0^1 z^{\mu n+\nu+1-1}(1-z)^{1-1}dz.$$

Applying definition of Beta function (5), we get our result.

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Shorab Wali Khan
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University,
Aligarh-202002, India
email: *swalikhan6@gmail.com*