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UNIQUENESS AND U-H-R STABILITY RESULTS FOR NONLINEAR DUFFING PROBLEM INVOLVING TWO SEQUENTIAL CAPUTO-HADAMARD FRACTIONAL DERIVATIVES

M. Houas, S. Ayadi and H. Boukabcha

Abstract. In this manuscript, we study the uniqueness and Ulam-stability type of solutions for nonlinear sequential Duffing problem with two Caputo-Hadamard-type fractional derivatives. The uniqueness of solutions is derived by Banach's fixed point theorem. Also, we prove the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability of considered problem. An example is provided to illustrate our results.

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1. Introduction and preliminaries

Differential equations of fractional order involving different fractional operators can be used for modeling phenomena in mechanics, biology, chemistry, control theory, etc. These equations have attracted great attention of several researchers, see for example [1, 3, 9, 12, 23, 25] and the references cited therein. Also nonlinear differential equations with fractional derivative are one of the most important mathematical tools used to model real world problems in several domains of science, see [10, 11, 19, 20, 24] and reference therein. The Duffing equation one of these nonlinear equations, which has become very important in the engineering sciences, see for example [4, 6, 15, 21]. The classical form of Duffing equation [5] is given by:

$$D^2 y(t) + \xi D^1 y(t) = f(t) - \varphi(t, y(t)), \ t \in \Omega := [0,1], \ \xi > 0,$$

with $y(0) = d_1, \ D^1 y(0) = d_2, d_i \in \mathbb{R}, (i = 1, 2), \ f$ and φ are continuous real functions. Recently, considerable attention has been given to the study of the uniqueness, existence and Ulam-stability of solutions for fractional version of the Duffing problem, see [2, 8, 16, 17, 22] and the references cited therein. In [7] the authors considered the fractional Duffing problem:

$$_{C}D^{\theta}y\left(t\right)+\xi\ _{C}D^{\gamma}y\left(t\right)=\sin\left(\epsilon t\right)-\upsilon_{1}y\left(t\right)-\upsilon_{2}y^{3}\left(t\right),\xi,\epsilon,\upsilon_{i}>0,i=1,2,$$

for each $t\in\Omega$, under conditions: $v\left(0\right)=d_{1}=0,\ _{C}D^{\delta}v\left(0\right)=d_{2}=0,d_{i}\in\mathbb{R},i=1,2,$ where $1<\theta<2$, $0<\gamma<1$ and $_{C}D^{\varkappa},\varkappa\in\{\theta,\gamma\}$ are the Caputo fractional derivatives. Also, in [18], the authors studied the following fractional Duffing problem:

$$_{C}D^{\theta}y\left(t\right)+\xi\ _{C}D^{\gamma}y\left(t\right)=f\left(t\right)-\varphi\left(t,y\left(t\right)\right),\ t\in\Omega,\ \xi>0,$$

with the conditions: $y\left(t_{0}\right)=y_{0},\ D^{1}y\left(t_{0}\right)=y_{1}$, where $\theta\in\left(1,2\right),\ \gamma\in\left(0,1\right)$ and $_{C}D^{\varkappa},\varkappa\in\{\theta,\gamma\}$ are of the Caputo. In this current manuscript, we study the uniqueness and the Ulam stability of solutions for the following fractional Duffing equation with two Caputo-Hadamard-type fractional derivatives:

$$\begin{cases} {}_{C.H}D^{\theta}\left[{}_{C.H}D^{\gamma}y\left(t\right)\right] \\ =f\left(t\right)-\xi\varphi\left(t,y\left(t\right),{}_{C.H}D^{r}y\left(t\right)\right)-\phi\left(t,y\left(t\right),{}_{H}I^{\alpha}y\left(t\right)\right) \\ y\left(1\right)=A,{}_{C.H}D^{\gamma}y\left(1\right)=B,\ \beta_{1}\ {}_{C.H}D^{\gamma}y\left(\lambda\right)=\beta_{2}\ {}_{C.H}D^{\gamma}y\left(e\right), \\ t\in\Omega:=\left[1,e\right],\alpha>0,\xi>0,1<\lambda<e,A,B,\beta_{i}\in\mathbb{R},i=1,2, \end{cases} \tag{1}$$

where $1<\theta<2$, $0<\gamma<1,r<\delta$ and $_{C.H}D^{\sigma}$ $\sigma\in\{\theta,\gamma,r\}$ are the Caputo-Hadamard fractional derivatives, $_{H}I^{\alpha}$ is the Hadamard fractional integral and $\varphi,\phi:\Omega\times\mathbb{R}\to\mathbb{R}$ and $f:\Omega\to\mathbb{R}$ are given continuous functions. The operator $^{H}I^{\rho}$ is the Hadamard fractional integral [14] given by:

$$_{H}I^{\rho}h\left(t\right)=\tfrac{1}{\Gamma(\rho)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{\rho-1}\frac{h\left(s\right)}{s}ds,\ \rho>0,$$

where $\Gamma\left(\rho\right)=\int_{0}^{\infty}e^{-x}x^{\rho-1}dx$. The operator $_{C.H}D^{\rho}$ is the Caputo-Hadamard fractional derivative [14] defined by:

$$_{C.H}D^{\rho}h(t)=\tfrac{1}{\Gamma(n-\rho)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{n-\rho-1}\delta^{n}\frac{h(s)}{s}ds,$$

where $n-1<\rho<n,n=[\rho]+1,\delta=t\frac{d}{dt},[\rho]$ denotes the integer part of ρ and $\log\left(.\right)=\log_{e}\left(.\right)$.

We recall the following lemma [13].

Lemma 1. *Let* $y \in C_{\delta}^{n}([a,b],\mathbb{R})$. *Then*

$$_{H}I^{\rho}\left(_{C.H}D^{\rho}y\right)(t) = y(t) - \sum_{i=0}^{n-1} c_i (\log t)^i, c_i \in \mathbb{R},$$

where $C_{\delta}^{n}([a,b],\mathbb{R}) = \left\{\psi : [a,b] \to \mathbb{R} : \delta^{n-1}\psi \in C([a,b],\mathbb{R})\right\}$.

Now, let us introduce the space

$$W = \{y : y \in C(\Omega,\mathbb{R}) \text{ and } _{C.H}D^{r}y \in C(\Omega,\mathbb{R})\},$$

endowed with the norm

$$\|y\|_W = \|y\| + \|_{C.H}D^{r}y\| = \sup_{t\in\Omega}|y(t)| + \sup_{t\in\Omega}|_{C.H}D^{r}y(t)|.$$

Then it is well known that $(W, \|.\|_W)$ is a Banach space.

Now, we prove anauxiliary lemma which is pivotal to define the solution for the problem (1).

Lemma 2. *Let* $\beta_1 \log(\lambda) \neq \beta_2$. *Given* $h \in C(\Omega,\mathbb{R})$, *the unique solution of the problem*

$$\begin{cases} _{C.H}D^{\theta}\left[_{C.H}D^{\gamma}y(t)\right] = h(t), \ t \in \Omega, \\ w(1) = A,_{C.H} D^{\gamma}y(1) = B, \beta_{1C.H}D^{\gamma}y(\lambda) = \beta_{2C.H}D^{\gamma}y(e), \\ 1 < \theta < 2, 0 < \gamma < 1, 1 < \lambda < e, A, B, \beta_i, i = 1,2, \end{cases} \tag{2}$$

is given by

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\theta+\gamma)}\int_1^t \left(\log(\frac{t}{s})\right)^{\theta+\delta-1}\frac{h(s)}{s}\,ds \\ &+\frac{\beta_2(\log(t))^{\gamma+1}}{(\beta_1\log(\lambda)-\beta_2)\Gamma(\gamma+2)\Gamma(\theta)}\int_1^e \left(\log(\frac{e}{s})\right)^{\theta-1}\frac{h(s)}{s}\,ds \\ &-\frac{\beta_1(\log(t))^{\gamma+1}}{(\beta_1\log(\lambda)-\beta_2)\Gamma(\gamma+2)\Gamma(\theta)}\int_1^\lambda \left(\log(\frac{\lambda}{s})\right)^{\theta-1}\frac{h(s)}{s}\,ds \\ &+\frac{(\beta_2-\beta_1)B(\log(t))^{\gamma+1}}{(\beta_1\log(\lambda)-\beta_2)\Gamma(\gamma+2)} + \frac{B(\log(t))^{\gamma}}{(\beta_1\log(\lambda)-\beta_2)\Gamma(\gamma+1)} + A. \end{aligned} \tag{3}$$

Proof. Using Lemma 1, we get

$$ {}_{C.H}D^{\gamma}\left[y(t)\right] = {}_{H}I^{\theta}\left[y(t)\right] + c_0 + c_1 \log(t). \tag{4}$$

It follows that

$$ y(t) = {}_{H}I^{\theta+\gamma}\left[y(t)\right] + \frac{c_0\left(\log(t)\right)^{\gamma}}{\Gamma(\gamma+1)} + \frac{c_1\left(\log(t)\right)^{\gamma+1}}{\Gamma(\gamma+2)} + c_2, \tag{5}$$

where $c_i, i = 0,1,2$ are arbitrary constants.

Using the boundary conditions (2), we finnd that

$$ c_0 = B,\ c_2 = A, $$

and

$$\begin{aligned} c_1 &= \frac{\beta_2}{\left(\beta_1\log(\lambda)-\beta_2\right)\Gamma(\theta)}\int_1^e \left(\log(\frac{e}{s})\right)^{\theta-1}\frac{h(s)}{s}\,ds \\ &\quad -\frac{\beta_1}{\left(\beta_1\log(\lambda)-\beta_2\right)\Gamma(\theta)}\int_1^{\lambda} \left(\log(\frac{\lambda}{s})\right)^{\theta-1}\frac{h(s)}{s}\,ds \\ &\quad +\frac{\left(\beta_2-\beta_1\right)B}{\left(\beta_1\log(\lambda)-\beta_2\right)}. \end{aligned}$$

Substituting the value of $c_i, i = 0,1,2$ in (5), we obtain (3).

2. Existence and uniqueness of solution

In this section, we will use the the contraction mapping principle to prove the uniqueness of solutions the above problem. In view of Lemma 2, we define an operator $G : W \to W$ as

$$\begin{aligned} Gy\left(t\right) &= \frac{1}{\Gamma(\theta+\gamma)}\int_1^t \left(\log(\frac{t}{s})\right)^{\theta+\gamma-1}\frac{\left(f\left(s\right)-\xi\varphi_y^{\bullet}\left(s\right)-\phi_y^{\bullet}\left(t\right)\right)}{s}\,ds \\ &\quad +\frac{\beta_2\left(\log(t)\right)^{\gamma+1}}{\left(\beta_1\log(\lambda)-\beta_2\right)\Gamma(\gamma+2)\Gamma(\theta)}\int_1^e \left(\log(\frac{e}{s})\right)^{\theta-1}\frac{\left(f\left(s\right)-\xi\varphi_y^{\bullet}\left(s\right)-\phi_y^{\bullet}\left(t\right)\right)}{s}\,ds \\ &\quad -\frac{\beta_1\left(\log(t)\right)^{\gamma+1}}{\left(\beta_1\log(\lambda)-\beta_2\right)\Gamma(\gamma+2)\Gamma(\theta)}\int_1^{\lambda} \left(\log(\frac{\lambda}{s})\right)^{\theta-1}\frac{\left(f\left(s\right)-\xi\varphi_y^{\bullet}\left(s\right)-\phi_y^{\bullet}\left(t\right)\right)}{s}\,ds \\ &\quad +\frac{\left(\beta_2-\beta_1\right)B\left(\log(t)\right)^{\gamma+1}}{\left(\beta_1\log(\lambda)-\beta_2\right)\Gamma(\gamma+2)} + \frac{B\left(\log(t)\right)^{\gamma}}{\left(\beta_1\log(\lambda)-\beta_2\right)\Gamma(\gamma+1)} + A. \end{aligned} \tag{6}$$

For computational convenience, we set

$$\begin{aligned}
\Theta &: = \frac{1}{\Gamma(\theta+\gamma+1)}+\frac{|\beta_2|+|\beta_1|(\log(\lambda))^{\theta}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+2)\Gamma(\theta+1)}, \\
\Theta^* &: = \frac{1}{\Gamma(\theta+\gamma-r+1)}+\frac{|\beta_2|+|\beta_1|(\log(\lambda)^{\theta}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma-r+2)\Gamma(\theta+1)}, \\
M &: = \frac{|\beta_2-\beta_1|\,|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\delta+2)}+\frac{|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\delta+1)}+|A|, \\
M^* &: = \frac{|\beta_2-\beta_1|\,|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma-r+2)}+\frac{|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma-r+1)}.
\end{aligned} \tag{7}$$

We give the following main result:

Theorem 3. *Let $\varphi,\phi:\Omega\times\mathbb{R}^2\to\mathbb{R}$ and $f:\Omega\to\mathbb{R}$ be continuous functions. In addition we suppose that:*

(C_1) : *There exists constant $k_1>0, k_2>0$ such that for all $t\in\Omega$ and $x_i, y_i\in\mathbb{R}, i=1,2,$ we have*

$$|\varphi(t,y_1,x_1)-\varphi(t,y_2,x_2)|\le k_1(|y_1-y_2|+|x_1-x_2|),$$

and

$$|\phi(t,u_1,v_1)-\phi(t,t,u_1,v_1)|\le k_2(|u_1-u_2|+|v_1-v_2|).$$

If

$$((\xi+1)\Gamma(\alpha+1)+1)(\Theta+\Theta^*)<\Gamma(\alpha+1)k^{-1}, \tag{8}$$

where $k=\max\{k_i, i=1,2\}$, Θ and Θ^* given by (7). Then the problem (1) has a unique solution.

Proof. We set $N=\max\{N_i, i=1,2,3\}$, where N_i are finite numbers given by $N_1=\sup_{t\in\Omega}|\varphi(t,0,0,0)|$, $N_2=\sup_{t\in\Omega}|\phi(t,0,0,0)|$ and $N_3=\sup_{t\in\Omega}|f(t)|$. Setting

$$\frac{N(\Theta+\Theta^*)(\xi+2)N+M+M^*}{1-(\Theta+\Theta^*)\dfrac{k[(\xi+1)\Gamma(\alpha+1)+1]}{\Gamma(\alpha+1)}}\le\mu,$$

we show that $GB_\mu\subset B_\mu$, where $B_\mu=\{y\in W:\|y\|_W\le\mu\}$. By (C_1), we can write

$$\begin{aligned}
\left|\varphi_y^{\bullet}(t)\right| &= |\varphi(t,y(t),{}_{C.H}D^r y(t))| \\
&\le |\varphi(t,y(t),{}_{C.H}D^r y(t))-\varphi(t,0,0)|+|\varphi(t,0,0)| \\
&\le k_1\|y\|_W+N_2\le k_1\mu+N,
\end{aligned} \tag{9}$$

and

$$\begin{aligned} \left|\phi_y^{\bullet}(t)\right| &= \left|\phi\left(t, y(t), {}_H I^{\alpha} y(t)\right)\right| \\ &\leq \left|\phi\left(t, y(t), {}_H I^{\alpha} y(t)\right) - \phi(t,0,0)\right| + \left|\phi(t,0,0)\right| \\ &\leq k_2\left(\|y\|_W + \frac{\|y\|_W}{\Gamma(\alpha+1)}\right) + N_2 \leq k_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\mu + N. \end{aligned} \tag{10}$$

For $y \in B_\mu$, we have

$$\begin{aligned} &\|G(y)\| \\ \leq\; & \frac{1}{\Gamma(\theta+\gamma)}\int_1^t \left(\log(\frac{t}{s})\right)^{\theta+\gamma-1} \frac{\left|\left(f(s) - \xi\varphi_y^{\bullet}(s) - \phi_y^{\bullet}(t)\right)\right|}{s}\, ds \\ & + \frac{|\beta_2|\,(\log(t))^{\delta+1}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+2)\Gamma(\theta)}\int_1^e \left(\log(\frac{e}{s})\right)^{\theta-1} \frac{\left|\left(f(s) - \xi\varphi_y^{\bullet}(s) - \phi_y^{\bullet}(t)\right)\right|}{s}\, ds \\ & + \frac{|\beta_1|\,(\log(t))^{\gamma+1}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+2)\Gamma(\theta)}\int_1^\lambda \left(\log(\frac{\lambda}{s})\right)^{\theta-1} \frac{\left|\left(f(s) - \xi\varphi_y^{\bullet}(s) - \phi_y^{\bullet}(t)\right)\right|}{s}\, ds \\ & + \frac{|\beta_2-\beta_1|\,|B|\,(\log(t))^{\gamma+1}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+2)} + \frac{|B|\,(\log(t))^{\gamma}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+1)} + |A|. \end{aligned}$$

Using (9) and (10), we get

$$\begin{aligned} &\|G(y)\| \\ \leq\; & \left[\frac{1}{\Gamma(\theta+\gamma+1)} + \frac{|\beta_2| + |\beta_1|\,(\log(\lambda))^{\theta}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+2)\Gamma(\theta+1)}\right] k\left(\xi+1+\frac{1}{\Gamma(\alpha+1)}\right)\mu \\ & + \left[\frac{1}{\Gamma(\theta+\gamma+1)} + \frac{|\beta_2| + |\beta_1|\,(\log(\lambda))^{\theta}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+2)\Gamma(\theta+1)}\right](\xi+2)\,N \\ & + \frac{|\beta_2-\beta_1|\,|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+2)} + \frac{|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma+1)} + |A| \\ =\; & \frac{k\left[(\xi+1)\,\Gamma(\alpha+1)+1\right]}{\Gamma(\alpha+1)}\Theta\mu + \Theta(\xi+2)\,N + M. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\left\|_{C.H} D^{r} G\left(y\right)\right\| \\
\leq\ & \frac{1}{\Gamma(\theta+\gamma-r)} \int_{1}^{t} \left(\log(\frac{t}{s})\right)^{\theta+\gamma-r-1} \frac{\left|\left(f\left(s\right)-\xi\varphi_{y}^{\bullet}\left(s\right)-\phi_{y}^{\bullet}\left(t\right)\right)\right|}{s} ds \\
&+\frac{|\beta_2|\left(\log(t)\right)^{\gamma-r+1}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma-r+2)\Gamma(\theta)} \int_{1}^{e} \left(\log(\frac{e}{s})\right)^{\theta-1} \frac{\left|\left(f\left(s\right)-\xi\varphi_{y}^{\bullet}\left(s\right)-\phi_{y}^{\bullet}\left(t\right)\right)\right|}{s} ds \\
&+\frac{|\beta_1|\left(\log(t)\right)^{\gamma-r+1}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma-r+2)\Gamma(\theta)} \int_{1}^{\lambda} \left(\log(\frac{\lambda}{s})\right)^{\theta-1} \frac{\left|\left(f\left(s\right)-\xi\varphi_{y}^{\bullet}\left(s\right)-\phi_{y}^{\bullet}\left(t\right)\right)\right|}{s} ds \\
&+\frac{|\beta_2-\beta_1|\,|B|\left(\log(t)\right)^{\gamma-r+1}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma-r+2)}+\frac{|B|\left(\log(t)\right)^{\gamma-r}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\gamma-r+1)}.
\end{aligned}$$

Thanks to (9) and (10), we can write

$$\begin{aligned}
&\left\|_{C.H} D^{r} G\left(y\right)\right\| \\
\leq\ & \left[\frac{1}{\Gamma(\theta+\delta-r+1)}+\frac{|\beta_2|+|\beta_1|\left(\log(\lambda\right)^{\theta}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\delta-r+2)\Gamma(\theta+1)}\right] k\left(\xi+1+\frac{1}{\Gamma\left(\alpha+1\right)}\right)\mu \\
&+\left[\frac{1}{\Gamma(\theta+\delta-r+1)}+\frac{|\beta_1|\left(\log(\lambda\right)^{\theta}}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\delta-r+2)\Gamma(\theta+1)}\right]\left(\xi+2\right)N \\
&+\frac{|\beta_2-\beta_1|\,|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\delta-r+2)}+\frac{|B|}{|\beta_1\log(\lambda)-\beta_2|\,\Gamma(\delta-r+1)} \\
=\ & \frac{k\left[\left(\xi+1\right)\Gamma\left(\alpha+1\right)+1\right]}{\Gamma\left(\alpha+1\right)}\Theta^{*}\mu+\Theta^{*}\left(\xi+2\right)N+M^{*}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left\|G\left(y\right)\right\|_{W} \\
=\ & \frac{k\left[\left(\xi+1\right)\Gamma\left(\alpha+1\right)+1\right]}{\Gamma\left(\alpha+1\right)}\left(\Theta+\Theta^{*}\right)\mu+\left(\Theta+\Theta^{*}\right)\left(\xi+2\right)N+M+M^{*}\leq\mu,
\end{aligned}$$

which implies that $GB_\mu \subset B_\mu$. For $x, y \in B_\mu$, we have

$$
\begin{aligned}
&\|G(y) - G(x)\| \\
\leq\ & \frac{1}{\Gamma(\theta+\gamma)} \int_1^t \left(\log(\frac{t}{s})\right)^{\theta+\gamma-1} \frac{\xi \left|\varphi_y^{\bullet}(s) - \varphi_x^{\bullet}(s)\right| + \left|\phi_y^{\bullet}(t) - \phi_x^{\bullet}(t)\right|}{s} ds \\
&+ \frac{|\beta_2| (\log(t))^{\gamma+1}}{|\beta_1 \log(\lambda) - \beta_2| \Gamma(\gamma+2)} \int_1^e \frac{\left(\log(\frac{e}{s})\right)^{\theta-1}}{\Gamma(\theta)} \frac{\xi \left|\varphi_y^{\bullet}(s) - \varphi_x^{\bullet}(s)\right| + \left|\phi_y^{\bullet}(t) - \phi_x^{\bullet}(t)\right|}{s} ds \\
&+ \frac{|\beta_1| (\log(t))^{\gamma+1}}{|\beta_1 \log(\lambda) - \beta_2| \Gamma(\gamma+2)} \int_1^\lambda \frac{\left(\log(\frac{\lambda}{s})\right)^{\theta-1}}{\Gamma(\theta)} \frac{\xi \left|\varphi_y^{\bullet}(s) - \varphi_x^{\bullet}(s)\right| + \left|\phi_y^{\bullet}(t) - \phi_x^{\bullet}(t)\right|}{s} ds \\
\leq\ & k \left(\frac{(\xi+1)\Gamma(\alpha+1)+1}{\Gamma(\alpha+1)}\right) \Theta \|y - x\|_W .
\end{aligned}
$$

Also for x, $y \in B_\mu$, we have

$$
\begin{aligned}
&\|{}_{C.H}D^r G(y) - {}_{C.H}D^r G(x)\| \\
\leq\ & \frac{1}{\Gamma(\theta+\gamma-r)} \int_1^t \left(\log(\frac{t}{s})\right)^{\theta+\gamma-r-1} \frac{\left|\left(f(s) - \xi\varphi_y^{\bullet}(s) - \phi_y^{\bullet}(t)\right)\right|}{s} ds \\
&+ \frac{|\beta_2| (\log(t))^{\gamma-r+1}}{|\beta_1 \log(\lambda) - \beta_2| \Gamma(\gamma-r+2)} \int_1^e \frac{\left(\log(\frac{e}{s})\right)^{\theta-1}}{\Gamma(\theta)} \frac{\left|\left(f(s) - \xi\varphi_y^{\bullet}(s) - \phi_y^{\bullet}(t)\right)\right|}{s} ds \\
&+ \frac{|\beta_1| (\log(t))^{\gamma-r+1}}{|\beta_1 \log(\lambda) - \beta_2| \Gamma(\gamma-r+2)} \int_1^\lambda \frac{\left(\log(\frac{\lambda}{s})\right)^{\theta-1}}{\Gamma(\theta)} \frac{\left|\left(f(s) - \xi\varphi_y^{\bullet}(s) - \phi_y^{\bullet}(t)\right)\right|}{s} ds \\
=\ & k \left(\frac{(\xi+1)\Gamma(\alpha+1)+1}{\Gamma(\alpha+1)}\right) \Theta^* \|y - x\|_W .
\end{aligned}
$$

From the definition of $\|.\|_W$, we have

$$
\begin{aligned}
\|G(y) - G(x)\|_W &= \|G(y) - G(x)\| + \|{}_{C.H}D^r G(y) -{}_{C.H} D^r G(x)\| \\
&\leq \frac{k\left[(\xi+1)\Gamma(\alpha+1)+1\right]}{\Gamma(\alpha+1)} (\Theta + \Theta^*) \|y - x\|_W .
\end{aligned}
$$

By (8), we can see that G is a contraction. Consequently, by the contraction mapping principle, problem (1) has a uniqueness solution.

3. Ulam-Hyers-Rassias stability

In this section, we consider the Ulam-stability type for the sequential fractional Duffing problem (1).

Definition 1. *The problem (1) is stable in Ulam-Hyers sense if there exists a real number* $\mu_{\varphi,\phi} > 0$ *such that for each* $\lambda > 0$ *and for each solution* $x \in W$ *of the inequality*

$$\left|{}_{C.H}D^{\theta}\left[{}_{C.H}D^{\gamma}x\left(t\right)\right] - \left(f\left(t\right) - \varphi_x^{\bullet}\left(t\right) - \phi_x^{\bullet}\left(t\right)\right)\right| \leq \lambda,\ t \in \Omega, \tag{11}$$

there exists a solution $u \in W$ *of fractional boundary value problem (1) with*

$$\|x-y\|_W \leq \mu_{\varphi,\phi}\lambda,\ t \in \Omega.$$

Definition 2. *The fractional boundary value problem (1) is generalized Ulam-Hyers stable if there exists* $h_{\varphi,\phi} \in C(\mathbb{R}_+,\mathbb{R}_+), h_{\varphi,\phi}(0) = 0,$ *such that for each solution* $x \in W$ *of the inequality (2) there exists a solution* $y \in W$ *of the fractional boundary value problem (1) with*

$$\|x-y\|_W \leq h_{\varphi,\phi}\left(\lambda\right),\ t \in \Omega.$$

Definition 3. *The fractional boundary value problem (1) is Ulam-Hyers-Rassias stable with respect to* $g \in W$ *if there exists a real number* $\mu_{\varphi,\psi} > 0$ *such that for each* $\lambda > 0$ *and for each solution* $x \in W$ *of the inequality*

$$\left|{}_{C.H}D^{\theta}\left[{}_{C.H}D^{\gamma}x\left(t\right)\right] - \left(f\left(t\right) - \varphi_x^{\bullet}\left(t\right) - \phi_x^{\bullet}\left(t\right)\right)\right| \leq \lambda g\left(t\right),\ t \in \Omega, \tag{12}$$

there exists a solution $y \in W$ *of problem (1) with*

$$\|x-y\|_W \leq \mu_{\varphi,\phi}\lambda g\left(t\right),\ t \in \Omega.$$

Definition 4. *The fractional boundary value problem (1) is generalized Ulam-Hyers-Rassias stable with respect to* $g \in W$ *if there exists a real number* $\mu_{\varphi,\psi,g} > 0$ *such that for each solution* $x \in W$ *of the inequality*

$$\left|{}_{C.H}D^{\theta}\left[{}_{C.H}D^{\gamma}x\left(t\right)\right] - \left(f\left(t\right) - \varphi_x^{\bullet}\left(t\right) - \phi_x^{\bullet}\left(t\right)\right)\right| \leq g\left(t\right)\ t \in \Omega, \tag{13}$$

there exists a solution $y \in W$ *of problem (1) with*

$$\left|v\left(t\right) - u\left(t\right)\right| \leq \mu_{\varphi,\phi,g}g\left(t\right),\ t \in \Omega.$$

Remark 1. *A function* $v \in W$ *is a solution of the inequality (11) if and only if there exists a function* $F : [1,e] \to \mathbb{R}$ *such that*

$|F\left(t\right)| \leq \lambda$, $t \in \Omega$,

${}_{C.H}D^{\theta}\left[{}_{C.H}D^{\gamma}x\left(t\right)\right] - \left(f\left(t\right) - \varphi_x^{\bullet}\left(t\right) - \phi_x^{\bullet}\left(t\right)\right) = F\left(t\right)$, $t \in \Omega$.

Theorem 4. *Let $\varphi, \phi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ be continuous functions. Assume that the assumption (C_1) and (8) hold, then problem (1) is Ulam-Hyers stable.*

Proof. Let $x \in W$ be a solution of the inequality (11), i.e.

$$\left|{}_{C.H}D^{\theta}\left[{}_{C.H}D^{\gamma}x\left(t\right)\right]-\left(f\left(t\right)-\varphi_{x}^{\bullet}\left(t\right)-\phi_{x}^{\bullet}\left(t\right)\right)\right|\leq\lambda, t\in\Omega,$$

and let us denote by $y \in W$ the unique solution of the fractional problem

$$\begin{cases} {}_{C.H}D^{\theta}\left[{}_{C.H}D^{\gamma}y\left(t\right)\right]=f\left(t\right)-\xi\varphi_{y}^{\bullet}\left(t\right)-\phi_{y}^{\bullet}\left(t\right) \\ y\left(1\right)=x\left(1\right),\ {}_{C.H}D^{\delta}y\left(1\right)={}_{C.H}D^{\delta}x\left(1\right), \\ {}_{C.H}D^{\gamma}y\left(\lambda\right)={}_{C.H}D^{\gamma}x\left(\lambda\right),\ {}_{C.H}D^{\gamma}y\left(e\right)={}_{C.H}D^{\gamma}y\left(e\right), \\ t\in\Omega,\ 1<\theta<2,\ 0<\gamma<1,\ \xi>0, \end{cases} \tag{14}$$

By integration of the inequality (11), we have

$$\begin{aligned} &\left|x(t)-{}_{H}I^{\theta+\gamma}\left[h_{x}(t)\right]-\frac{c_{0}\left(\log(t)\right)^{\gamma}}{\Gamma(\gamma+1)}-\frac{c_{1}\left(\log(t)\right)^{\gamma+1}}{\Gamma(\gamma+2)}-c_{2}\right| \\ \leq\ &\frac{\lambda}{\Gamma(\theta+\gamma+1)}\left(\log(t)\right)^{\theta+\gamma}, \end{aligned}$$

where $h_x(t) = f\left(t\right)-\varphi_{x}^{\bullet}\left(t\right)-\phi_{x}^{\bullet}\left(t\right)$. By Lemma 2, we can write

$$|x(t)-Gx(t)|\leq\frac{\lambda}{\Gamma(\theta+\gamma+1)}\left(\log(t)\right)^{\theta+\gamma},\ t\in\Omega,$$

and

$$|{}_{C.H}D^{r}x(t)-{}_{C.H}D^{r}Gx(t)|\leq\frac{\lambda}{\Gamma(\theta+\gamma-r+1)}\left(\log(t)\right)^{\theta+\gamma-r},\ t\in\Omega,$$

which imply that

$$\left\|x-G\left(x\right)\right\|_{W}\leq\frac{\lambda}{\Gamma(\theta+\gamma+1)}+\frac{\lambda}{\Gamma(\theta+\gamma-r+1)}.$$

On the other hand, we have

$$\begin{aligned}\|x-y\|_W &\leq \|x-G(x)\|_W+\|G(x)-y\|_W \\ &\leq \|x-G(x)\|_W+\|G(x)-G(y)\|_W \\ &\leq \frac{\lambda}{\Gamma(\theta+\gamma+1)}+\frac{\lambda}{\Gamma(\theta+\gamma-r+1)} \\ &\quad +\frac{k\left[(\xi+1)\Gamma(\alpha+1)+1\right]}{\Gamma(\alpha+1)}\left(\Theta+\Theta^*\right)\|x-y\|_W .\end{aligned}$$

Thus,

$$\|x-y\|_W \leq \frac{\dfrac{1}{\Gamma(\theta+\gamma+1)}+\dfrac{1}{\Gamma(\theta+\gamma-r+1)}}{1-\dfrac{k\left[(\xi+1)\Gamma(\alpha+1)+1\right]}{\Gamma(\alpha+1)}\left(\Theta+\Theta^*\right)}\lambda,$$

if we put

$$\mu_{\varphi,\phi}:=\frac{\dfrac{1}{\Gamma(\theta+\gamma+1)}+\dfrac{1}{\Gamma(\theta+\gamma-r+1)}}{1-\dfrac{k\left[(\xi+1)\Gamma(\alpha+1)+1\right]}{\Gamma(\alpha+1)}\left(\Theta+\Theta^*\right)},$$

then

$$\|x-y\|_W \leq \mu_{\varphi,\phi}\lambda.$$

This shows that the problem (1) is Ulam-Hyers stability.

Theorem 5. *Let $\varphi,\phi:\Omega\times\mathbb{R}^2\to\mathbb{R}$ and $f:\Omega\to\mathbb{R}$ be continuous functions and suppose that the condition (C_1) and (8) hold,. Suppose there exist $\omega_g>0$ and ϱ_g such that*

$$_HI^{\theta+\gamma}\left[g(t)\right]\leq\omega_g g(t) \text{ and } {}_HI^{\theta+\gamma-r}\left[g(t)\right]\leq\varrho_g g(t), \tag{15}$$

for any $t\in\Omega$, where $g\in C([1,e],\mathbb{R}_+)$ is nondecreasing. Then the fractional Duffing problem (1) is Ulam-Hyers-Rassias stable.

Proof. Let $x\in W$ be a solution of the inequality (13), i.e.

$$|x(t)-Gx(t)|\leq {}_HI^{\theta+\gamma}\left[g(t)\right]\leq\frac{1}{\Gamma(\theta+\gamma)}\omega_g g(t),$$

and

$$|{}_{C.H}D^r x(t)-{}_{C.H}D^r Gx(t)|\leq {}_HI^{\theta+\gamma-r}\left[g(t)\right]\leq\frac{1}{\Gamma(\theta+\gamma-r)}\varrho_g g(t).$$

Then we get

$$\begin{aligned}\|x-y\|_W &\leq \|x-Tx\|_W+\|Tx-y\|_W \\ &\leq \|x-Tx\|+\|{}_{C.H}D^r x-{}_{C.H}D^r Tx\|+\|Tx-Ty\|_W\,,\end{aligned}$$

where $y\in W$ the unique solution of the problem (14). Thanks to $(C_i)_{i=1,2}$, we can write

$$\begin{aligned}\|x-y\|_W &\leq \left(\frac{\omega_g}{\Gamma(\theta+\gamma)}+\frac{\varrho_g}{\Gamma(\theta+\gamma-r)}\right)g(t) \\ &\quad+\frac{k\left[(\xi+1)\,\Gamma\left(\alpha+1\right)+1\right]}{\Gamma\left(\alpha+1\right)}\left(\Theta+\Theta^*\right)\|x-y\|_W\,,\end{aligned}$$

which implies that

$$\|x-y\|_W\leq\frac{\dfrac{\omega_g}{\Gamma(\theta+\gamma)}+\dfrac{\varrho_g}{\Gamma(\theta+\gamma-r)}}{1-\dfrac{k\left[(\xi+1)\,\Gamma\left(\alpha+1\right)+1\right]}{\Gamma\left(\alpha+1\right)}\left(\Theta+\Theta^*\right)}g(t),$$

If we take

$$\mu_{\varphi,\phi,g}:\frac{\dfrac{\omega_g}{\Gamma(\theta+\gamma)}+\dfrac{\varrho_g}{\Gamma(\theta+\gamma-r)}}{1-\dfrac{k\left[(\xi+1)\,\Gamma\left(\alpha+1\right)+1\right]}{\Gamma\left(\alpha+1\right)}\left(\Theta+\Theta^*\right)},$$

then

$$\|x-y\|_W\leq\mu_{\varphi,\phi,g}\,g(t),\ \ t\in\Omega.$$

So, the problem (1) is generalized Ulam-Hyers-Rassias stable.

4. Application

Consider the following nonlinear fractional Duffing equation with Hadamard-Caputo type fractional derivatives

$$\begin{cases} {}_{C.H}D^{\frac{5}{3}}\left[{}_{C.H}D^{\frac{1}{2}}y(t)\right] \\ +\dfrac{1}{10\pi^2}\left[\dfrac{1}{3\sqrt{8+t}}\left(\dfrac{|y(t)|}{e^{\pi}(1+|y(t)|)}+\dfrac{\arctan|{}_{C.H}D^{\frac{1}{3}}y(t)|}{1+\arctan|{}_{H}^{C}D^{\frac{1}{3}}y(t)|}+e^{-1}\right)\right] \\ \dfrac{e^{-t}}{5\sqrt{8+t^2}}\sin\left(t+y(t)+\ {}_{H}I^{\frac{3}{2}}y(t)\right)=\dfrac{1}{3}e^{t+1},\ t\in[1,e], \\ z(1)=\dfrac{2}{5}, {}_{H}^{C}D^{\delta}z(1)=\dfrac{\sqrt{6e}}{5},\ \dfrac{5}{17}\ {}_{C.H}D^{\delta}z\left(\dfrac{7}{4}\right)-\dfrac{11}{12}\ {}_{C.H}D^{\delta}z(e)=0, \end{cases} \tag{16}$$

For this example, we have: $\theta=\frac{5}{3}, \gamma=\frac{1}{2}, r=\frac{1}{3}, \alpha=\frac{3}{2}, \zeta=\dfrac{1}{10\pi^2}, A=\dfrac{2}{5}, B=\dfrac{\sqrt{6e}}{5}, \beta_1=\dfrac{5}{17}, \beta_2=\dfrac{11}{12}, \lambda=\dfrac{7}{4}$. So, it is easy to see that $\beta_1\log(\lambda)\neq\beta_2$.

On the other hand,

$$\begin{aligned} \varphi(t,x,y) &= \frac{1}{3\sqrt{8+t}}\left(\frac{|x|}{e^{\pi}(1+|x|)}+\frac{\arctan|y|}{1+\arctan|y|}+e^{-1}\right), \\ \psi(t,x,y) &= \frac{e^{-t}}{5\sqrt{8+t^2}}\sin(t+x+y),\ \phi(t)=\frac{1}{3}e^{t+1}. \end{aligned}$$

For $x_i, y_i\in\mathbb{R}, i=1,2$ and $t\in\Omega$, we have

$$\begin{aligned} |\varphi(t,x_1,y_1)-\varphi(t,x_2,y_2)| &\leq \frac{1}{9}(|x_1-x_2|+|y_1-y_2|), \\ |\psi(t,x_1,y_1)-\psi(t,x_2,y_2)| &\leq \frac{e^{-1}}{15}(|x_1-x_2|+|y_1-y_2|). \end{aligned}$$

So, we can take

$$k_1=\frac{1}{9}, k_2=\frac{e^{-1}}{15},\ k=\max(k_1,k_2)=\frac{1}{9}, \|\phi\|=13.4646.$$

We also have

$$\begin{aligned} \Theta_1 &\simeq 1.1101,\ \Theta_2\simeq 0.1977,\ \Theta_1^*\simeq 1.4196,\ \Theta_2^*\simeq 0.2673, \\ \mu_{\varphi,\psi} &\simeq 1.405,\ \mu_{\varphi,\psi,g}\simeq 0.389\,82. \end{aligned}$$

It follows that

$$((\xi+1)\Gamma(\alpha+1)+1)(\Theta+\Theta^{*}) = 3.396 < \Gamma(\alpha+1)k^{-1} = 11.964.$$

By Theorem 3, we conclude that the problem (16) has a unique solution, and from Theorem 9, problem (16) is Ulam-Hyers stable with

$$\|x-y\|_{W} \leq 1.405\,1\lambda,\ \lambda>0.$$

If we take $g(t)=t^{\frac{1}{2}}$, then we obtain

$${}_{H}I^{\frac{5}{3}+\frac{1}{2}}\left[g(t)=t^{\frac{1}{2}}\right] \leq \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{3}\right)}t^{\frac{1}{2}} = \omega_{g}t^{\frac{1}{2}},$$

and

$${}_{H}I^{\frac{5}{3}+\frac{1}{2}-\frac{1}{3}}\left[g(t)=t^{\frac{1}{2}}\right] \leq \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{10}{3}\right)}t^{\frac{1}{2}} = \varrho_{g}t^{\frac{1}{2}},$$

Hence, the condition (15) is satisfied with $g(t)=t^{\frac{1}{2}}$ and $\omega_{g}=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{3}\right)}, \varrho_{g}=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{10}{3}\right)}t^{\frac{1}{2}}$. It follows from Theorem 10, problem (16) is Ulam-Hyers-Rassias stable with

$$\|x-y\|_{W} \leq 0.389\,82\lambda t^{\frac{1}{2}},\ \lambda>0,\ t\in[1,e].$$

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Houas Mohamed
Laboratory FIMA, Faculty of Sciences and Technology,
Khemis Miliana University,
Khemis Miliana, Algeria
email: *m.houas.st@univ-dbkm.dz*

Ayadi Souad
Laboratory ACE, Faculty of Sciences and Technology,
Khemis Miliana University,
Khemis Miliana, Algeria

Boukabcha Hocine
Laboratory LEIS, Faculty of Sciences and Technology,
Khemis Miliana University,
Khemis Miliana, Algeria