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DIFFERENTIAL SUBORDINATIONS OBTAINED BY USING GENERALIZED SĂLĂGEAN INTEGRO-DIFFERENTIAL OPERATOR

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Abstract. In this paper, we introduce the $\mathcal{L}^n_{\lambda\delta}f$ generalized Sălăgean integro-differential operator, using the Al-Oboudi operator and the generalized Sălăgean integral operator. We investigate differential subordinations, and generalize some previously known results.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $m \in \mathbb{N} = \{1, 2, \cdots\}$, let

$$\mathcal{H}[a, m] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_m z^m + \cdots, z \in \mathbb{U}\}$$

and

$$\mathcal{A}_m = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = z + a_{m+1} z^{m+1} + \cdots, z \in \mathbb{U}\},$$

with $\mathcal{A}_1 = \mathcal{A}$.

Definition 1. *[4, p.4] Let $f, F \in \mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there exists a function $w \in \mathcal{H}(\mathbb{U})$, with $w(0) = 0$ and $\big|w(z)\big| < 1, z \in \mathbb{U}$, such that $f(z) = F\big[w(z)\big], z \in \mathbb{U}$.*

Remark 1. *[4, p.4] If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.*

Definition 2. *[4, p.16] Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination*

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \prec h(z), z \in \mathbb{U}, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1).

To prove our main results we need the following lemmas.

Lemma 1. *[4, p.71] Let h be convex in $\mathbb{U}$ with $h(0) = a$, $\gamma \neq 0$ and $\Re\gamma \geq 0$. If $p \in \mathcal{H}[a, m]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{mz^{\frac{\gamma}{m}}} \int_0^z h(t)t^{\frac{\gamma}{m}-1}dt.$$

The function q is convex and is the best dominant.

Lemma 2. *[5, p.419] Let q be a convex function in $\mathbb{U}$ and let*

$$h(z) = q(z) + m\alpha zq'(z),$$

where $\alpha > 0$ and $m \in \mathbb{N}$. If $p \in \mathcal{H}[q(0), m]$ and

$$p(z) + \alpha zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z),$$

and this result is sharp.

Definition 3. *[1] For a function $f \in \mathcal{A}, \delta \geq 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the Al-Oboudi differential operator $D_\delta^n f$ is defined by*

$$D_\delta^0 f(z) = f(z),$$

$$D_\delta^1 f(z) = (1-\delta)f(z) + \delta zf'(z) = D_\delta f(z),$$

$$D_\delta^n f(z) = D_\delta\big(D_\delta^{n-1} f(z)\big), z \in \mathbb{U}.$$

Remark 2. *D_δ^n is a linear operator and for $f \in \mathcal{A}$,*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we have

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} \big[1 + (k-1)\delta\big]^n a_k z^k, z \in \mathbb{U} \tag{2}$$

and

$$D_\delta^{n+1} f(z) = (1-\delta) D_\delta^n f(z) + \delta z \big(D_\delta^n f(z)\big)', z \in \mathbb{U}. \tag{3}$$

When $\delta = 1$, we get Sălăgean's differential operator [8].

Remark 3. *Differentiating* (3)*, we obtain*

$$\big(D_\delta^{n+1} f(z)\big)' = \big(D_\delta^n f(z)\big)' + \delta z \big(D_\delta^n f(z)\big)'', z \in \mathbb{U}. \tag{4}$$

Definition 4. *[6] For a function $f \in \mathcal{A}, \delta > 0$ and $n \in \mathbb{N}_0$, the operator $I_\delta^n f$ is defined by*

$$I_\delta^0 f(z) = f(z),$$
$$I_\delta^1 f(z) = \frac{1}{\delta} z^{1-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-2} f(t) dt = I_\delta f(z),$$
$$I_\delta^n f(z) = I_\delta\big(I_\delta^{n-1} f(z)\big), z \in \mathbb{U}.$$

Remark 4. *If $f \in \mathcal{A}$ and $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, then*

$$I_\delta^n f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1}{1 + (k-1)\delta}\right]^n a_k z^k, z \in \mathbb{U}, \tag{5}$$

and

$$\delta z \big(I_\delta^n f(z)\big)' = I_\delta^{n-1} f(z) - (1-\delta) I_\delta^n f(z), z \in \mathbb{U}. \tag{6}$$

When $\delta = 1$, we get Sălăgean's integral operator [8].

Remark 5. *Using* (6)*, we have*

$$\big(I_\delta^n f(z)\big)' = \big(I_\delta^{n+1} f(z)\big)' + \delta z \big(I_\delta^{n+1} f(z)\big)'', z \in \mathbb{U}. \tag{7}$$

Motivated by [3] and [7], we introduce the following operator.

Definition 5. *Let* $n \in \mathbb{N}_0, \delta \geq 0$ *and* $\lambda \geq 0$ *with* $\delta \neq \frac{\lambda-1}{\lambda}$. *For* $f \in \mathcal{A}$, *let*

$$\mathcal{L}_{\lambda\delta}^{n} f(z) = \frac{1}{1-\lambda+\lambda\delta}\big[(1-\lambda)D_{\delta}^{n} f(z) + \lambda\delta I_{\delta}^{n} f(z)\big], z \in \mathbb{U}, \tag{8}$$

where the opertors $D_{\delta}^{n} f$ *and* $I_{\delta}^{n} f$ *are given by Definition 3 and Definition 4, respectively.*

Remark 6. *We have*

$$\mathcal{L}_{0\delta}^{n} f(z) = D_{\delta}^{n} f(z),$$
$$\mathcal{L}_{1\delta}^{n} f(z) = I_{\delta}^{n} f(z),$$
$$\mathcal{L}_{\lambda\delta}^{0} f(z) = \mathcal{L}_{\lambda 0}^{n} f(z) = f(z),$$

and

$$\mathcal{L}_{\lambda 1}^{n} f(z) = (1-\lambda)D^{n} f(z) + \lambda I^{n} f(z) \quad \text{(see [7])}.$$

Remark 7. *For* $f \in \mathcal{A}, f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ *by using* (2) *and* (5), *we have*

$$\mathcal{L}_{\lambda\delta}^{n} f(z) = z + \frac{1}{1-\lambda+\lambda\delta}\sum_{k=2}^{\infty}\Bigg[(1-\lambda)\big(1+(k-1)\delta\big)^{n} + \frac{\lambda\delta}{\big(1+(k-1)\delta\big)^{n}}\Bigg] a_k z^k, z \in \mathbb{U}. \tag{9}$$

2. Main results

Theorem 3. *If* $0 \leq \alpha < 1, f \in \mathcal{A}_m$ *and*

$$\Re\Bigg[\big(\mathcal{L}_{\lambda\delta}^{n+1} f(z)\big)' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\delta z\Big(\big(I_{\delta}^{n+1} f(z)\big)'' + \big(I_{\delta}^{n} f(z)\big)''\Big)\Bigg] > \alpha, z \in \mathbb{U}, \tag{10}$$

then

$$\Re\big(\mathcal{L}_{\lambda\delta}^{n} f(z)\big)' > \gamma, z \in \mathbb{U},$$

where

$$\gamma = \gamma(\alpha) = 2\alpha - 1 + \frac{2(1-\alpha)}{\delta m}\int_{0}^{1}\frac{t^{\frac{1}{\delta m}-1}}{1+t}dt.$$

Proof. Let $f \in \mathcal{A}_m$.
If
$$h(z) = \frac{1+(2\alpha-1)z}{1+z}, z \in \mathbb{U},$$
then (10) is equivalent to
$$\left(\mathcal{L}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\delta z\Big(\left(I_\delta^{n+1} f(z)\right)'' + \left(I_\delta^{n} f(z)\right)''\Big) \prec h(z), z \in \mathbb{U}. \tag{11}$$
Using (4), (7) and (8), we obtain
$$\left(\mathcal{L}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\delta z\Big(\left(I_\delta^{n+1} f(z)\right)'' + \left(I_\delta^{n} f(z)\right)''\Big)$$
$$= \frac{(1-\lambda)\left(D_\delta^{n+1} f(z)\right)' + \lambda\delta\left(I_\delta^{n+1} f(z)\right)'}{1-\lambda+\lambda\delta}$$
$$+\frac{\lambda\delta}{1-\lambda+\lambda\delta}\Big(\left(I_\delta^{n+1} f(z)\right)' + \delta z\left(I_\delta^{n+1} f(z)\right)'' + \delta z\left(I_\delta^{n} f(z)\right)'' - \left(I_\delta^{n+1} f(z)\right)'\Big)$$
$$= \frac{1-\lambda}{1-\lambda+\lambda\delta}\left(D_\delta^{n+1} f(z)\right)' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\left(I_\delta^{n+1} f(z)\right)'$$
$$+\frac{\lambda\delta}{1-\lambda+\lambda\delta}\Big[\left(I_\delta^{n} f(z)\right)' + \delta z\left(I_\delta^{n} f(z)\right)'' - \left(I_\delta^{n+1} f(z)\right)'\Big]$$
$$= \frac{1-\lambda}{1-\lambda+\lambda\delta}\Big[\left(D_\delta^{n} f(z)\right)' + \delta z\left(D_\delta^{n} f(z)\right)''\Big] + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\left(I_\delta^{n} f(z)\right)'$$
$$+\frac{\lambda\delta}{1-\lambda+\lambda\delta}\delta z\left(I_\delta^{n} f(z)\right)''$$
$$= \frac{1-\lambda}{1-\lambda+\lambda\delta}\left(D_\delta^{n} f(z)\right)' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\left(I_\delta^{n} f(z)\right)'$$
$$+\delta z\left(\frac{1-\lambda}{1-\lambda+\lambda\delta}\left(D_\delta^{n} f(z)\right)'' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\left(I_\delta^{n} f(z)\right)''\right)$$
$$= \left(\mathcal{L}_{\lambda\delta}^{n} f(z)\right)' + \delta z\left(\mathcal{L}_{\lambda\delta}^{n} f(z)\right)'', z \in \mathbb{U}. \tag{12}$$
From (11) and (12), we have
$$\left(\mathcal{L}_{\lambda\delta}^{n} f(z)\right)' + \delta z\left(\mathcal{L}_{\lambda\delta}^{n} f(z)\right)'' \prec h(z), z \in \mathbb{U}. \tag{13}$$
Let
$$p(z) = \left(\mathcal{L}_{\lambda\delta}^{n} f(z)\right)', z \in \mathbb{U}. \tag{14}$$

Using (9), we get

$$p(z) = 1 + \frac{1}{1-\lambda+\lambda\delta}\sum_{k=m+1}^{\infty}\left[\big(1-\lambda\big)\big(1+(k-1)\delta\big)^n + \frac{\lambda\delta}{\big(1+(k-1)\delta\big)^n}\right]ka_kz^{k-1}$$

$$= 1 + b_mz^m + b_{m+1}z^{m+1} + \cdots, z \in \mathbb{U},$$

and from (13), we have

$$p(z) + \delta zp'(z) \prec h(z), z \in \mathbb{U}.$$

Applying Lemma 1, we obtain

$$p(z) \prec q(z) \prec h(z), z \in \mathbb{U},$$

where

$$q(z) = \frac{1}{\delta m z^{\frac{1}{\delta m}}}\int_0^z h(t)t^{\frac{1}{\delta m}-1}dt$$

$$= \frac{1}{\delta m z^{\frac{1}{\delta m}}}\int_0^z \left[2\alpha - 1 + 2(1-\alpha)\frac{1}{1+t}\right]t^{\frac{1}{\delta m}-1}dt$$

$$= \frac{2\alpha-1}{\delta m z^{\frac{1}{\delta m}}}\int_0^z t^{\frac{1}{\delta m}-1}dt + \frac{2(1-\alpha)}{\delta m z^{\frac{1}{\delta m}}}\int_0^z \frac{t^{\frac{1}{\delta m}-1}}{1+t}dt$$

$$= 2\alpha - 1 + \frac{2(1-\alpha)}{\delta m z^{\frac{1}{\delta m}}}\int_0^z \frac{t^{\frac{1}{\delta m}-1}}{1+t}dt, z \in \mathbb{U}.$$

The function q is convex, it is the best dominant, $q(\mathbb{U})$ is symmetric with respect to the real axis, so we obtain

$$\Re(\mathcal{L}^n_{\lambda\delta}f(z))' = \Re p(z) > \Re q(1) = \gamma(\alpha) = 2\alpha - 1 + \frac{2(1-\alpha)}{\delta m}\int_0^1 \frac{t^{\frac{1}{\delta m}-1}}{1+t}dt.$$

Example 1. *For $m = 1, \lambda = \frac{1}{2}, \delta = \frac{1}{2}, n = 0$ and $\alpha = \frac{1}{2}$ we obtain that the inequality*

$$\Re\left(f'(z) + \frac{zf''(z)}{2}\right) > \frac{1}{2}, z \in \mathbb{U},$$

implies

$$\Re f'(z) > 2 - 2ln2, z \in \mathbb{U}.$$

Theorem 4. *Let q be a convex function, $q(0)=1$ and let h be a function such that*

$$h(z)=q(z)+m\delta zq'(z), m\in\mathbb{N}, \delta>0, z\in\mathbb{U}.$$

If $f\in\mathcal{A}_m$ verifies the following subordination

$$\left(\mathcal{L}^{n+1}_{\lambda\delta}f(z)\right)'+\frac{\lambda\delta}{1-\lambda+\lambda\delta}\delta z\Big(\left(I^{n+1}_{\delta}f(z)\right)''+\left(I^{n}_{\delta}f(z)\right)''\Big)\prec h(z), z\in\mathbb{U}, \tag{15}$$

then

$$\left(\mathcal{L}^{n}_{\lambda\delta}f(z)\right)'\prec q(z), z\in\mathbb{U}.$$

The result is sharp.

Proof. Using (12) and (14), the subordination (15) is equivalent to

$$p(z)+\delta zp'(z)\prec h(z)=q(z)+m\delta zq'(z), z\in\mathbb{U}.$$

Applying Lemma 2, we obtain

$$p(z)\prec q(z), z\in\mathbb{U},$$

that is,

$$(\mathcal{L}^{n}_{\lambda\delta}f(z))'\prec q(z), z\in\mathbb{U},$$

and the result is sharp.

Remark 8. *Taking $m=1$ and $\delta=1$, we obtain Theorem 3 from [7].*

Remark 9. *Taking $\lambda=0$, we obtain Theorem 2.2 from [2].*

Theorem 5. *Let q be a convex function, $q(0)=1$ and let h be a function such that*

$$h(z)=q(z)+mzq'(z), m\in\mathbb{N}, z\in\mathbb{U}.$$

If $f\in\mathcal{A}_m$ verifies the following subordination

$$\left(\mathcal{L}^{n}_{\lambda\delta}f(z)\right)'\prec h(z), z\in\mathbb{U}, \tag{16}$$

then

$$\frac{\mathcal{L}^{n}_{\lambda\delta}f(z)}{z}\prec q(z), z\in\mathbb{U}.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{L}^n_{\lambda\delta} f(z)}{z}, z \in \mathbb{U}. \tag{17}$$

Differentiating (17), we get

$$(\mathcal{L}^n_{\lambda\delta} f(z))' = p(z) + zp'(z), z \in \mathbb{U}.$$

The subordination (16) becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + mzq'(z), z \in \mathbb{U}.$$

Applying Lemma 2, we obtain

$$p(z) \prec q(z), z \in \mathbb{U},$$

that is,

$$\frac{\mathcal{L}^n_{\lambda\delta} f(z)}{z} \prec q(z), z \in \mathbb{U},$$

and the result is sharp.

Remark 10. *Taking $m = 1$ and $\delta = 1$, we obtain Theorem 1 from [7].*

Remark 11. *Taking $\lambda = 0$, we obtain Theorem 2.3 from [2].*

Theorem 6. *Let q be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + mzq'(z), m \in \mathbb{N}, z \in \mathbb{U}.$$

If $f \in \mathcal{A}_m$ verifies the following subordination

$$\left(\frac{z\mathcal{L}^{n+1}_{\lambda\delta} f(z)}{\mathcal{L}^n_{\lambda\delta} f(z)} \right)' \prec h(z), z \in \mathbb{U}, \tag{18}$$

then

$$\frac{\mathcal{L}^{n+1}_{\lambda\delta} f(z)}{\mathcal{L}^n_{\lambda\delta} f(z)} \prec q(z), z \in \mathbb{U}.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{L}^{n+1}_{\lambda\delta} f(z)}{\mathcal{L}^n_{\lambda\delta} f(z)}, z \in \mathbb{U}.$$

We get

$$p(z) + zp'(z) = \big(zp(z)\big)' = \left(\frac{z\mathcal{L}^{n+1}_{\lambda\delta}f(z)}{\mathcal{L}^{n}_{\lambda\delta}f(z)}\right)', z \in \mathbb{U}.$$

The subordination (18) becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + mzq'(z), z \in \mathbb{U}.$$

Applying Lemma 2, we obtain

$$p(z) \prec q(z), z \in \mathbb{U},$$

that is,

$$\frac{\mathcal{L}^{n+1}_{\lambda\delta}f(z)}{\mathcal{L}^{n}_{\lambda\delta}f(z)} \prec q(z), z \in \mathbb{U},$$

and the result is sharp.

Remark 12. *Taking $m = 1$ and $\delta = 1$, we obtain Theorem 2 from [7].*

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