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HANKEL DETERMINANT FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO Q-DIFFERENCE OPERATOR ASSOCIATED WITH GENERALIZED TELEPHONE NUMBER

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Abstract. In the present paper, the second Hankel determinant for certain subclasses of analytic functions with respect to $q-$difference operator associated with generalized telephone number in terms of subordination relation are investigated. The early few coefficients are obtained through the series expansion and later used to determine the optimum bound of second Hankel determinant for the two subclasses of functions defined.

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1. Introduction

Starlike and convex functions denoted by $\mathcal{S}^*$ and $\mathcal{K}$ with representation $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0$ are the usual subclasses of A of the form

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f'(0) - 1 = f(0) = 0$. With slight modification and differentiation of the above two subclasses of functions, starlike with respect to symmetric points, starlike with respect to conjugate points, convex with respect to symmetric points indicated by $\mathcal{T}_s^*, \mathcal{T}_c^*, \mathcal{K}_s^c$ with geometric quantities $Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0, Re\left(\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}\right) > 0$

and $Re\left(\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0$ are respectively derived which belong to the class of Caratheodory function $\mathcal{P}$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{2}$$

with the conditions that $Rep(z) > 0$ and $p(0) = 1$ (see [24, 29, 31]).
Precisely in 2013, a modification was made on starlike function by Babalola [4] to define a class $\mathcal{L}_\lambda(\beta)$ with a representation

$$Re\frac{z(f'(z))^\lambda}{f(z)} > \beta \qquad (\beta \in [0,1), \quad \lambda \geq 1, \quad z \in \mathbb{U}), \tag{3}$$

which are analytic in the unit disk $\mathbb{U}$ and also satisfies the normalization conditions. In that paper some characterization properties for the above defined class were studied (see [22]).
For two functions f and g, analytic in $\mathbb{U}$, one will say that the function f is subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\omega(z)$ which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad and \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)).$$

By making use of subordination between two analytic functions, Ma and Minda [15] first introduced and studied the class of starlike function denoted by $\mathcal{S}^*(\phi)$ which satisfy

$$\mathcal{S}^*(\phi) := \left\{ f \in A, \frac{zf'(z)}{f(z)} \prec \phi(z) \right\} \tag{4}$$

where $\phi(z)$ is analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi \in \mathcal{P}$.
The coefficient bounds of the univalent functions give details information about geometric properties of functions. In the last century, a lot of findings have been done to know the upper bounds for a_2, a_3, a_4 and $|a_2a_4 - a_3^2|$ for the function of the form (1).

Hankel determinant can be used to find the upper bounds of non-linear functional $|a_2a_4 - a_3^2|$. The determinant of the form

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (n, q \in \mathbb{N} = 1, 2, 3...)$$

has its source from $f \in A$ were defined by Noonan and Thomas [21] as it played an important role in the study of singularities (see [7, 8]). According to the existing literature, the determinant of univalent functions satisfy $|H_q(n)| < Kn^{(-\frac{1}{2}+\beta)q+\frac{3}{2}}, (n = 1, 2, ..., q = 2, 3, ...)$ where $\beta > \frac{1}{4000}$ and K depends only on q was investigated by Pommenrenke [25]. Later Hayman [12] showed that $|H_2(n)| < Jn^{\frac{1}{2}}, (n = 1, 2, ...)$ where J is the absolute constant for areally mean univalent functions. The work of Hayman [12] prompted the scholars of now-a-days to study the real depth of the determinant and their results are too voluminous to discuss (see [3, 17, 20, 28, 32, 33]).

Researchers have devoted much time to study q-analysis and $(p, q)-$analysis which are powerful tools in Geometric Functions Theory. Now-a-days the attention has been given to this analysis because of their usefulness in Mathematics and Physics such as areas like calculus, optimal control problems, multiplier transformation and so on.

Jackson ([10, 11]) has initiated the study of q-analysis. This analysis has given birth to many operators like $q-$Alexander, $q-$Bernadi, $q-$Ruscheweyh, $q-$Catas, $q-$hypergeometric and many others. The scholars in recent time have tested these aforementioned operators in different directions with different perspectives and the results obtained are available in literature. For details, see ([1, 2, 9, 16, 18, 19, 23, 26, 27, 30, 31]).

Jackson's q-derivative of a function $f(z) \in A$ given by (1) defined on a subset of the complex space $\mathbb{C}$ is defined as follows:

$$\begin{aligned} D_qf(z) &= \frac{f(z) - f(qz)}{(1-q)z} \quad (0 < q < 1,\ z \neq 0), \\ &= 1 + \sum_{k=2}^{\infty} \frac{1-q^k}{1-q} a_k z^{k-1} \\ &= 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \end{aligned} \tag{5}$$

where $[k]_q = \frac{1-q^k}{1-q}$. Note that $D_qf(0) = f'(0), D_q^2f(z) = D_q(D_qf(z))$.

As $q \to 1^- \implies [k]_q \to k$, hence one will have

$$lim_{q\to 1^-} D_q f(z) = f'(z) \qquad (z \in \mathbb{U}). \tag{6}$$

Bednarz and Wolowiec-Musial [5] introduced a new generalization telephone numbers $T_\beta(n) = T_\beta(n-1) + \beta(n-1)T_\beta(n-2)$ with initial conditions $T_\beta(0) = T_\beta(1) = 1$ for integers $n \geq 2$ and $\beta \geq 1$ for the series

$$e^{x+\frac{\beta x^2}{2}} = \sum_{n=0}^{\infty} T_\beta(n)\frac{x^n}{n!} \quad (\beta \geq 1). \tag{7}$$

Setting parameter $\beta = 1$, one will have classical telephone numbers T_n(see [6]). $T_\beta(n)$ for some values of n are $T_\beta(0) = T_\beta(1) = 1,\ T_\beta(2) = 1+\beta,\ T_\beta(3) = 1+3\beta,\ T_\beta(4) = 3\beta^2+6\beta+1,\ T_\beta(5) = 15\beta^2+10\beta+1$ and $T_\beta(6) = 15\beta^3+45\beta+15\beta+1$ and so on. The authors considered the function $\phi(z) = e^{z+\frac{\beta z^2}{2}}$ with its domain in the unit disk $\mathbb{U}$. Motivated essentially by earlier researchers (see [4, 5, 6, 9, 10, 15, 22]), we introduce two subclasses of analytic functions involving λ-pseudo -q-operator associated with generalized telephone number as follows:

Definition 1: A function f of the form (1) is said to be in the class $\mathcal{ST}_q^*(\lambda, \beta)$ $(\lambda \geq 1,\ \beta \geq 1)$ if it satisfy the following subordination condition:

$$\frac{z(D_q f(z))^\lambda}{f(z)} \prec \phi(z). \tag{8}$$

Definition 2: A function f of the form (1) belongs to the class $\mathcal{M}_q^*(\lambda, \beta)$ $(\lambda \geq 1,\ \beta \geq 1)$ if the following subordination

$$\frac{\left(z(D_q f(z))^\lambda\right)'}{D_q f(z)} \prec \phi(z). \tag{9}$$

holds where the function $\phi(z)$ defined as above.
In this paper, we investigate the early few coefficients through the series expansion and later used to determine the optimum bound of second Hankel determinant $|H_2(2)| = |a_2a_4 - a_3^2|$ for the two subclasses of functions defined.

For the purpose of our investigation, we need the following lemma.

Lemma 1 (Libera and Zlotkiewicz [13, 14]): Let the function $p \in \mathcal{P}$ be given by the power series (2). Then

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{10}$$

for some x, $|x| \leq 1$ and

$$4p_3 = p_1^3 + 2(4-p_1^2)p_1 x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z, \tag{11}$$

for some value of z, $|z| < 1$.

2. Main Results

Theorem 1: Let the function $f \in A$ given by (1) belongs to the function class $\mathcal{ST}_q^*(\lambda, \beta)$ $(\lambda \geq 1,\ \beta \geq 1)$. Then

$$|a_2a_4 - a_3^2| \leq \eta_1 T^2 + \eta_2 T + \frac{2}{(\lambda[3]_q - 1)^2}, \tag{12}$$

where

$$\eta_1 = \Gamma_1 - \Gamma_2 - \frac{1}{16(\lambda[2]_q - 1)(\lambda[4]_q - 1)} + \frac{1}{16(\lambda[3]_q - 1)^2}, \tag{13}$$

$$\eta_2 = 4\Gamma_2 + \frac{1}{4(\lambda[2]_q - 1)(\lambda[4]_q - 1)} - \frac{1}{16(\lambda[3]_q - 1)^2} \tag{14}$$

and

$$T = -\frac{\eta_2}{2\eta_1}. \tag{15}$$

Proof: Let $f \in \mathcal{ST}_q^*(\lambda, \beta)$. Hence by Definition 1, there exist an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{z(D_q f(z))^\lambda}{f(z)} = \phi(\omega(z)) = e^{\omega(z) + \frac{\beta\omega^2(z)}{2}}. \tag{16}$$

Define the function $p(z)$ given by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + ... \quad (z \in \mathbb{U}). \tag{17}$$

Clearly, $p \in \mathcal{P}$. From (17), it follows that

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{p_1}{2}z + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right)z^3 + ... \tag{18}$$

By using (18) in the right hand side of (16) we obtain

$$\phi(\omega(z)) = 1 + \frac{p_1 z}{2} + \frac{4p_2 + (\beta-1)p_1^2}{8}z^2 + \frac{24p_3 + 12(\beta-1)p_1p_2 - (3\beta-1)p_1^3}{48}z^3 + \ldots \tag{19}$$

The left hand side of (16) shows that

$$\begin{aligned}\frac{z(D_qf(z))^\lambda}{f(z)} &= 1 + (\lambda[2]_q - 1)\,a_2 z + \left(\frac{\left((\lambda^2-\lambda)[2]_q^2 - 2\lambda[2]_q + 2\right)}{2}a_2^2 + (\lambda[3]_q - 1)a_3\right)z^2 \\ &+ \Bigg(\frac{(\lambda^3 - 3\lambda^2 + 2\lambda)[2]_q^3 - (3\lambda^2 - 3\lambda)[2]_q^2 + 6\lambda[2]_q - 6}{6}a_2^3 \\ &+ \left((\lambda^2-\lambda)[2]_q[3]_q - \lambda([3]_q + [2]_q) + 2\right)a_2a_3 + (\lambda[4]_q - 1)a_4\Bigg)z^3 + \ldots\end{aligned} \tag{20}$$

Using (19) and (20) in (16) and equating the coefficients of z, z^2 and z^3 on both sides we get

$$a_2 = \frac{p_1}{2(\lambda[2]_q - 1)}, \tag{21}$$

$$a_3 = \frac{p_2}{2(\lambda[3]_q - 1)} + \frac{(\beta-1)p_1^2}{8(\lambda[3]_q - 1)} - \frac{\left((\lambda^2-\lambda)[2]_q^2 - 2\lambda[2]_q + 2\right)p_1^2}{8(\lambda[2]_q - 1)^2(\lambda[3]_q - 1)}, \tag{22}$$

and

$$\begin{aligned}a_4 &= \frac{p_3}{2(\lambda[4]_q - 1)} + \frac{(\beta-1)p_1p_2}{4(\lambda[4]_q - 1)} - \frac{(3\beta-1)p_1^3}{48(\lambda[4]_q - 1)} - \frac{\left((\lambda^3 - 3\lambda^2 + 2\lambda)[2]_q^3 - (3\lambda^2 - 3\lambda)[2]_q^2 + 6\lambda[2]_q - 6\right)p_1^3}{48(\lambda[2]_q - 1)^3(\lambda[4]_q - 1)} \\ &- \frac{\left((\lambda^2-\lambda)[2]_q[3]_q - \lambda([3]_q + [2]_q) + 2\right)p_1p_2}{4(\lambda[2]_q - 1)(\lambda[3]_q - 1)(\lambda[4]_q - 1)} - \frac{(\beta-1)\left((\lambda^2-\lambda)[2]_q[3]_q - \lambda([3]_q + [2]_q) + 2\right)p_1^3}{16(\lambda[2]_q - 1)(\lambda[3]_q - 1)(\lambda[4]_q - 1)} \\ &+ \frac{\left((\lambda^2-\lambda)[2]_q[3]_q - \lambda([3]_q + [2]_q) + 2\right)\left((\lambda^2-\lambda)[2]_q^2 - 2\lambda[2]_q + 2\right)p_1^3}{16(\lambda[2]_q - 1)^3(\lambda[3]_q - 1)(\lambda[4]_q - 1)}\end{aligned} \tag{23}$$

From (21), (22) and (23), one will have

$$\begin{aligned}a_2a_4 - a_3^2 &= \frac{p_1p_3}{4(\lambda[2]_q - 1)(\lambda[4]_q - 1)} + \frac{(\beta-1)p_1^2p_2}{8(\lambda[2]_q - 1)(\lambda[4]_q - 1)} - \frac{(3\beta-1)p_1^4}{96(\lambda[2]_q - 1)(\lambda[4]_q - 1)} \\ &- \frac{\left((\lambda^3 - 3\lambda^2 + 2\lambda)[2]_q^3 - (3\lambda^2 - 3\lambda)[2]_q^2 + 6\lambda[2]_q - 6\right)p_1^4}{96(\lambda[2]_q - 1)^4(\lambda[4]_q - 1)} - \frac{\left((\lambda^2-\lambda)[2]_q[3]_q - \lambda([3]_q + [2]_q) + 2\right)p_1^2p_2}{8(\lambda[2]_q - 1)^2(\lambda[3]_q - 1)(\lambda[4]_q - 1)} \\ &+ \frac{\left((\lambda^2-\lambda)[2]_q[3]_q - \lambda([3]_q + [2]_q) + 2\right)\left((\lambda^2-\lambda)[2]_q^2 - 2\lambda[2]_q + 2\right)p_1^4}{32(\lambda[2]_q - 1)^4(\lambda[3]_q - 1)(\lambda[4]_q - 1)} \\ &- \frac{p_2^2}{4(\lambda[3]_q - 1)^2} - \frac{(\beta-1)p_1^2p_2}{8(\lambda[3]_q - 1)^2} + \frac{\left((\lambda^2-\lambda)[2]_q^2 - 2\lambda[2]_q + 2\right)p_1^2p_2}{8(\lambda[2]_q - 1)^2(\lambda[3]_q - 1)^2} - \frac{(\beta-1)^2p_1^4}{64(\lambda[3]_q - 1)^2} \\ &+ \frac{(\beta-1)\left((\lambda^2-\lambda)[2]_q^2 - 2\lambda[2]_q + 2\right)p_1^4}{32(\lambda[2]_q - 1)^2(\lambda[3]_q - 1)^2} - \frac{\left((\lambda^2-\lambda)[2]_q^2 - 2\lambda[2]_q + 2\right)^2p_1^4}{64(\lambda[2]_q - 1)^4(\lambda[3]_q - 1)^2} \\ &- \frac{(\beta-1)\left((\lambda^2-\lambda)[2]_q[3]_q - \lambda([3]_q + [2]_q) + 2\right)p_1^4}{32(\lambda[2]_q - 1)^2(\lambda[3]_q - 1)(\lambda[4]_q - 1)}\end{aligned} \tag{24}$$

Substitute the values of p_2 and p_3 from (10) and (11) of Lemma 1 in (24), it follows that

$$\begin{aligned} a_2a_4-a_3^2 = &\Bigg(\frac{(\beta-1)\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{32(\lambda[2]_q-1)^2(\lambda[3]_q-1)^2}-\frac{\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)^2}{64(\lambda[2]_q-1)^4(\lambda[3]_q-1)^2}-\frac{(\beta-1)^2}{64(\lambda[3]_q-1)^2}\\ &-\frac{(\beta-1)\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)}{32(\lambda[2]_q-1)^2(\lambda[3]_q-1)(\lambda[4]_q-1)}+\frac{\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{32(\lambda[2]_q-1)^4(\lambda[3]_q-1)(\lambda[4]_q-1)}\\ &-\frac{\left((\lambda^3-3\lambda^2+2\lambda)[2]_q^3-(3\lambda^2-3\lambda)[2]_q^2+6\lambda[2]_q-6\right)}{96(\lambda[2]_q-1)^4(\lambda[4]_q-1)}-\frac{(3\beta-1)}{96(\lambda[2]_q-1)(\lambda[4]_q-1)}+\frac{(\beta-1)}{16(\lambda[2]_q-1)(\lambda[4]_q-1)}\\ &-\frac{\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)(\lambda[4]_q-1)}-\frac{(\beta-1)}{16(\lambda[3]_q-1)^2}+\frac{\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)^2}\\ &-\frac{1}{16(\lambda[3]_q-1)^2}+\frac{1}{16(\lambda[2]_q-1)\lambda[4]_q-1}\Bigg)p^4+\Bigg(\frac{(\beta-1)}{16(\lambda[2]_q-1)(\lambda[4]_q-1)}-\frac{\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)(\lambda[4]_q-1)}\\ &-\frac{(\beta-1)}{16(\lambda[3]_q-1)^2}+\frac{\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)^2}+\frac{1}{8(\lambda[2]_q-1)(\lambda[4]_q-1)}-\frac{1}{8(\lambda[3]_q-1)^2}\Bigg)p^2x(4-p^2)\\ &-\frac{p^2x^2(4-p^2)}{16(\lambda[2]_q-1)(\lambda[4]_q-1)}+\frac{p(4-p^2)(1-|x|^2)z}{8(\lambda[2]_q-1)(\lambda[4]_q-1)}-\frac{x^2(4-p^2)}{16(\lambda[3]_q-1)^2} \end{aligned} \tag{25}$$

Since $|p_1| \leq 2$ by (2), without restriction, we may assume $p_1 = p \in [0,2]$. Then using the triangle inequality to both sides of (25) with $|x| = \rho$ to obtain

$$\begin{aligned} \left|a_2a_4-a_3^2\right| \leq \Gamma_1p^4+\Gamma_2p^2\rho(4-p^2)+\frac{p^2\rho^2(4-p^2)}{16(\lambda[2]_q-1)(\lambda[4]_q-1)}+\frac{p(4-p^2)(1-\rho^2)}{8(\lambda[2]_q-1)(\lambda[4]_q-1)}\\ +\frac{\rho^2(4-p^2)^2}{16(\lambda[3]_q-1)^2}=F(p,\rho)(say), \end{aligned}$$

where

$$\begin{aligned} \Gamma_1 = &\frac{(\beta-1)\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{32(\lambda[2]_q-1)^2(\lambda[3]_q-1)^2}-\frac{\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)^2}{64(\lambda[2]_q-1)^4(\lambda[3]_q-1)^2}-\frac{(\beta-1)^2}{64(\lambda[3]_q-1)^2}\\ &-\frac{(\beta-1)\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)}{32(\lambda[2]_q-1)^2(\lambda[3]_q-1)(\lambda[4]_q-1)}+\frac{\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{32(\lambda[2]_q-1)^4(\lambda[3]_q-1)(\lambda[4]_q-1)}\\ &-\frac{\left((\lambda^3-3\lambda^2+2\lambda)[2]_q^3-(3\lambda^2-3\lambda)[2]_q^2+6\lambda[2]_q-6\right)}{96(\lambda[2]_q-1)^4(\lambda[4]_q-1)}\\ &-\frac{(3\beta-1)p_1^4}{96(\lambda[2]_q-1)(\lambda[4]_q-1)}+\frac{(\beta-1)}{16(\lambda[2]_q-1)(\lambda[4]_q-1)}-\frac{\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)(\lambda[4]_q-1)}-\frac{(\beta-1)}{16(\lambda[3]_q-1)^2}\\ &+\frac{\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)^2}-\frac{1}{16(\lambda[3]_q-1)^2}+\frac{1}{16(\lambda[2]_q-1)\lambda[4]_q-1}. \end{aligned}$$

and

$$\begin{aligned} \Gamma_2 = &\frac{(\beta-1)}{16(\lambda[2]_q-1)(\lambda[4]_q-1)}-\frac{\left((\lambda^2-\lambda)[2]_q[3]_q-\lambda([3]_q+[2]_q)+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)(\lambda[4]_q-1)}-\frac{(\beta-1)}{16(\lambda[3]_q-1)^2}\\ &+\frac{\left((\lambda^2-\lambda)[2]_q^2-2\lambda[2]_q+2\right)}{16(\lambda[2]_q-1)^2(\lambda[3]_q-1)^2}+\frac{1}{8(\lambda[2]_q-1)(\lambda[4]_q-1)}-\frac{1}{8(\lambda[3]_q-1)^2}. \end{aligned}$$

Now

$$\frac{\partial F}{\partial \rho} = \Gamma_2 p^2(4-p^2) + \frac{p^2\rho(4-p^2)}{8(\lambda[2]_q-1)(\lambda[4]_q-1)} - \frac{p(4-p^2)\rho}{4(\lambda[2]_q-1)(\lambda[4]_q-1)} + \frac{\rho(4-p^2)^2}{8(\lambda[3]_q-1)^2}. \tag{26}$$

It is clear that $\frac{\partial F}{\partial \rho} > 0$. This implies that $F(p,\rho)$ is an increasing function on the closed interval $[0,1]$. This implies that maximum value of $F(p,\rho)$ occurs at $\rho = 1$. Therefore,

$$\max_{0\leq\rho\leq 1} F(p,\rho) = F(p,1) = G(p)\ (say),$$

where

$$G(p) = \Gamma_1 p^4 + \Gamma_2 p^2(4-p^2) + \frac{p^2(4-p^2)}{16(\lambda[2]_q-1)(\lambda[4]_q-1)} + \frac{(4-p^2)^2}{16(\lambda[3]_q-1)^2} = \eta_1 p^4 + \eta_2 p^2 + \frac{2}{(\lambda[3]_q-1)^2} \tag{27}$$

where

$$\eta_1 = \Gamma_1 - \Gamma_2 - \frac{1}{16(\lambda[2]_q-1)(\lambda[4]_q-1)} + \frac{1}{16(\lambda[3]_q-1)^2}$$

and

$$\eta_2 = 4\Gamma_2 + \frac{1}{4(\lambda[2]_q-1)(\lambda[4]_q-1)} - \frac{1}{16(\lambda[3]_q-1)^2}$$

Now

$$G'(p) = 4\eta_1 p^3 + 2\eta_2 p \tag{28}$$

$$G''(p) = 12\eta_1 p^2 + 2\eta_2. \tag{29}$$

For optimum value of $G(p)$, consider $G'(p) = 0$. From (28), one will get

$$p^2 = -\frac{\eta_2}{2\eta_1} = \frac{-4\Gamma_2 - \frac{1}{4(\lambda[2]_q-1)(\lambda[4]_q-1)} + \frac{1}{16(\lambda[3]_q-1)^2}}{2\Gamma_1 - 2\Gamma_2 - \frac{1}{8(\lambda[2]_q-1)(\lambda[4]_q-1)} + \frac{1}{8(\lambda[3]_q-1)^2}} = T \tag{30}$$

Substituting the value of p^2 from (30) in (29), it can be shown that

$$G''(p) = -4\eta_2 < 0. \tag{31}$$

Therefore by the second derivative test, $G(p)$ has maximum value at p, where p^2 is given by (30). Substituting the obtained value of p^2 in the expression (27), which gives the maximum value of $G(p)$. From (26) and (27) we have

$$|a_2a_4 - a_3^2| \leq \eta_1 T^2 + \eta_2 T + + \frac{2}{(\lambda[3]_q-1)^2}.$$

This complete the proof of the Theorem 1.

Theorem 2: Let the function $f \in A$ given by (1) belongs to the function class $\mathcal{M}_q^*(\lambda, \beta)$ $(\lambda \geq 1,\ \beta \geq 1)$. Then

$$|a_2a_4 - a_3^2| \leq \eta_3 T_1^2 + \eta_4 T_1 + \frac{1}{(3\lambda-1)^2[3]_q^2}, \tag{32}$$

where

$$\eta_3 = \Gamma_3 - \Gamma_4 - \frac{1}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q} + \frac{1}{16(3\lambda-1)^2[3]_q^2}, \tag{33}$$

$$\eta_4 = 4\Gamma_4 + \frac{1}{4(2\lambda-1)(4\lambda-1)[2]_q[4]_q} - \frac{1}{2(3\lambda-1)^2[3]_q^2} \tag{34}$$

and

$$T_1 = -\frac{\eta_4}{2\eta_3}. \tag{35}$$

Proof: Let the function $f \in A$ be in the class $\mathcal{M}_q^*(\lambda, \beta)$. Hence by Definition 2, there exist an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{(z(D_qf(z))^\lambda)'}{D_qf(z)} = \phi(\omega(z)). \tag{36}$$

The left hand side of (36) yields

$$\begin{aligned}\frac{(z(D_qf(z))^\lambda)'}{D_qf(z)} &= 1 + (2\lambda-1)[2]_q a_{a_2} + \left(\frac{3\lambda(\lambda-1)-4\lambda+2}{2}[]_q^2 a_2^2 + (3\lambda-1)[3]_q a_3\right) z^2 \\ &\left(\frac{(4\lambda(\lambda-1)(\lambda-1)) + 9\lambda(\lambda-1) + 12\lambda - 6}{6}[2]_q^3 a_2^3 + (4\lambda(\lambda-1) - 5\lambda + 2)[2]_q[3]_q a_2 a_3\right. \\ &\left. + (4\lambda-1)[4]_q a_4\right) z^3 + \cdots .\end{aligned} \tag{37}$$

Using (19) and (37) in (36) and equating the coefficients of z, z^2 and z^3 on both sides we get

$$a_2 = \frac{p_1}{2(2\lambda-1)[2]_q}, \tag{38}$$

$$a_3 = \frac{p_2}{2(3\lambda-1)[3]_q} + \frac{(\beta-1)p_1^2}{8(3\lambda-1)[3]_q} - \frac{(3\lambda(\lambda-1)-4\lambda+2)}{8(2\lambda-1)^2(3\lambda-1)[3]_q}p_1^2, \tag{39}$$

and

$$\begin{aligned}a_4 = {} & \frac{p_3}{2(4\lambda-1)[4]_q} + \frac{(\beta-1)p_1p_2}{4(4\lambda-1)[4]_q} - \frac{(3\beta-1)p_1^3}{48(4\lambda-1)[4]_q} - \frac{(4\lambda^3-21\lambda^2+29\lambda-6)p_1^3}{48(2\lambda-1)^3(4\lambda-1)[4]_q} - \frac{(4\lambda^2-9\lambda+2)p_1p_2}{4(2\lambda-1)(3\lambda-1)(4\lambda-1)[4]_q} \\ & - \frac{(\beta-1)(4\lambda^2-9\lambda+2)p_1^3}{16(2\lambda-1)(3\lambda-1)(4\lambda-1)[4]_q} + \frac{(4\lambda^2-9\lambda+2)(3\lambda^2-7\lambda+2)p_1^3}{16(2\lambda-1)^3(3\lambda-1)(4\lambda-1)[4]_q}.\end{aligned} \tag{40}$$

From (38),(39) and (40), one will have

$$\begin{aligned}a_2a_4-a_3^2 &= \frac{p_1p_3}{4(2\lambda-1)(4\lambda-1)[2]_q[4]_q}+\frac{(\beta-1)p_1^2p_2}{8(2\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(3\beta-1)p_1^4}{96(2\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(4\lambda^3-21\lambda^2+29\lambda-6)p_1^4}{96(2\lambda-1)^4(4\lambda-1)[2]_q[4]_q}\\
&-\frac{(4\lambda^2-9\lambda+2)p_1^2p_2}{8(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(\beta-1)(4\lambda^2-9\lambda+2)p_1^4}{32(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}+\frac{(4\lambda^2-9\lambda+2)(3\lambda^2-7\lambda+2)p_1^4}{32(2\lambda-1)^4(3\lambda-1)(4\lambda-1)[2]_q[4]_q}\\
&-\frac{p_2^2}{4(3\lambda-1)^2[3]_q^2}-\frac{(\beta-1)p_1^2p_2}{8(3\lambda-1)^2[3]_q^2}+\frac{(3\lambda^2-7\lambda+2)p_1^2p_2}{8(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}-\frac{(\beta-1)^2p_1^4}{64(3\lambda-1)^2[3]_q^2}+\frac{(\beta-1)(3\lambda^2-7\lambda+2)p_1^4}{32(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}\\
&-\frac{(3\lambda^2-7\lambda+2)^2p_1^4}{64(2\lambda-1)^4(3\lambda-1)^2[3]_q^2}\end{aligned} \tag{41}$$

Substitute the values of p_2 and p_3 from (10) and (11) of Lemma 1 in (41), it follows that

$$\begin{aligned}a_2a_4-a_3^2 &= \Bigg(\frac{(\beta-1)(3\lambda^2-7\lambda+2)}{32(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}-\frac{(3\lambda^2-7\lambda+2)^2}{64(2\lambda-1)^4(3\lambda-1)^2[3]_q^2}-\frac{(\beta-1)^2}{64(3\lambda-1)^2[3]_q^2}\\
&+\frac{(4\lambda^2-9\lambda+2)(3\lambda^2-7\lambda+2)}{32(2\lambda-1)^4(3\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(\beta-1)(4\lambda^2-9\lambda+2)}{32(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(3\beta-1)}{96(2\lambda-1)(4\lambda-1)[2]_q[4]_q}\\
&-\frac{(4\lambda^3-21\lambda^2+29\lambda-6)}{96(2\lambda-1)^4(4\lambda-1)[2]_q[4]_q}+\frac{(\beta-1)}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(4\lambda^2-9\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}\\
&-\frac{(\beta-1)}{16(3\lambda-1)^2[3]_q^2}+\frac{(3\lambda^2-7\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}-\frac{1}{16(3\lambda-1)^2[3]_q^2}+\frac{1}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}\Bigg)p^4\\
&+\Bigg(\frac{(\beta-1)}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(4\lambda^2-9\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(\beta-1)}{16(3\lambda-1)^2[3]_q^2}+\frac{(3\lambda^2-7\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}\\
&-\frac{1}{8(3\lambda-1)^2[3]_q^2}+\frac{1}{8(2\lambda-1)(4\lambda-1)[2]_q[4]_q}\Bigg)p^2x(4-p^2)-\frac{p^2x^2(4-p^2)}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}+\frac{p(1-|x^2|)(4-p^2)z}{8(2\lambda-1)(4\lambda-1)[2]_q[4]_q}\\
&-\frac{x^2(4-p^2)}{16(3\lambda-1)^2[3]_q^2}.\end{aligned} \tag{42}$$

Since $|p_1|\le 2$ by (2) , without restriction we may assume $p_1=p\in[0,2]$. Then using the triangle inequality to both sides of (42) with $|x|=\rho$ to obtain

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\le \Gamma_3p^4+\Gamma_4p^2\rho(4-p^2)+\frac{p^2\rho^2(4-p^2)}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}+\frac{p(1-\rho^2)(4-p^2)}{8(2\lambda-1)(4\lambda-1)[2]_q[4]_q}+\frac{\rho^2(4-p^2)}{16(3\lambda-1)^2[3]_q^2} \\
&= F_1(p,\rho)(say),\end{aligned} \tag{43}$$

where

$$\begin{aligned}\Gamma_3 = \Bigg(&\frac{(\beta-1)(3\lambda^2-7\lambda+2)}{32(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}-\frac{(3\lambda^2-7\lambda+2)^2}{64(2\lambda-1)^4(3\lambda-1)^2[3]_q^2}-\frac{(\beta-1)^2}{64(3\lambda-1)^2[3]_q^2}+\frac{(4\lambda^2-9\lambda+2)(3\lambda^2-7\lambda+2)}{32(2\lambda-1)^4(3\lambda-1)(4\lambda-1)[2]_q[4]_q}\\
&-\frac{(\beta-1)(4\lambda^2-9\lambda+2)}{32(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(3\beta-1)}{96(2\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(4\lambda^3-21\lambda^2+29\lambda-6)}{96(2\lambda-1)^4(4\lambda-1)[2]_q[4]_q}\\
&+\frac{(\beta-1)}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(4\lambda^2-9\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(\beta-1)}{16(3\lambda-1)^2[3]_q^2}+\frac{(3\lambda^2-7\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}\\
&-\frac{1}{16(3\lambda-1)^2[3]_q^2}+\frac{1}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}\Bigg)\end{aligned} \tag{44}$$

and

$$\begin{aligned}\Gamma_4 = \Bigg(&\frac{(\beta-1)}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(4\lambda^2-9\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)(4\lambda-1)[2]_q[4]_q}-\frac{(\beta-1)}{16(3\lambda-1)^2[3]_q^2}+\frac{(3\lambda^2-7\lambda+2)}{16(2\lambda-1)^2(3\lambda-1)^2[3]_q^2}\\
&-\frac{1}{8(3\lambda-1)^2[3]_q^2}+\frac{1}{8(2\lambda-1)(4\lambda-1)[2]_q[4]_q}\Bigg).\end{aligned} \tag{45}$$

Now

$$\frac{\partial F_1}{\partial \rho} = \Gamma_4 p^2(4-p^2) + \frac{p^2\rho(4-p^2)}{8(2\lambda-1)(4\lambda-1)[2]_q[4]_q} - \frac{p\rho(4-p^2)}{4(2\lambda-1)(4\lambda-1)[2]_q[4]_q} + \frac{\rho(4-p^2)^2}{8(3\lambda-1)^2[3]_q^2}. \quad (46)$$

It is clear that $\frac{\partial F_1}{\partial \rho} > 0$. This implies that $F_1(p, \rho)$ is an increasing function on the closed interval $[0, 1]$. This implies that maximum value of $F_1(p, \rho)$ occurs at $\rho = 1$. Therefore,

$$\max_{0\leq\rho\leq 1} F_1(p, \rho) = F_1(p, 1) = G_1(p) \ (say)$$

Now

$$G_1(p) = \Gamma_3 p^4 + \Gamma_4 p^2(4-p^2) + \frac{p^2(4-p^2)}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q} + \frac{(4-p^2)}{16(3\lambda-1)^2[3]_q^2} = \eta_3 p^4 + \eta_4 p^2 + \frac{1}{(3\lambda-1)^2[3]_q^2}, \quad (47)$$

where

$$\eta_3 = \Gamma_3 - \Gamma_4 - \frac{1}{16(2\lambda-1)(4\lambda-1)[2]_q[4]_q} + \frac{1}{16(3\lambda-1)^2[3]_q^2}$$

and

$$\eta_4 = 4\Gamma_4 + \frac{1}{4(2\lambda-1)(4\lambda-1)[2]_q[4]_q} - \frac{1}{2(3\lambda-1)^2[3]_q^2}.$$

Now

$$G_1'(p) = 4\eta_3 p^3 + 2\eta_4 p \quad (48)$$

$$G_1''(p) = 12\eta_3 p^2 + 2\eta_4. \quad (49)$$

For optimum value of $G_1(p)$, consider $G_1'(p) = 0$. From (48), one will get

$$p^2 = -\frac{\eta_4}{2\eta_3} = \frac{-4\Gamma_4 - \frac{1}{4(2\lambda-1)(4\lambda-1)[2]_q[4]_q} + \frac{1}{2(3\lambda-1)^2[3]_q^2}}{2\Gamma_3 - 2\Gamma_4 - \frac{1}{8(2\lambda-1)(4\lambda-1)[2]_q[4]_q} + \frac{1}{16(3\lambda-1)^2[3]_q^2}} = T_1. \quad (50)$$

Substituting the value of p^2 from (50) in (49), it can be shown that

$$G_1''(p) = -4\eta_2 < 0. \quad (51)$$

Therefore by the second derivative test, $G_1(p)$ has maximum value at p, where p^2 is given by (50). Substituting the obtained value of p^2 in the expression (47), which gives the maximum value of $G_1(p)$. From (43) and (47) we have

$$|a_2a_4 - a_3^2| \leq \eta_3 T_1^2 + \eta_4 T_1 + \frac{1}{(3\lambda-1)^2[3]_q^2}.$$

This complete the proof of the Theorem 2.

Concluding Remark: Making use of subordination and q-calculus, authors have introduced two subclasses of analytic functions and obtained second Hankel determinant for the above said classes. The bounds for Fekete-Szegö inequality $|a_3 - \mu a_2^2|$ for both real and complex parameters μ and third Hankel determinant can be also be investigated.

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