Acta Universitatis Apulensis No. 70/2022
ISSN: 1582-5329 pp. 47-63
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2022.70.06

HIGHER ORDER LOGARITHMIC KLEIN-GORDON EQUATION: GLOBAL EXISTENCE, DECAY AND NONEXISTENCE

E. Pişkin, R. Aksoy

Abstract. In this work, we study a higher order Klein-Gordon equation with logarithmic nonlinearity. Firstly, we established the global existence of solution by potential well method. In addition, we obtain exponential decay and global nonexistence of solutions.

2010 *Mathematics Subject Classification*: 35A01, 35B44, 35G20.

Keywords: Higher order Klein-Gordon, Global existence, Decay, Nonexistence, Logarithmic nonlinearity

1. Introduction

In this paper, we consider the following higher order Klein-Gordon equation with logarithmic source term

$$\begin{cases} u_{tt} + \mathcal{P}u + u + u_t = 2u\ln|u|, & x \in \Omega,\ t > 0, \\ \frac{\partial^i u(x,t)}{\partial \upsilon^i} = 0, \quad i = 0,1,2,...,m-1, & x \in \partial\Omega, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1}$$

where $\mathcal{P} = (-\Delta)^m$, $m \geq 1$ is positive integer, Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, υ denotes the unit outward normal vector on $\partial\Omega$, and $\frac{\partial^i}{\partial \upsilon^i}$ denotes the i-th order normal derivation.

The model equation (1) arises in logarithmic quantum mechanics, nuclear physics, optics, supersymmetry and geophysics [5, 6, 7, 21].

When $m = 1$, (1) becomes

$$u_{tt} - \Delta u + u + u_t = u\ln|u|^2. \tag{2}$$

In 2020, Ye [36] proved the existence, exponential decay and blow up of solutions of the equation (2). Hu et al. [33] studied the following equation

$$u_{tt} - \Delta u + u + u_t = u\ln|u|^k. \tag{3}$$

They studied exponential growth and decay of solutions for the equation (3).

In [13], Gorka studied the following Klein-Gordon equation

$$u_{tt} - u_{xx} + u = \varepsilon u \ln |u|^2 .$$

Ye and Li [38] considered the following Klein-Gordon equation

$$u_{tt} - \Delta u + u = u \ln |u| .$$

They obtained global existence and blow up of solutions. Hiramatsu et al. [16] studied the following Klein-Gordon equation

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln u. \tag{4}$$

They proved the dynamics of Q-balls in theoretical physics. Later, Han [15] studied global existence of weak solutions (4). Pişkin and Çalışır [29] investigated the following Petrovsky equation

$$u_{tt} + \Delta^2 u + \Delta^2 u_t = u \ln |u|^2 .$$

They proved energy decay and blow up at infinite time of solutions. Recently, some authors studied the hyperbolic or parabolic type equations with logarithmic nonlinearity (see [3, 4, 8, 9, 10, 11, 17, 19, 20, 25, 30, 31, 26, 27, 28, 37, 39]).

The main purpose of this paper is to proved the global existence, the decay and the global nonexistence of solution to the higher order Klein-Gordon equation with logarithmic source term (1).

This paper is organized as follows: In Section 2, we present some notations and lemmas. In Section 3, we prove the global existence and decay of solutions. In Section 4, we prove the global nonexistence of solutions.

2. Preliminaries

In this section, we denote

$$\|u\| = \|u\|_{L^2(\Omega)} , \qquad \|u\|_p = \|u\|_{L^p(\Omega)} ,$$

for $1 < p < \infty$. Also, let $L^p(\Omega)$ denote the Lebesgue spaces and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ the Sobolev spaces (see [1, 32], for details).

Next, we define the potential energy functional and Nehari functional of problem (1)

$$J(u) = \frac{1}{2}\left\|\mathcal{P}^{\frac{1}{2}} u\right\|^2 + \|u\|^2 - \frac{1}{2}\int_\Omega u^2 \ln |u|^2 dx, \tag{5}$$

$$I(u) = \left\|\mathcal{P}^{\frac{1}{2}} u\right\|^2 + \|u\|^2 - \int_\Omega u^2 \ln |u|^2 dx, \tag{6}$$

and the total energy functional

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \|u\|^2 - \frac{1}{2}\int_\Omega u^2 \ln|u|^2 dx \\ &= \frac{1}{2}\|u_t\|^2 + J(u) \end{aligned} \tag{7}$$

for $u \in H_0^m(\Omega)$, $t \geq 0$ and

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\left\|\mathcal{P}^{\frac{1}{2}}u_0\right\|^2 + \|u_0\|^2 - \frac{1}{2}\int_\Omega u_0^2 \ln|u_0|^2 dx \tag{8}$$

is the initial total energy.

As in Payne and Sattinger [24], The mountain pass value of $J(u)$ (also known as potential well depth) is defined as

$$d = \inf\left\{\sup_{\lambda\geq 0} J(\lambda u) : u \in H_0^m(\Omega)/\{0\}\right\}. \tag{9}$$

Now, we define the so called Nehari manifold (see [23, 24, 34, 35]) as follows

$$\mathcal{N} = \{u \in H_0^m(\Omega)/\{0\} : K(u) = 0\}$$

$\mathcal{N}$ separates the two unbounded sets

$$\mathcal{N}^+ = \{u \in H_0^m(\Omega)/\{0\} : K(u) > 0\} \cup \{0\}$$

$$\mathcal{N}^- = \{u \in H_0^m(\Omega)/\{0\} : K(u) < 0\}.$$

Then, the stable set $\mathcal{W}$ and the unstable set $\mathcal{U}$ as follows

$$\mathcal{W} = \{u \in H_0^m(\Omega)/\{0\} : J(u) \leq d\} \cap \mathcal{N}^+$$

$$\mathcal{U} = \{u \in H_0^m(\Omega)/\{0\} : J(u) \leq d\} \cap \mathcal{N}^-.$$

It is readily seen that the potential well depht d defined in (9) may also be characterized as

$$d = \inf_{u\in\mathcal{N}} J(u). \tag{10}$$

Definition 1. *The function $u(x,t)$ is a weak solution of (1) on $[0,T]$, if*

$$u \in C([0,T], H_0^m(\Omega)),\ u_t \in C([0,T], L^2(\Omega))$$

and u satisfies

$$\int_\Omega u_{tt}\varphi dx + \int_\Omega \mathcal{P}^{\frac{1}{2}}u\mathcal{P}^{\frac{1}{2}}\varphi dx + \int_\Omega u_t\varphi dx + \int_\Omega u\varphi dx = \int_\Omega u\ln|u|^2\varphi dx$$

for each test function $\varphi \in H_0^m(\Omega)$ and for almost all $t \in [0,T]$.

The proof of the following lemma can be done as in [17].

Lemma 1. *Let $u(x,t)$ be a solution of the problem (1). Then $E(t)$ is a non-increasing function for $t>0$ and*

$$E'(t) = -\|u_t\|^2 \le 0.$$

Lemma 2. *[1, 32]. Let r be a number with*

$$\begin{cases} 2 \le r < +\infty, & \text{if} \quad n \le 2m, \\ 2 \le r \le \frac{2n}{n-2m}, & \text{if} \quad n > 2m. \end{cases}$$

Then there is constant C depending on Ω and r such that

$$\|u\|_r \le C \left\| \mathcal{P}^{\frac{1}{2}} u \right\|, \ \forall u \in H_0^m(\Omega).$$

Lemma 3. *[12, 14]. If $u \in H_0^1(\Omega)$, then for each $a>0$, one has the inequality*

$$\int_\Omega u^2 \ln|u| dx \le \|u\|^2 \ln\|u\| + \frac{\alpha^2}{2\pi}\|\nabla u\|^2 - \frac{n}{2}(1+\ln\alpha)\|u\|^2.$$

Lemma 4. *If $u \in H_0^m(\Omega)$, then for each $a>0$,*

$$\int_\Omega u^2 \ln|u| dx \le \|u\|^2 \ln\|u\| + \frac{c_p\alpha^2}{2\pi}\left\|\mathcal{P}^{\frac{1}{2}} u\right\|^2 - \frac{n}{2}(1+\ln\alpha)\|u\|^2.$$

Proof. By using the embedding theorem ($\|\nabla u\|^2 \le c_p \left\|\mathcal{P}^{\frac{1}{2}} u\right\|^2$), we arrive at

$$\int_\Omega u^2 \ln|u| dx \le \|u\|^2 \ln \| \ u \ \| + \frac{c_p\alpha^2}{2\pi}\left\|\mathcal{P}^{\frac{1}{2}} u\right\|^2 - \frac{n}{2}(1+\ln\alpha)\|u\|^2,$$

where c_p constant.

We conclude this section by stating a local existence result of the problem (1), which can be established by similar way as done in combination of the arguments in [2, 18, 22].

Theorem 5. *(Local existence). Assume that $u_0 \in H_0^m(\Omega)$, $u_1 \in L^2(\Omega)$. Then there exists $T>0$ such that the problem (1) has a unique local solution $u(x,t)$ which satisfies*

$$u \in C\left([0,T); H_0^m(\Omega)\right), \quad u_t \in C([0,T); L^2(\Omega)).$$

Moreover, at least one of the following statements holds true:

i. $\|u_t\|^2 + \left\|\mathcal{P}^{\frac{1}{2}} u\right\|^2 + \|u\|^2 \longrightarrow \infty$ *as* $t \longrightarrow T^-$;
ii. $T = +\infty$.

3. Global existence and decay of solutions

In this section, we establish the global existence and decay of solutions of (1).

Lemma 6. *Let $u \in H_0^m(\Omega)$ and $\|u\| \neq 0$. Then*

$$I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty, \end{cases}$$

where

$$\lambda^* = \exp\left(\frac{\left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \|u\|^2 - 2\int_\Omega u^2 \ln u dx}{2\|u\|^2} \right).$$

Proof. From (5) it implies

$$J(\lambda u) = \frac{\lambda^2}{2} \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \lambda^2 \|u\|^2 - \lambda^2 \int_\Omega u^2 \ln \lambda u dx.$$

A direct computation on above equality, we have

$$\frac{d}{d\lambda} J(\lambda u) = \lambda \left(\left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \|u\|^2 - 2\ln\lambda \|u\|^2 - 2\int_\Omega u^2 \ln u dx \right). \tag{11}$$

Let $\frac{d}{d\lambda} J(\lambda u) = 0$, then we have

$$\lambda^* = \exp\left(\frac{\left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \|u\|^2 - 2\int_\Omega u^2 \ln u dx}{2\|u\|^2} \right).$$

It follows from (6) that

$$I(\lambda u) = \lambda^2 \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \lambda^2 \|u\|^2 - 2\lambda^2 \int_\Omega u^2 \ln u dx - 2\lambda^2 \ln\lambda \|u\|^2. \tag{12}$$

By (11) and (12), the conclusion in Lemma 6 is valid.

Lemma 7. *Assume that $u \in H_0^m(\Omega)$. The depth of potential well d is defined as*

$$d = \frac{1}{2} \left(\frac{\pi}{c_p} \right)^{\frac{n}{2}} e^n. \tag{13}$$

Proof. By definition of $I(u)$ and using Lemma 4, we get

$$\begin{aligned} I(u) &= \left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \|u\|^2 - \int_\Omega u^2 \ln|u|^2 dx \\ &\geq \left(1 - \frac{c_p \alpha^2}{\pi}\right)\left(\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \|u\|^2\right) + \left[n(1+\ln\alpha) - 2\ln\|u\|\ \right]\|u\|^2 \end{aligned} \quad (14)$$

for any $\alpha > 0$. Taking $\alpha = \sqrt{\frac{\pi}{c_p}}$, we obtain from (14) that

$$I(u) \geq \left[n(1+\ln\alpha) - 2\ln\|u\|\ \right]\|u\|^2 . \quad (15)$$

We have from Lemma 6 that

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2} I(\lambda^* u) + \frac{1}{2}\|\lambda^* u\|^2 . \quad (16)$$

We obtain from (15) and Lemma 6 that

$$0 = I(\lambda^* u) \geq \left[n(1+\ln\alpha) - 2\ln\|\lambda^* u\|\right]\|\lambda^* u\|^2 ,$$

then

$$\|\lambda^* u\|^2 \geq \alpha^n e^n \quad (17)$$

It follows from (16) and (17) that

$$\sup_{\lambda\geq 0} J(\lambda u) \geq \frac{1}{2}\alpha^n e^n \quad (18)$$

By (9) and (18), we get

$$d = \frac{1}{2}\left(\frac{\pi}{c_p}\right)^{\frac{n}{2}} e^n.$$

Lemma 8. *Let $E(0) < d$. If $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$, then $u(t) \in \mathcal{N}^+$ for each $t \in [0, T)$.*

Proof. From (7) ve Lemma 1, we obtain

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + J(u) \\ &\leq \frac{1}{2}\|u_1\|^2 + J(u_0) \\ &= E(0) < d \end{aligned}$$

for $\forall t \in [0,T)$, which implies that

$$J(u) < d. \tag{19}$$

Assume that there exists a number $t^* \in [0,T)$ such that $u(t) \in \mathcal{N}^+$ on $[0,t^*)$ and $u(t^*) \notin \mathcal{N}^+$. Then, in virtue of continuity of $u(t)$,we see $u(t^*) \in \partial\mathcal{N}^+$. From the definition of $\mathcal{N}^+$ and the continuity of $I(u)$ with respect to t, we have

$$I(u(t^*)) = 0. \tag{20}$$

Suppose that (20) holds, then we get from (18) and (15) that

$$\|u(t^*)\|^2 \geq 2d. \tag{21}$$

By (5), (6), (20) and (21), we have

$$J(u(t^*)) = \frac{1}{2}\|u(t^*)\|^2 + \frac{1}{2}I(u(t^*)) \geq d,$$

which is contradictive with (19). Hence, the case (20) is impossible. Consequently, we conclude that $u(t) \in \mathcal{N}^+$ on $[0,T)$.

Theorem 9. *(Global existence). Assume that $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$ and $E(0) < d$. Then the local solution furnished in Theorem 5 is a global solution and T may be taken arbitrarily large.*

Proof. It suffices to show that

$$\|u_t\|^2 + \left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \|u\|^2$$

is bounded independently of t. Under the hypotheses Theorem 9, we get from Lemma 8 that $u \in \mathcal{W}$ on $[0,T)$. So, the following formula holds on $[0,T)$ by Lemma 4

$$\begin{aligned} J(u) &= \frac{1}{2}\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \|u\|^2 - \frac{1}{2}\int_\Omega u^2 \ln|u|^2 dx \\ &\geq \frac{1}{2}\left(1 - \frac{c_p\alpha^2}{\pi}\right)\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \left(1 - \ln\|u\| + \frac{n}{2}(1+\ln\alpha)\right)\|u\|^2. \end{aligned} \tag{22}$$

By (5), (6) and $u \in \mathcal{W}$, we have

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}I(u) \geq \frac{1}{2}\|u\|^2, \tag{23}$$

which implies that

$$\|u\|^2 \leq 2J(u) \leq 2d. \tag{24}$$

It follows from (22) and (24), we obtain

$$J(u) \geq \frac{1}{2}\left(1-\frac{c_p\alpha^2}{\pi}\right)\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2+\left(1-\frac{1}{2}\ln 2d+\frac{n}{2}(1+\ln\alpha)\right)\|u\|^2. \tag{25}$$

By Lemma 7 and $0<\alpha<\sqrt{\frac{\pi}{c_p}}$, we have

$$1-\frac{c_p\alpha^2}{\pi}\geq 0,\ 1-\frac{1}{2}\ln 2d+\frac{n}{2}(1+\ln\alpha)>0.$$

Thus, we have from (25) that

$$J(u)\geq C_1\left(\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2+\|u\|^2\right), \tag{26}$$

where

$$C_1=\min\left\{\frac{1}{2}-\frac{c_p\alpha^2}{2\pi},\ 1-\frac{1}{2}\ln 2d+\frac{n}{2}(1+\ln\alpha)\right\}.$$

We have from (26) that

$$\frac{1}{2}\parallel u_t\parallel^2+C_1\left(\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2+\|u\|^2\right)\leq\frac{1}{2}\|u_t\|^2+J(u)=E(t)\leq E(0)<d, \tag{27}$$

which implies that

$$\|u_t\|^2+\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2+\|u\|^2\leq\frac{d}{C_2}<\infty,$$

where $C_2=\min\{C_1,1\}$. The above inequality and the continuation principle lead to the global existence of solution u for the problem (1).

Theorem 10. *(Decay). Suppose that* $E(0)<\frac{1}{2}\left(\frac{\pi}{c_p}\right)^{\frac{n}{2}}e^n\beta\leq d$, *where* β *is a positive number which satisfies* $0<\beta\leq 1$. *If* $u_0\in\mathcal{W}$, $u_1\in L^2(\Omega)$, *then there exist two positive constants* κ *and* k *independent of* t *such that the global solution has the following exponential decay property*

$$0<E(t)\leq\kappa e^{-kt},\ \forall t\geq 0.$$

Proof. By Lemma 8, we see that $u(t)\in\mathcal{N}^+$ for all $t\geq 0$. Thus, we have $0<E(t)<d$ for all $t\geq 0$. In order to prove the decay of solution. We define

$$F(t)=E(t)+\varepsilon\int_\Omega u_t u dx, \tag{28}$$

where $\varepsilon>0$ will be determined later.

It is easy to prove that there exist two positive constants ξ_1 and ξ_2 depending on ε such that

$$\xi_1 E(t) \le F(t) \le \xi_2 E(t), \tag{29}$$

for $\forall t \ge 0$. In fact, we get from (27) and (28) that

$$\begin{aligned} F(t) &\le E(t) + \frac{\varepsilon}{2}\left(\|u_t\|^2 + \|u\|^2\right) \\ &\le \left(1 + \varepsilon + \frac{\varepsilon}{2C_1}\right) E(t) \\ &= \xi_2 E(t). \end{aligned} \tag{30}$$

On the other hand, by (27) and (28), we obtain the following inequality

$$\begin{aligned} F(t) &\ge E(t) - \frac{\varepsilon}{2}\|u_t\|^2 - \frac{\varepsilon}{2}\|u\|^2 \\ &\ge \frac{1}{2}(1-\varepsilon)\|u_t\|^2 + J(u) - \frac{\varepsilon}{2C_1}E(t). \end{aligned} \tag{31}$$

By choosing ε small enough such that $0 < \varepsilon \le \min\left\{1, \frac{2C_1}{2C_1+1}\right\}$, it follows from (31) that

$$\begin{aligned} F(t) &\ge \left(1 - \varepsilon - \frac{\varepsilon}{2C_1}\right) E(t) \\ &= \xi_1 E(t). \end{aligned} \tag{32}$$

From (30) and (32), the inequality (29) is valid.

We now differentiate (28), by using the equation (1) and Lemma 1, to obtain

$$F^{'}(t) = (\varepsilon - 1)\|u_t\|^2 - \varepsilon\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 - \varepsilon\|u\|^2 - \varepsilon\int_\Omega u_t u dx + \varepsilon\int_\Omega u^2 \ln|u|^2 dx. \tag{33}$$

For any $\zeta > 0$, we have from Young's inequality that

$$\left|\int_\Omega u_t u dx\right| \le \frac{1}{4\zeta}\|u_t\|^2 + \zeta\|u\|^2. \tag{34}$$

Therefore, inserting (34) into (33), we obtain

$$F^{'}(t) \le \left(\varepsilon + \frac{\varepsilon}{4\zeta} - 1\right)\|u_t\|^2 - \varepsilon\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \varepsilon(\zeta - 1)\|u\|^2 + \varepsilon\int_\Omega u^2 \ln|u|^2 dx. \tag{35}$$

By using (7) and (35), for any positive constant η, we have

$$\begin{aligned} F'(t) &\leq -\eta\varepsilon E(t) + \left[\varepsilon\left(1+\frac{\eta}{2}+\frac{1}{4\eta}\right)-1\right]\|u_t\|^2 \\ &+\varepsilon\left(\frac{\eta}{2}-1\right)\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 + \varepsilon(\eta+\zeta-1)\|u\|^2 \\ &+\varepsilon\left(1-\frac{\eta}{2}\right)\int_\Omega u^2\ln|u|^2dx. \end{aligned} \tag{36}$$

Now, choosing $0<\eta\leq 1$, and by Lemma 3 and (24), we get

$$\begin{aligned} F'(t) &\leq -\eta\varepsilon E(t) + \left[\varepsilon\left(1+\frac{\eta}{2}+\frac{1}{4\eta}\right)-1\right]\|u_t\|^2 \\ &-\varepsilon\left(1-\frac{\eta}{2}\right)\left(1-\frac{\alpha^2}{\pi}\right)\left\|\mathcal{P}^{\frac{1}{2}}u\right\|^2 \\ &+\varepsilon\left\{\eta+\zeta-1+\left(1-\frac{\eta}{2}\right)\left[\ln(2J(t))-n(1+\ln\alpha)\right]\right\}\|u\|^2. \end{aligned} \tag{37}$$

By $0<\eta\leq 1$ and $J(t)<E(0)<\frac{1}{2}\left(\frac{\pi}{c_p}\right)^{\frac{n}{2}}e^n\beta\leq d$, we select the constant α to meet $\sqrt{\frac{\pi}{c_p}}\beta^{\frac{1}{n}}\leq\alpha\leq\sqrt{\frac{\pi}{c_p}}$, and take $\zeta>0$ small sufficiently such that

$$\begin{aligned} \zeta &< 1-\eta+\left(\frac{\eta}{2}-1\right)\left[\ln(2J(t))-n(1+\ln\alpha)\right] \\ &< 1-\eta+\left(\frac{\eta}{2}-1\right)\left[\ln\left(\left(\frac{\pi}{c_p}\right)^{\frac{n}{2}}e^n\beta\right)-n(1+\ln\alpha)\right] \\ &= 1-\eta+\left(\frac{\eta}{2}-1\right)\ln\left(\frac{\left(\frac{\pi}{c_p}\right)^{\frac{n}{2}}\beta}{\alpha}\right). \end{aligned}$$

Then, we obtain

$$F'(t)\leq -\eta\varepsilon E(t)+\left[\varepsilon\left(1+\frac{\eta}{2}+\frac{1}{4\eta}\right)-1\right]\|u_t\|^2. \tag{38}$$

Now, choosing ε so small enough that

$$\varepsilon\left(1+\frac{\eta}{2}+\frac{1}{4\eta}\right)-1<0,$$

then the inequality (38) implies that

$$F'(t)\leq -\eta\varepsilon E(t), \forall t\geq 0. \tag{39}$$

We conclude from (29) and (39) that

$$F'(t) \leq -kF(t), \forall t \geq 0, \tag{40}$$

where $k = \eta\varepsilon/\xi_2 > 0$.

Integrating the differential inequality (40) from 0 to t gives the following exponential decay estimate for function $F(t)$

$$F(t) \leq F(0)e^{-kt}, \forall t \geq 0. \tag{41}$$

Consequently, we obtain from (29) once again that

$$E(t) \leq \kappa e^{-kt}, \forall t \geq 0,$$

where $\kappa = F(0)/\xi_1$.This completes the proof of Theorem 10.

4. Global nonexistence of solutions

In this section, we establish the global nonexistence of solutions of (1).

Lemma 11. *Let $u(t)$ be a solution of (1) which is given by Theorem 5. If $u_0 \in \mathcal{U}$ and $E(0) < d$, then $u(t) \in \mathcal{U}$ and $E(t) < d$, for all $t \geq 0$.*

Proof. It follows from the conditions in Lemma 11 and Lemma 1 that

$$E(t) \leq E(0) < d, \forall t \in [0, T).$$

Therefore, we have from (7) that

$$J(u) \leq E(t) < d, \forall t \in [0, T). \tag{42}$$

Next, let us assume by contradiction that there exists $t^* \in [0, T)$ such that $u(t^*) \notin \mathcal{U}$, then by continuity, we have $I(u(t^*)) = 0$. This implies that $u(t^*) \in \mathcal{N}$. We get from (10) that $J(u(t^*)) \geq d$, which is contradiction with (42). Consequantly, the conclusion in Lemma 11 holds.

Theorem 12. *(Global nonexistence) Suppose that $u_0 \in \mathcal{U}, u_1 \in L^2(\Omega)$ satisfies $\int_\Omega u_0(x)u_1(x)dx \neq 0$ and*

$$0 < E(0) < \min\left\{d, \frac{3}{4}\left(\frac{\pi}{c_p}\right)^{\frac{n}{2}} e^n\right\}.$$

Then the solution $u(t)$ in Theorem 5 of the problem (1) blows up in finite $T_* < +\infty$, this means that

$$\lim_{t \longrightarrow T_*^-} \|u(t)\|^2 = +\infty.$$

Proof. By $u_0 \in \mathcal{U}$, $E(0) < d$ and Lemma 11, we obtain $u \in \mathcal{U}$ for all $t \in [0,T]$. Thus, we get

$$I(u) = \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \|u\|^2 - \int_\Omega u^2 \ln |u|^2 dx < 0, \ \forall t \in [0,T]. \tag{43}$$

We have from (43) and Lemma 4 that

$$\left(1 - \frac{c_p \alpha^2}{\pi}\right) \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \|u\|^2 + \left[n(1+\ln\alpha) - \ln\|u\|^2\right] \|u\|^2 < 0. \tag{44}$$

We conclude from $\alpha = \sqrt{\frac{\pi}{c_p}}$ and (44) that

$$n(1+\ln\alpha) - \ln\|u\|^2 < 0,$$

which implies that

$$\|u(t)\|^2 > 2d, \forall t \in [0,T]. \tag{45}$$

Assume by contradiction that the solution $u(t)$ is global. Then for any $T > 0$, we define $G(t) : [0,T] \longrightarrow [0,+\infty]$ by

$$G(t) = \|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds + (T-t)\|u_0\|^2. \tag{46}$$

Noting that $G(t) > 0$ for all $t \in [0,T]$. By the continuity of the function $G(t)$, there exists $\mu > 0$ (independent of the choice of T) such that

$$G(t) \geq \mu > 0, \forall t \in [0,T]. \tag{47}$$

By differentiating on both sides of (46), we get

$$\begin{aligned} G^{'}(t) &= 2\int_\Omega u u_t dx + \|u(t)\|^2 - \|u_0\|^2 \\ &= 2\int_\Omega u u_t dx + 2\int_0^t \int_\Omega u(s) u_t(s) dx ds. \end{aligned} \tag{48}$$

Taking the derivative of the function $G^{'}(t)$ in (48), we obtain

$$G^{''}(t) = 2\|u_t\|^2 + 2\int_\Omega u_{tt} u dx + 2\int_\Omega u_t u dx. \tag{49}$$

We get from (1) and (49) that

$$G^{''}(t) = 2\left[\|u_t(t)\|^2 + \int_\Omega u^2 \ln|u|^2 dx - \left\| \mathcal{P}^{\frac{1}{2}} u(t) \right\|^2 - \|u(t)\|^2\right]. \tag{50}$$

We have from (46), (48) and (50) that

$$\begin{aligned} G(t)G''(t)-\frac{3}{2}[G'(t)]^2 &= 2G(t)\left[\|u_t(t)\|^2+\int_\Omega u^2\ln|u|^2dx\right] \\ &\quad -2G(t)\left[\left\|\mathcal{P}^{\frac{1}{2}}u(t)\right\|^2+\|u(t)\|^2\right] \\ &\quad -6[G(t)-(T-t)\|u_0\|^2]\times\left[\|u_t(t)\|^2+\int_0^t\|u_t(s)\|^2ds\right] \\ &\quad +6K(t) \end{aligned} \tag{51}$$

where

$$\begin{aligned} K(t) &= \left[\|u(t)\|^2+\int_0^t\|u(s)\|^2ds\right]\times\left[\|u_t(t)\|^2+\int_0^t\|u_t(s)\|^2ds\right] \\ &\quad -\left[\int_\Omega uu_tdx+\int_0^t\int_\Omega u(s)u_t(s)dxds\right]^2. \end{aligned} \tag{52}$$

By using Schwarz inequality, we have

$$\left(\int_\Omega uu_tdx\right)^2\leq\|u(t)\|^2\|u_t(t)\|^2, \tag{53}$$

$$\left(\int_0^t\int_\Omega uu_tdxds\right)^2\leq\int_0^t\|u(s)\|^2ds\int_0^t\|u_t(s)\|^2ds, \tag{54}$$

and

$$2\int_0^t\int_\Omega u(s)u_t(s)dxds\int_\Omega uu_tdx\leq\|u_t(t)\|^2\int_0^t\|u(s)\|^2ds+\|u(t)\|^2\int_0^t\|u_t(s)\|^2ds. \tag{55}$$

These inequalities (52)-(55) entail $K(t)\geq 0$ for all $t\in[0,T]$. Therefore, we reach the following differential inequality from (51) that

$$G(t)G''(t)-\frac{3}{2}[G'(t)]^2\geq G(t)\chi(t),\forall t\in[0,T], \tag{56}$$

where

$$\begin{aligned} \chi(t) &= 2\left[\|u_t(t)\|^2+\int_\Omega u^2\ln|u|^2dx-\left\|\mathcal{P}^{\frac{1}{2}}u(t)\right\|^2-\|u(t)\|^2\right] \\ &\quad -6\left[\|u_t(t)\|^2+\int_0^t\|u_t(s)\|^2ds\right]. \end{aligned} \tag{57}$$

We have from (7) and Lemma 4 that

$$\begin{aligned}\chi(t) &\geq -8E(t)+2\left(1-\frac{c_p\alpha^2}{\pi}\right)\left\|\mathcal{P}^{\frac{1}{2}}u(t)\right\|^2+6\left\|u(t)\right\|^2\\ &+2\left[n\left(1+\ln\alpha\right)-\ln\left\|u(t)\right\|^2\right]\left\|u(t)\right\|^2-6\int_0^t\left\|u_t(s)\right\|^2ds. \qquad (58)\end{aligned}$$

By (13), (45) and $\alpha=\sqrt{\frac{\pi}{c_p}}$, we have from (58) that

$$\chi(t)\geq -8E(t)+6\left\|u(t)\right\|^2-6\int_0^t\left\|u_t(s)\right\|^2ds. \qquad (59)$$

By Lemma 1, we get

$$\chi(t)\geq -8E(0)+6\left\|u(t)\right\|^2+2\int_0^t\left\|u_t(s)\right\|^2ds. \qquad (60)$$

Hence, we conclude from (45) and $E(0)<d$ that

$$\begin{aligned}\chi(t) &\geq -8E(0)+12d\\ &= 8\left[d-E(0)\right]+4d>0. \qquad (61)\end{aligned}$$

Therefore, there exists $\gamma>0$ which is independent of T such that

$$\chi(t)\geq\gamma>0,\forall t\geq 0. \qquad (62)$$

It follows from (47), (56) and (62) that

$$G(t)G^{''}(t)-\frac{3}{2}[G^{'}(t)]^2\geq\mu\gamma>0,\forall t\in[0,T]. \qquad (63)$$

By the differential inequality (63), we have

$$G(t)\geq\frac{G(0)}{\left(1-\frac{G^{'}(0)}{2G(0)}t\right)^2}. \qquad (64)$$

Hence, there exists T_* such that

$$0<T_*<\frac{2G(0)}{G^{'}(0)}\leq T, \qquad (65)$$

and we have

$$\lim_{t\longrightarrow T_*^-}G(t)=+\infty. \qquad (66)$$

From the definition (46) of $G(t)$, (66) means that

$$\lim_{t\longrightarrow T_*^-}\left\|u(t)\right\|^2=+\infty.$$

Thus we can not suppose that the solution of (1) is global.

References

[1] R. A. Adams, J.J.F. Fournier, *Sobolev Spaces*, Acad. Press, New York (2003).

[2] K. Agre, M. A. Rammaha, *Systems of nonlinear wave equations with damping and source terms*, Differ. Integral Equ. 19(11) (2006), 1235-1270.

[3] M.M. Al-Gharabli, S.A. Messaoudi, *Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term*, J. Evol. Equ. 18(1) (2018), 105-125.

[4] M.M. Al-Gharabli, S.A. Messaoudi, *The existence and the asymptotic behavior of a plate equation with frictional damping and a logarithmic source term*, J. Math. Anal. Appl. 454 (2017), 1114-1128.

[5] K. Bartkowski, P. Gorka, *One-dimensional Klein–Gordon equation with logarithmic nonlinearities*, J. Phys. A. 41(35) (2008), 1-11.

[6] I. Bialynicki-Birula, J. Mycielski, *Nonlinear wave mechanics*, Ann. Phys. 100(1–2) (1976), 62-93.

[7] I. Bialynicki-Birula, J. Mycielski, *Wave equations with logarithmic nonlinearities*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 23(4) (1975), 461-466.

[8] S. Boulaaras, A. Draifia, M. Alnegga, *Polynomial decay rate for Kirchhoff type in viscoelasticity with logarithmic nonlinearity and not necassarily decreasing kernel*, Symmetry 11(2) (2019), 1-24.

[9] S. Boulaaras, A. Draifia, K. Zennir, *General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity*, Math. Methods Appl. Sci. 42(14) (2019), 4795-4814.

[10] T. Cazenave, A. Haraux, *Equations d'evolution avec non linéarité logarithmique*, Ann. Fac. Sci. Toulouse 2(1) (1980), 21–51.

[11] Y. Chen, R. Xu, *Global well-posedness of solutions for fourth order dispersive wave equation with nonlinear weak damping, linear strong damping and logarithmic nonlinearity*, Nonlinear Anal. 192 (2020),1-39.

[12] H. Chen, P. Luo, G. Liu, *Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl. 422 (2015), 84-98.

[13] P. Gorka, *Logarithmic Klein–Gordon equation*, Acta Phys. Pol. B. 40(1) (2009), 59-66.

[14] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97(4) (1975), 1061-1083.

[15] X. Han, *Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics*, Bull. Korean Math. Soc. 50 (2013), 275-283.

[16] T. Hiramatsu, F. Takahashi, M. Kawasaki, *Numerical study of Q-ball formation in gravity mediation*, J. Cosmol. Astropart. Phys. (2010).

[17] N. Irkıl, E. Pişkin, *Global existence and decay of solutions for a higher-order Kirchhoff-type systems with logarithmic nonlinearities*, Quaest. Math. 45(4) (2022), 523-546.

[18] N. Irkıl, E. Pişkin, *Local existence for class of nonlinear higher-order wave equation with logarithmic source term*, Asia Math. 5(3) (2021), 89-100.

[19] C.N. Le, X.T. Le, *Global solution and blow up for a class of Pseudo p-Laplacian evolution equations with logarithmic nonlinearity*, Comput. Math. Appl. 151(1) (2017), 1-21.

[20] L. Ma, Z. B. Fang, *Energy decay estimates and infnite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source*, Math. Methods Appl. Sci. 41 (2018), 2639-2653.

[21] D. Martino, M. Falanga, C. Godano, G. Lauro, *Logarithmic Schrödinger-like equation as a model for magma transport*, Europhys. Lett. 63(3) (2003), 472–475.

[22] S. A. Messaoudi, *Global existence and nonexistence in a system of Petrovsky*, J. Math. Anal. Appl. 265 (2002), 296-308.

[23] Z. Nehari, *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. (1960), 101-123.

[24] L. E. Payne, D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. (1975), 273-303.

[25] E. Pişkin, S. Boulaaras, N. Irkıl, *Qualitative analysis of solutions for the p-Laplacian hyperbolic equation with logarithmic nonlinearity*, Math. Methods Appl. Sci. 44 (2021), 5654-4672.

[26] E. Pişkin, T. Cömert, *Blow-up of solutions for a parabolic Kirchhoff type equation with logarithmic nonlinearity*, Gulf J. Math. 9(2) (2020), 21-30.

[27] E. Pişkin, T. Cömert, *Qualitative analysis of solutions for a parabolic type Kirchhoff equation with logarithmic nonlinearity*, Open J. Discrete Appl. Math. 4(2) (2021), 1-10.

[28] E. Pişkin, T. Cömert, *Existence and decay of solutions for a parabolic type Kirchhoff equation with logarithmic nonlinearity*, Adv. Stud.: Euro-Tbil. Math. J. 15(1) (2022), 111-128.

[29] E. Pişkin, Z. Çalışır, *Decay and blow up at infinite time of solutions for a logarithmic Petrovsky equation*, Tbil. Math. J. 13(4) (2020), 113-127.

[30] E. Pişkin, N. Irkıl, *Well-posedness results for a sixth-order logarithmic Boussinesq equation*, Filomat 33(13) (2019), 3985-4000.

[31] E. Pişkin, N. Irkıl, *Local Existence and Blow up for p-Laplacian Equation with Logarithmic Nonlinearity*, Miskolc Math. Notes 23(1) (2022), 231-251.

[32] E. Pişkin, B. Okutmuştur, *An Introduction to Sobolev Spaces*, Bentham Sci. (2021).

[33] H. Qingying, Z. Hongwei, L. Gongwei, *Asymptotic behavior for a class of logarthmic wave equations with linear damping*, Appl. Math. Optim. (2019), 131-144.

[34] D. H. Sattinger, *On global solution of nonlinear hyperbolic equations*, Arch. Rational Mech. Anal. 28 (1968), 226-244.

[35] M. Willem, *Minimax theorems, Progress in nonlinear differential equations and their applications*, Birkhauser Boston (1996).

[36] Y. Ye, *Global solution and blow up of logarithmic Klein-Gordon equation*, Korean Math. Soc. (2020), 281-294.

[37] Y. Ye, *Logarithmic viscoelastic wave equation in threedimensional space*, Appl. Anal. 100 (2021), 2210-2226.

[38] Y. Ye, L. Li, *Global existence and blow-up of solutions for logarithmic Klein-Gordon equation*, AIMS Mathematics 6(7) (2017), 2076-2091.

[39] H. Zhang, G. Liu, Q. Hu, *Initial boundary value problem for class wave equation with logarithmic source term*, Acta Math. Scientia 36(6) (2016), 1124-1136.

Erhan Pişkin
Dicle University, Department of Mathematics
Diyarbakır, Turkey
email: *episkin@dicle.edu.tr*

Ruken Aksoy
Dicle University, Institute of Natural and Applied Sciences,
Department of Mathematics
Diyarbakır, Turkey
email: *ruken.5214@hotmail.com*