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ON THE TRANSVERSAL INTERSECTION OF THE SURFACES GENERATED BY TIMELIKE BERTRAND CURVES

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Abstract. In this paper, we study the intersection curve of the normal and tangent surfaces of a timelike Bertrand curves. We express the curvature of the intersection curve for the transversal intersection . Moreover, we investigate local properties and some characteristic features of the intersection curve and give some results for all two cases.

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1. Introduction

At the beginning of the 1900s, Bertrand had carried out a work on special types of curves, and then these curves were referred to by his name. He had worked on above mentioned special curves. Furthermore, he had investigated their applications in the areas such as thermodynamics and analytical mechanics [2]. Bertrand curve is defined as a particular curve, whose principal normal vector is linear dependent with another special curve's principal normal vector. These two special curves are called Bertrand pair. Additionally, there is a linear relation between the curvature and the torsion. Analysis of the surface- surface intersection problem is an essential process required in modeling and designing complex objects and shapes in CAD/CAM systems, computer animation, and NC machining[7],[1],[6],[4]. Parametric and implicit surfaces are the most commonly used types of surfaces in geometric modeling systems. These two types of surfaces contributed to three types of surface-surface intersection problems: parametric-parametric, implicit - implicit, and parametric – implicit. In the Euclidean space, there are numerous works on the differential geometry of surface-surface intersection and intersection curves. Most of the papers are subjected to examining the properties of the intersection curve for parametric-parametric intersection problems. Faux and Pratt's work is focused on the parametric-parametric intersection problem. They compute the curvature of the intersection curve [3].Willmore in his book examined the implicit-implicit intersection problem. He introduced a method to obtain the Frenet-Serret apparatus of the intersection curve [10]. Hartmann was interested in all three types of intersection problems. He described how to compute the curvature for all of these intersection problems [5].

In three-dimensional Minkowski space, there is not much literature about the surface-surface intersection problem. In this paper, we examined the transversal intersection of the normal and tangent surfaces of timelike Bertrand curve pairs with a geometric perspective. For a timelike curve γ, the tangent, principal normal, and binormal surfaces of γ are special ruled surfaces generated by its Frenet vectors T, N and B, respectively. We study the intersection problem for the tangent and principal normal surfaces of timelike Bertrand curve γ and its timelike Bertrand mate. We investigate the characterizations of the intersection curve for each case of the surface-surface intersection. First, we express the curvature vector and compute the curvature of the intersection curve. Lastly, for all cases, we obtain some results which characterize the intersection curve.

2. Preliminaries

$\mathbb{R}_1^3$ Minkowski space is the $\mathbb{R}^3$ vector space endowed with the scalar product given by

$$\langle a, b\rangle = -a_1b_1 + a_2b_2 + a_3b_3 \tag{1}$$

here $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{R}^3$.
Borrowing the terminology from the theory of relativity, a vector a in $\mathbb{R}_1^3$ is

Spacelike if $a = 0$ or $\langle a, a\rangle > 0$,
Timelike if $\langle a, a\rangle < 0$,
Lightlike if $a \neq 0$ and $\langle a, a\rangle = 0$.

The zero vector is usually considered to be spacelike but in some of the literature it is taken to be lightlike. In the same manner, for any curve $\gamma(t)$ in $\mathbb{R}_1^3$ we say that

Spacelike if $\langle \gamma'(t), \gamma'(t)\rangle > 0$,
Timelike if $\langle \gamma'(t), \gamma'(t)\rangle < 0$,
Lightlike if $\gamma'(t) \neq 0$ and $\langle \gamma'(t), \gamma'(t)\rangle = 0$.

here $\gamma'(t)$ is the velocity vector of the curve $\gamma(t)$. For a vector $a \in \mathbb{R}_1^3$, the norm of a is defined by

$$\|a\| = \sqrt{|\langle a, a\rangle|} \tag{2}$$

As a result, a is called a unit vector if $\langle a, a\rangle = \pm 1$. For any vectors $a = (a_1, a_2, a_3), a = (b_1, b_2, b_3) \in \mathbb{R}^3$, the vector (cross) product of a and b is defined by

$$a \times b = (a_3b_2 - a_2b_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \tag{3}$$

$\{T(t), N(t), B(t)\}$ represents the moving Frenet frame along the curve $\gamma(t)$ here T, N, and B are the tangent, the principal normal and the binormal vectors of the curve $\gamma(t)$, respectively. Let γ be a unit speed timelike space curve with curvature κ, torsion τ. Then, Frenet formulas for the curve $\gamma(t)$ are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{4}$$

here $\{T, N, B\}$ are the Frenet vector fields of γ where T is timelike and N, B are spacelike vector fields.

For given any two vectors $u, v \in \mathbb{R}^3_1$, if u and v are positive (negative) timelike vectors then there is a unique nonnegative real number θ such that

$$\langle u, v \rangle = \|u\| \, \|v\| \cosh\theta \tag{5}$$

moreover, θ is called The Lorentzian timelike angle between u and v.
If u be a spacelike vector and v be a positive timelike vector then there is a unique nonnegative real number θ such that

$$\langle u, v \rangle = \|u\| \, \|v\| \sinh\theta \tag{6}$$

therefore, θ is called The Lorentzian timelike angle between u and v,[9].

Let γ be a timelike curve. If there exists a timelike curve γ^* whose principal normal vector linear dependent with that of original curve γ, then γ is said tobe a timelike Bertrand curve. The pair (γ, γ^*) is said to be a timelike Bertrand pair [8]. We denote the moving Frenet frame along the curve γ by $\{T, N, B\}$ and along the curve γ^* by $\{T^*, N^*, B^*\}$. In this case the tangent vector fields and binormal vector fields are related by

$$\begin{bmatrix} T \\ B \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T^* \\ B^* \end{bmatrix} \tag{7}$$

Let $\varphi^A = \varphi^A(u, v)$ and $\varphi^B = \varphi^B(u, v)$ be two parametric surfaces and $c = c(s)$ the transversal intersection curve of the surfaces φ^A and φ^B. This means that the tangent vector of the transversal intersection curve lies on the tangent planes of these surfaces. Therefore, it can be obtained as the vector product of the unit surface normal vectors at $p = c(s)$

$$t = \frac{N^A \times N^B}{\|N^A \times N^B\|}$$

where N^A and N^B are the unit normal vectors to the surfaces φ^A and φ^B, respectively.

3. Transversal Intersection of Tangent Surfaces

In this section, we compute the curvature and the torsion of the transversal intersection curve of tangent surfaces of Bertrand curve pairs. Let γ and γ^* are Timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their tangent surfaces, respectively. Let N^A be the unit surface normal of the tangent surface $\varphi^A(s,u)$ and N^B be the unit surface normal of the tangent surface $\varphi^B(s^*,u)$, we can compute N^A and N^B as;

$$N^A = \frac{\varphi_s^A \times \varphi_u^A}{\|\varphi_s^A \times \varphi_u^A\|} = \pm B \tag{8}$$

and

$$N^B = \frac{\varphi_{s^*}^B \times \varphi_u^B}{\left\|\varphi_{s^*}^B \times \varphi_u^B\right\|} = \pm B^* \tag{9}$$

where B is the binormal frenet vector of γ and B^* is the binormal frenet vector of γ^*. Let $c = c(s)$ the transversal intersection curve of both tangent surfaces φ^A and φ^B. Thus, the tangent vector of the transversal intersection curve $c = c(s)$ at $p = c(s)$ can be obtained as

$$t = \frac{N^A \times N^B}{\|N^A \times N^B\|} = \pm N^* \tag{10}$$

here N^A and N^B are the unit normal vectors to the surfaces φ^A and φ^B, respectively.

Result 1. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their tangent surfaces, respectively then, the curve $c = c(s)$ is evolute of γ or γ^* .*

Let investigate the angle between surfaces $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$. The angle between the surfaces $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ is the angle between the unit surface normal vectors N^A and N^B. If η denote the angle between N^A and N^B, then we can write;

$$\cosh\eta = \langle N^A, N^B \rangle = \pm\cosh\theta \tag{11}$$

Result 2. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their tangent surfaces, respectively. The angle between the surfaces $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ is equal to the angle θ, where θ is the angle between the tangent vectors T of γ and T^* of γ^* .*

Since, the curvature vector c'' of the transversal intersection curve at p is orthogonal to t, it must lie in the plane spanned by N^A and N^B . Thus, we can write it as

$$c'' = \lambda_1 N^A + \lambda_2 N^B \tag{12}$$

where λ_1 and λ_2 are the coefficients that we need to compute. Due to the inner product of the curvature vector c'' with the unit surface normals N^A and N^B, we obtain the following linear equations system;

$$\begin{aligned} \kappa_n^A &= \left\langle c'', N^A \right\rangle = \lambda_1 + \lambda_2 \cosh\theta \\ \kappa_n^B &= \left\langle c'', N^B \right\rangle = \lambda_1 \cosh\theta + \lambda_2 \end{aligned}. \tag{13}$$

When this linear system is solved and put the coefficients in the Eq 12, we can express the curvature vector of the intersection curve as

$$c'' = \frac{-\kappa_n^A + \kappa_n^B \cosh\theta}{\sinh^2\theta} N^A + \frac{-\kappa_n^B + \kappa_n^A \cosh\theta}{\sinh^2\theta} N^B, \tag{14}$$

Theorem 3. *Let α and α^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their tangent surfaces, respectively. Then the curvature κ of the curve $c = c(s)$ is given by*

$$\kappa = \frac{1}{\sinh\theta}\left\{\left|\left(\kappa_n^A\right)^2 + \left(\kappa_n^B\right)^2 - 2\kappa_n^A\kappa_n^B\cosh\theta\right|\right\}^{\frac{1}{2}} \tag{15}$$

where θ is the angle between the tangent vectors T of γ and T^* of γ^*.

Result 4. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their tangent surfaces, respectively. Then the curvature κ of the curve $c = c(s)$ is given by*

$$\kappa = \sqrt{|\langle c'', c''\rangle|} = \sqrt{\left|\tau^{*2} - \kappa^{*2}\right|}$$

in terms of the curvature κ^* and the torsion τ^* of the curve γ^* .

Theorem 5. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their tangent surfaces, respectively. Then the torison τ of the curve $c = c(s)$ is given by*

$$\tau = \pm\frac{-\kappa^{*\prime}\tau^* + \tau^{*\prime}\kappa^*}{\tau^{*2} - \kappa^{*2}} \tag{16}$$

in terms of the curvature κ^* and the torsion τ^* of the curve γ^* and their derivatives.

4. Transversal Intersection of Normal Surfaces

In this section, we compute the curvature of the transversal intersection curve of principal normal surfaces of timelike Bertrand curve pairs. Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their principal normal surfaces, respectively. Let N^A be the unit surface normal of the principal normal surface $\varphi^A(s,u)$ and N^B be the unit surface normal of the principal normal surface $\varphi^B(s^*,u)$, we can compute N^A and N^B as;

$$N^A = \frac{1}{\sqrt{\left|(1+u\kappa)^2 - u^2\tau^2\right|}}\left(u\tau T + (1+u\kappa)B\right) \tag{17}$$

and

$$N^B = \frac{1}{\sqrt{\left|(1+u\kappa^*)^2 - u^2\tau^{*2}\right|}}\left(u\tau^* T^* + (1+u\kappa^*)B^*\right) \tag{18}$$

where τ is the torsion, B is the binormal frenet vector of α and τ^* is the torsion, B^* is the binormal frenet vector of γ^*, respectively. Let $c = c(s)$ the transversal intersection curve of both principal normal surfaces φ^A and φ^B. Thus, the tangent vector of the transversal intersection curve $c = c(s)$ at $p = c(s)$ can be obtained as

$$t = \tfrac{1}{\omega_1\omega_2}\left[\left\{-u^2\tau\tau^* + (1+u\kappa)(1+u\kappa^*)\right\}\sinh\theta + u\left\{(1+u\kappa)\tau^* - (1-u\kappa^*)\tau\right\}\cosh\theta\right]N^* \tag{19}$$

where $\omega_1 = \sqrt{\left|(1+u\kappa)^2 - u^2\tau^2\right|}$ and $\omega_2 = \sqrt{\left|(1+u\kappa^*)^2 - u^2\tau^{*2}\right|}$, moreover N^A and N^B are the unit normal vectors to the surfaces φ^A and φ^B, respectively.

Result 6. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their principal normal surfaces, respectively then, the curve $c = c(s)$ is evolute of γ or γ^* .*

Let investigate the angle between surfaces $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$. The angle between the surfaces $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ is the angle between the unit surface normal vectors N^A and N^B. If η denote the angle between N^A and N^B, then we can write;

$$\cosh\eta = \omega_1\omega_2\left[\left\{u^2\tau\tau^* + (1-u\kappa)(1-u\kappa^*)\right\}\cosh\theta + u\left\{(1-u\kappa)\tau^* - (1-u\kappa^*)\tau\right\}\sinh\theta\right] \tag{20}$$

where $\omega_1 = \sqrt{\left|(1+u\kappa)^2 - u^2\tau^2\right|}$ and $\omega_2 = \sqrt{\left|(1+u\kappa^*)^2 - u^2\tau^{*2}\right|}$.

Since, the curvature vector c'' of the transversal intersection curve at p is orthogonal to t, it must lie in the plane spanned by N^A and N^B . Consequently, we can write it as

$$c'' = \lambda_1 N^A + \lambda_2 N^B \tag{21}$$

where λ_1 and λ_2 are the coefficients that we need to compute. Due to the inner product of the curvature vector c'' with the unit surface normals N^A and N^B, we obtain the following linear equations system;

$$\begin{aligned} \kappa_n^A &= \left\langle c'', N^A\right\rangle = \lambda_1 + \lambda_2\cosh\eta \\ \kappa_n^B &= \left\langle c'', N^B\right\rangle = \lambda_1\cosh\eta + \lambda_2 \end{aligned}. \tag{22}$$

When this linear system is solved and put the coefficients in the Eq 21, we can express the curvature vector c'' of the intersection curve as

$$c'' = \frac{\left(\kappa_n^A - \kappa_n^B \cosh\eta\right)}{\sinh^2\eta} N^A + \frac{\left(\kappa_n^B - \kappa_n^A \cosh\eta\right)}{\sinh^2\eta} N^B \tag{23}$$

Theorem 7. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their principal normal surfaces, respectively. Then the curvature κ of the curve $c = c(s)$ is given by*

$$\kappa = \frac{1}{\sinh\eta}\left\{\left|\left(\kappa_n^A\right)^2 + \left(\kappa_n^B\right)^2 - 2\kappa_n^A\kappa_n^B\cosh\eta\right|\right\}^{\frac{1}{2}} \tag{24}$$

where

$$\sinh\eta = \tfrac{-1}{\omega_1\omega_2}\left|\left\{u^2\tau\tau^* + (1-u\kappa)(1-u\kappa^*)\right\}\sinh\theta + u\left\{\tau(1-u\kappa^*) - \tau^*(1-u\kappa)\right\}\cosh\theta\right|$$

Lemma 8. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their principal normal surfaces, respectively. If κ_g^A and κ_g^B are the geodesic curvatures of $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$, respectively, then we have the following relationship;*

$$\frac{\kappa_g^A}{\kappa_g^B} = -\frac{\left(\kappa_n^B - \kappa_n^A\cosh\eta\right)}{\left(\kappa_n^A - \kappa_n^B\cosh\eta\right)}. \tag{25}$$

in terms of κ_n^A and κ_n^B are the geodesic curvatures of $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$, respectively.

Result 9. *Let γ and γ^* are timelike Bertrand curve pair and $\varphi^A(s,u)$ and $\varphi^B(s^*,u)$ their principal normal surfaces, respectively. Then $c = c(s)$ is a geodesic curve of the surface $\varphi^A(s,u)$ if and only if $c = c(s)$ is a geodesic curve of the surface $\varphi^B(s^*,u)$.*

Example 1. *Lets think about the Timelike Bertrand curve $\alpha(s) = \left(\sqrt{2}s, \sin(s), \cos(s)\right)$ and its Timelike Bertrand mate $\alpha^*(s) = \left(\sqrt{2}s, -\sin(s), -\cos(s)\right)$. Their Frenet frames are*

$$\begin{aligned} T &= \left(\sqrt{2}, \cos(s), -\sin(s)\right) \\ N &= \left(0, -\sin(s), -\cos(s)\right) \\ B &= \left(1, \sqrt{2}\cos(s), -\sqrt{2}\sin(s)\right) \end{aligned}$$

and

$$\begin{aligned} T^* &= \left(\sqrt{2}, -\cos(s), \sin(s)\right) \\ N^* &= \left(0, \sin(s), \cos(s)\right) \\ B^* &= \left(1, -\sqrt{2}\cos(s), \sqrt{2}\sin(s)\right) \end{aligned}$$

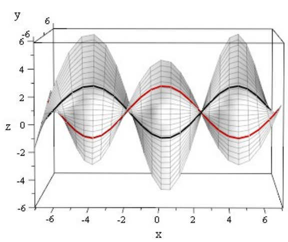
repectively. Therefore their normal surfaces are

$$X(s,t) = \left(\sqrt{2}s, (t-1)\sin(s), (t-1)\cos(s)\right)$$

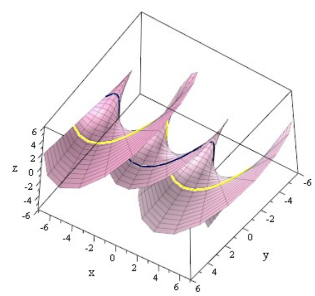
and

$$X^*(s,t) = \left(\sqrt{2}s, (1-t)\sin(s), (1-t)\cos(s)\right)$$

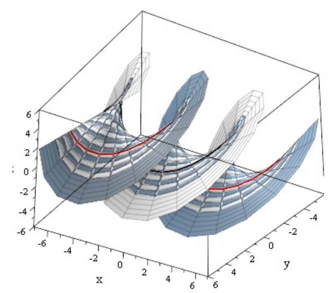
For the appropriate values of t parameter we can obtain two timelike intersection curves as $c_1(s) = \left(\sqrt{2}s, 2\sin(s), 2\cos(s)\right)$ and $c_2(s) = \left(\sqrt{2}s, -2\sin(s), -2\cos(s)\right)$. Also the intersection curve $c_1(s)$ is the evolute of the $\alpha^*(s)$ and the intersection curve $c_2(s)$ is the evolute of the $\alpha(s)$.



(a) $X(s,t), c_1(s), c_2(s)$



(b) $X^*(s,t), c_1(s), c_2(s)$



(c) $X(s,t), X^*(s,t), c_1(s), c_2(s)$

5. Conclusion

In this paper,We have investigated the transversal intersection problem for the tangent surfaces and the principal normal surfaces of the Bertrand curve pair. We have expressed the curvature of the intersection curve for the transversal intersection . Moreover, we have investigated local properties and some characteristic features of the intersection curve and gave some results for all two cases.

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