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SOME CLASSES OF INTERIOR CO-IDEALS IN Γ-SEMIGROUP WITH APARTNESS

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Abstract. The logical and working framework of this report is Bishop's Constitutive Mathematics, which includes Intuitionist logic. In this paper, we introduce the notions of interior co-ideals, weak interior co-ideals and quasi-interior co-ideals in Γ-semigroups with apartness as duals of the concept of interior ideals, weak interior ideals and quasi-interior ideals in Γ-semigroups respectively. In addition to the above, the article analyzes the fundamental properties of newly introduced concepts as well as the relationships between these three classes of co-ideals in Γ-semigroups with apartness.

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1. Introduction

The logical and working framework of this report is Bishop's Constitutive Mathematics (**Bish**) (in sense of books [1, 2, 6, 7]), which includes Intuitionist logic (**IL**) (for example [20]). Since in **IL**, the principle of excluding the third is not a valid formula, in the constructive algebra of Bishop's orientation many concepts (including some structures) have their own constructive dual. In that sense, the dual of the equality relation is the apartness relation. It is any consistent, symmetric and co-transitive relation extensive with respect to a given equality:

$(\forall x \in S)\neg(x \neq x)$ (consistency),

$(\forall x, y \in S)(x \neq y \implies y \neq x)$ (symmetric),

$(\forall x, y, z \in S)(x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity),

$(\forall x, y, z \in S)((x = y \wedge y \neq z) \implies x \neq z)$. (extendibility with the equality).

If the set on which equality and co-equality are observed is an algebraic structure, then we say that it is an algebraic structure with apartness. In addition to the above, constructive duals of classical substructures can be created in algebraic structures with apartness. For example, in a semigroup with apartness, the dual of the concept of ideals and its characteristics can be observed.

In [10], the concept of Γ-semigroup with apartness was introduced and some of its important features were described such as the concepts of co-ideals and co-congruences on it. In articles [12, 13], the concepts of co-filters and co-ideals are introduced and analyzed in a Γ-semigroup with apartness ordered under a co-order relation.

In this paper, we introduce the notions of interior co-ideals, weak interior co-ideals and quasi-interior co-ideals in Γ-semigroups with apartness as duals of the concept of interior ideals, weak interior ideals and quasi-interior ideals in Γ-semigroups. In addition to the above, the article analyzes the fundamental properties of newly introduced concepts as well as the relationships between these three classes of co-ideals in Γ-semigroups with apartness.

2. Preliminaries

2.1. Bishop's constructive environment

In order for a potential reader to follow our presentation in this report without difficulty, we will first repeat some of the basic specific concepts within this principled-logical orientation:

- Let $(S, =, \neq)$ a set wit apartness. A subset B of S is strongly extensional in S if holds

$$(\forall x, y \in S)(x \in B \implies (x \neq y \lor y \in B));$$

- For a function $f : S \longrightarrow T$ between sets with apartness it is said to be strictly extensional if holds

$$(\forall x, y \in S)(f(x) \neq f(y) \implies x \neq y);$$

- Let $(S, =, \neq)$ be a set with apartness. A total strongly extensional function $w : S \times S \longrightarrow S$ is an internal binary operation in S. The pair $((S, =, \neq), w)$ is a groupoid with apartness. For $(S, w) =: ((S, =, \neq), w)$ we say that it is a semigroup with apartness if for operation w holds

$$(\forall x, y, z \in S)(w(x, w(y, z)) = w(w(x, y), z)).$$

It should be noted that in addition to the validity of the standard implications

$$(\forall x, y, u, v \in S)((x, y) = (u, v)) \implies w(x, y) = w(u, v)$$

and the following implication is valid

$$(\forall x, y, u, v \in S)(w(x, y) \neq w(u, v) \implies (x, y) \neq (u, v)).$$

- Let S and T be sets with apartness. The apartness relation on $S \times T$ is determined as follows

$$(\forall x, y \in S)(\forall u, v \in T)((x, u) \neq (y, v) \iff (x \neq y \vee u \neq v)).$$

2.2. Γ-semigroup with apartness

The notion of Γ-semigroups as a generalization of semigroup was introduced in 1984 by Sen [16] and developed by Sen and Saha in 1986 ([17]).

The concept of Γ-semigroup with apartness was created in 2019 by this author in the article [10].

Definition 1 ([10], Definition 2.1)**.** *Let $(S, =, \neq)$ and $(\Gamma, =, \neq)$ be two non-empty sets with apartness. S is called a Γ-semigroup with apartness if there exist a strongly extensional total mapping*

$$w_S : S \times \Gamma \times S \ni (x, a, y) \longmapsto w_S(x, a, y) := xay \in S$$

satisfying the condition

$$w_S(w_S(x, a, y), b, z) =: (xay)bz =_S xa(ybz) := w_S(x, a, w_S(y, b, z))$$

for any $x, y, z \in S$ and $a, b \in \Gamma$.

We recognize immediately that the following implications

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \neq_S ubv \implies (x \neq_S u \vee a \neq_\Gamma b \vee y \neq_S v)),$$

and

$$(\forall x, y \in S)(\forall a, b \in \Gamma)(xay \neq_S xby \implies a \neq_\Gamma b)$$

are valid, because w_S is a strongly extensional function.

We will write (S, Γ, w_S) if S is a Γ-semigroup with apartness with the operation $w_S : S \times \Gamma \times S \longrightarrow S$. Sometimes, for simplicity, and when there can be no confusion, we will briefly write S for Γ-semigroup with apartness (S, Γ, w_S).

Example 1. *If $S = \mathbb{N}$ is a multiplicative semigroup of natural numbers, then S is a $\mathbb{N}$-semigroup with apartness, where the apartness is defined by*

$$(\forall x, y \in \mathbb{N})(x = y \iff \neg(x = y)).$$

Example 2. *Let* $\mathbb{N}$ *be a the set of all natural numbers and* $\Gamma = 4\mathbb{N}$. *Ternary operation* $w_{\mathbb{N}}$ *is defined as* $w_{\mathbb{N}}(x, a, y) = x + a + y$, *where* $+$ *is the usual addition of integers. Then* $\mathbb{N}$ *is an* Γ*-semigroup with apartness.*

Example 3. *Let* $S = M_{m\times n}$ *be the set of all* $m\times n$ *matrices over the real filed* $\mathbb{R}$, *and* $\Gamma = M_{n\times m}$ *be the set of all* $n \times m$ *matrices over the same field. Then for* $A, B \in S$, *the product* $A\cdot B$ *can not be defined i.e.,* S *is not a semigroup under the usual matrix multiplication. But, for all* $A, B, C \in S$ *and* $P, Q \in \Gamma$ *we have* $APB \in S$ *and since the matrix multiplication is associative, we have* $(APB)QC = AP(BQC)$. *Hence* S *is a* Γ*-semigroup with apartness.*

Recall that in the field of real numbers $\mathbb{R}$ *apartness is determined by*

$$(\forall x, y \in \mathbb{R})(x \neq y \iff (\exists k \in \mathbb{N})(|x - y| > \frac{1}{k})).$$

An inhabited subset J of a Γ-semigroup with apartness (S, Γ, w_S) is called right ideal of S if $J\Gamma S \subseteq J$. An inhabited subset J of a Γ-semigroup (S, Γ, w_S) is called a left ideal of S if $S\Gamma J \subseteq J$. A subset J is called ideal of S if it is both a left and a right ideal of S. Their constructive duals were introduced ([10], Definition 2.3, Definition 2.4) as follows:

- A strongly extensional subset A of Γ-semigroup with apartness S is called a co-Γ-subsemigroup of S if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)(xay \in A \implies (x \in A \vee y \in A));$$

- A strongly extensional subset B of a Γ-semigroup with apartness S is said to be a right Γ-coideal of S if the following implication holds

$$(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in B \implies y \in B);$$

- A strongly extensional subset B of a Γ-semigroup with apartness S is said to be a left Γ-coideal of S if the following implication is valid

$$(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in B \implies x \in B);$$

- A strongly extensional subset B of a Γ-semigroup with apartness S is said to be a (two side) Γ-coideal of S if the following implication is valid

$$(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in B \implies (x \in B \wedge y \in B)).$$

Let us see that every kind of co-ideal of a Γ-semigroup with apartness S is a co-subsemigroup of S. In the general case, the opposite does not have to be the case.

Example 4. *Let S and Γ be as in the Example 2. The subset $2\mathbb{N}+1$ (the set of all odd natural numbers) is a co-subsemigroup of S. Indeed, if $x+a+y$ is an odd natural number, then one of numbers x and y must be an odd natural number (but, not both), because otherwise, and the sum $x+a+y$ was even. This co-subsemigroup is not a (left, right) co-ideal in S.*

Example 5. *Let S and Γ be as in the Example 1. The subset $2\mathbb{N}+1$ (the set of all odd natural numbers) is a co-ideal in S. Indeed, if xay is an odd natural number, then each of the numbers x, y and a is also an odd natural number.*

Example 6. *Let S and Γ be as in the Example 2. The subset $2\mathbb{N}$ (the set of all even natural numbers) is a co-ideal of S. Indeed, if $x+a+y$ is an even natural number, then each of numbers x and y must be an even natural number, because otherwise, and the sum $x+a+y$ was an odd.*

In the following, for ease of writing, co-Γ-subsemigroup and (left, right) Γ-coideal will be written without Γ.

3. The results

3.1. Interior co-ideals

The concept of interior ideal of a semigroup S has been introduced by S. Lajos in [5] as a subsemigroup J of a semigroup S such that $SJS \subseteq J$. The interior ideals of semigroups have been also studied by G. Szász in [18, 19].

In Γ-semigroup S this concept is created as follows: A non-empty subset J of Γ-semigroup S is called an interior ideal of S if it is a sub-semigroup of S and the following holds $S\Gamma J\Gamma S \subseteq J$. In other words, the following is true:
$J \neq \emptyset$;
$(\forall x,y \in S)(\forall a \in \Gamma)((x \in J \wedge y \in J) \implies xay \in J)$;
$(\forall u,v,x \in S)(\forall a,b \in \Gamma)(x \in J \implies uaxbv \in J)$.

In the following definition we create the concept of interior co-ideal of a Γ-semigroup with apartness as a constructive dual of the concept of interior ideals in such a semigroup.

Definition 2. *A strongly extensional subset K of a Γ-semigroup with apartness S is an interior co-ideal of S if it is a co-subsemigroup of S and the following holds*

$$(\forall u,v,x \in S)(\forall a,b \in \Gamma)(uaxbv \in K \implies x \in K).$$

Example 7. *Let $S = \{0,1,2,3,4\}$ and '$\cdot$' defined on S as follows:*

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	3	2
3	0	3	3	3	3
4	0	4	2	3	4

Put $\Gamma = S$. A mapping $S \times \Gamma \times S \longrightarrow S$ is defined as xay = usual product of $x, y, a \in S$. Then S forms a Γ-semigroup. By direct verification one can establish that the sets $\{1, 2, 3, 4\}$, $\{1, 2, 4\}$ and $\{1\}$ are interior co-ideals in Γ-semigroup S.

The following theorem connects the concept of co-ideals and the concept of interior co-ideals in a Γ-semigroup with apartness.

Theorem 1. *Every co-ideal is an interior co-ideal in a Γ-semigroup.*

Proof. Let K be a co-ideal in a Γ-semigroup with apartness S. It suffices to prove that for K holds the conditions in Definition 2 because K is a co-subsemigroup of S by assumption. Let $u, v, x \in S$ and $a, b \in \Gamma$ be such that $uaxbv \in K$. Then $uax \in K$ since K is a right co-ideal in S. Also, from $uax \in K$ it follows $x \in K$ since K is a left co-ideal in R. So, K is an interior co-ideal in S.

The inverse of the previous theorem is not valid as the following example shows:

Example 8. *Let $S = \{a, b, c, d\}$ and operations '$\cdot$' defined on S as follows:*

$\cdot$	a	b	c	d
a	a	c	d	d
b	c	d	d	d
c	b	d	d	d
d	d	d	d	d

Put $\Gamma = S$. A mapping $S \times \Gamma \times S \longrightarrow S$ is defined as xay = usual product of $x, y, a \in S$. Then S forms a Γ-semigroup. By direct verification one can establish that the sets $\{a, c\}$ and $\{a, b\}$ are interior co-ideals in Γ-semigroup S but they are neither left co-ideals nor right co-ideals in S.

Our second theorem connects the terms 'interior ideal' and 'interior co-ideal' in a Γ-semiring with apartness S. In addition, the proof of this theorem illustrates the importance of the assumption of strictly extensionality of the subset K in S.

Theorem 2. *If $K\,(\neq S)$ is an interior co-ideal of a Γ-semiring with apartness S, then the set*

$$K^{\lhd} = \{x \in S : x \lhd K\} = \{x \in S : (\forall u \in K)(x \neq u)\}$$

is an interior ideal of S.

Proof. It should be shown that the set $K^{\lhd}$ satisfies the following conditions:
$K^{\lhd} \neq \emptyset$,
$(\forall x, y \in S)(\forall a \in \Gamma)(x \in K^{\lhd} \wedge y \in K^{\lhd} \implies xay \in K^{\lhd})$ and
$(\forall x, u, v \in S)(\forall a, b \in \Gamma)(x \in K^{\lhd} \implies uaxbv \in K^{\lhd})$.
The condition $K \neq S$ ensures that the set $K^{\lhd}$ is inhabited.

Let $x, y, t \in S$ and $a \in \Gamma$ be arbitrary elements such that $t \in K$, $x \lhd K$ and $y \lhd K$. Then $t \neq xay$ or $xay \in K$ by strongly extensionality of K in S. The second option $xay \in K$ would give $x \in K \vee y \in K$, which contradicts the assumptions. Therefore, it must be $xay \neq t \in K$. This means that $xay \lhd K$ holds.

Let $x, u, v, t \in S$ and $a, b \in \Gamma$ be arbitrary elements such that $t \in K$, $x \lhd K$. Then $t \neq uaxbv$ or $uaxbv \in K$ by strongly extensionality of K in S. The second option $uaxbv \in K$ would give $x \in K$ by Definition 2, which contradicts the assumptions. Therefore, it must be $uaxbv \neq t \in K$. This means that $uaxbv \lhd K$ holds.

The family $\mathfrak{Intc}(S)$ of all internal co-ideals of the Γ-semigroup with apartness S is not empty because $S \in \mathfrak{Intc}(S)$ and $\emptyset \in \mathfrak{Intc}(S)$. Additionally, the following applies:

Theorem 3. *The family $\mathfrak{Intc}(S)$ of all interior co-ideals of a Γ-semigroup with apartness S forms a complete lattice.*

Proof. Let $\{K_i\}_{i \in I}$ be a family of interior co-ideals of a Γ-semigroup with apartness S.

(a) (i) Let $u, v \in S$ be such that $u \in \bigcup_{i \in I} K_i$. Then there exists an index $k \in I$ such that $u \in K_k$. Thus $u \neq v$ or $v \in K_k \subseteq \bigcup_{i \in I} K_i$ by strongly extensionality of the co-ideal K_k in S. This means that the set $\bigcup_{i \in I} K_i$ is a strongly extensional subset in S.

(ii) Let $x, y \in S$ and $a \in \Gamma$ be such that $xay \in \bigcup_{i \in I} K_i$. Then there exists an index $k \in I$ such that $xay \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i \in I} K_i$ or $y \in K_k \subseteq \bigcup_{i \in I} K_i$. This means that the set $\bigcup_{i \in I} K_i$ is a co-subsemigroup of S.

(iii) Let $u, v, x \in S$ and $a \in \Gamma$ be such that $uaxbv \in \bigcup_{i \in I} K_i$. Then there exists an index $k \in I$ such that $uaxbv \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i \in I} K_i$.

Based on (i), (ii) and (iii), we conclude that the set $\bigcup_{i \in I} K_i$ is an interior co-ideal of S.

(b) Let X be the family of all interior co-ideals of Γ-semigroup with apartness S contained in $\bigcap_{i\in I} K_i$. Then $\cup X$ is the maximal interior co-ideal of S contained in $\bigcap_{i\in I} K_i$, according to (a).

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup X$, then $(\mathfrak{Intc}(S), \sqcup, \sqcap)$ a is a complete lattice.

Corollary 4. *For any subset X of Γ-semigroup with apartness S there is the maximal co-ideal contained in X.*

Proof. The proof of this Corollary is obtained directly from part (b) of the evidence in the previous theorem.

Corollary 5. *For any element $x \in S$ there is the maximal interior co-ideal K_x in Γ-semigroup with apartness S such that $x \lhd K_x$.*

Proof. One should take $X = \{u \in S : u \neq x\}$ and apply the previous corollary.

3.2. Weak interior co-ideals

Weak-interior ideals in a quasi-ordered semigroup are discussed in [14] by D. A. Romano. The concept of weak interior ideals in Γ-semigroup was introduced in [8] as follows:

- A non-empty subset J of a Γ-semigroup S is said to be left weak interior ideal of S if J is a Γ-subsemigroup of S and $S\Gamma J\Gamma J \subseteq J$.

-A non-empty subset J of a Γ-semigroup S is said to be right weak-interior ideal of S if J is a Γ-subsemigroup of S and $J\Gamma J\Gamma S \subseteq J$.

- A non-empty subset J of a Γ-semigroup S is said to be weak interior ideal of S if J is a left weak interior ideal and a right weak interior ideal of S.

A weak interior ideal of a Γ-semigroup S need not be an interior ideal of Γ-semigroup S (see, for example, [8], Remark 3.1 and Example 3.1). In the following theorem, some important properties of weak interior ideals in Γ-semigroup are given:

Theorem 6 ([8], Theorem 3.1)**.** *Let S be a Γ-semigroup. Then the following holds:*

- Every left ideal is a left weak interior ideal of S.

- Every right ideal is a right weak interior ideal of S.

- Let S be a Γ-semigroup and J be a Γ-subsemigroup of S. If $S\Gamma S\Gamma J \subseteq J$ and $J\Gamma S\Gamma S \subseteq J$, then J is a weak interior ideal of S.

- Every ideal is a weak interior ideal of S.

Constructive dual of classical concept of (left, right) weak interior ideals in Γ-semigroup with apartness is created in the following way:

Definition 3. *Let K be a strongly extensional subset of a Γ-semigroup with apartness S.*

(1) *K is a left weak interior co–ideal of S if K is a co-Γ-subsemigroup of S and holds*

$$(\forall x, y, u \in S)(\forall a, b \in \Gamma)(uaxby \in K \implies (x \in K \vee y \in K));$$

(2) *K is a right weak interior co-ideal of S if if K is a co-Γ-subsemigroup of S and holds*

$$(\forall x, y, v \in S)(\forall a, b \in \Gamma)(xaybv \in K \implies (x \in K \vee y \in K));$$

(3) *K is a weak interior co-ideal of S if if K is a co-Γ-subsemigroup of S and K is a left weak interior co-ideal and a right weak interior co-ideal of S.*

Example 9. *Let $\mathbb{Q}$ be a field of rational numbers, $S := \{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \,|\, b, d \in \mathbb{Q}\}$ be the semigroup of matrices over the filed $\mathbb{Q}$ and $\Gamma = S$. The ternary operation in S over $\mathbb{Q}$ is the standard multiplication of matrices. Then S is a Γ-semigroup. Then $K =: \{\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \,|\, d \in \mathbb{Q} \wedge d \neq 0\}$ is a left weak interior co-ideal of the Γ-semigroup S and K is neither a left co-ideal nor a right co-ideal, not a weak interior co-ideal and not an interior co-ideal of the Γ-semigroup S.*

The following theorem proves that the concept of left weak interior co-ideals in Γ-semigroup with apartness is well determined.

Theorem 7. *Let $K\,(\neq S)$ be a left weak interior co-ideal of a Γ-semigroup with apartness S. The the set $K^{\lhd}$ is a left weak interior ideal in S.*

Proof. Let $K\,(\neq S)$ be a left weak interior co-ideal in a Γ-semigroup with apartness. It needs to be proven:

$K^{\lhd} \neq \emptyset$,

$(\forall x, y \in S)(\forall a \in \Gamma)((x \in K^{\lhd} \wedge y \in K^{\lhd}) \implies xay \in K^{\lhd})$, and

$(\forall x, y, z \in S)(\forall a, b \in \Gamma)((y \in K^{\lhd} \wedge z \in K^{\lhd}) \implies xaybz \in K^{\lhd})$.

The condition $K \neq S$ ensures that the set $K^{\lhd}$ is inhabited.

Let $a \in \Gamma$ and $x, y, t \in S$ be arbitrary elements such that $x \lhd K$, $y \lhd K$ and $t \in K$. Then $xay \in K$ or $xay \neq t \in K$ by strongly extensionality of K in S. The first option would give $x \in K$ or $y \in K$ since the set K is a co-subsemigroup of S which is contrary to the assumptions $x \lhd K$ and $y \lhd K$. Therefore, it must be $xay \neq t \in K$. This means $xay \lhd K$. Then $xaybz \in K$ or $xaybz \neq t \in K$. The first option would give $x \in K$ or $y \in K$ since the set K is a left weak interior co-ideal of S.

Let $x, y, z, t \in S$ and $a, b \in \Gamma$ be arbitrary elements such that $y \lhd K$, $z \lhd K$ and $t \in K$. The first option would give $y \in K$ or $z \in K$ since K is a left weak interior co-ideal of S which contradicts the hypotheses $y \lhd K$ and $z \lhd K$. Therefore, it must be $xaybz \neq t \in K$. This means $xabyz \lhd K$.

The connection between the concept of right co-ideals and the concept of left weak interior co-ideals in Γ-semigroup with apartness is described in the following theorem.

Theorem 8. *Every right co-ideal of a* Γ*-semigroup S with apartness is a left weak interior co-ideal of S.*

Proof. Let K be a right co-ideal of a Γ-semigroup with apartness S and let $.x, y, z \in S$ and $a, b \in \Gamma$ such that $(xay)bz = xaybz \in K$. Then $z \in K$ because K is a right co-ideal of S. So, K is a left weak interior co-ideal of S.

The relationship between interior co-ideal and weak interior co-ideal in Γ-semigroup with apoartness is described by the following theorem.

Theorem 9. *Every interior co-ideal of a* Γ*-semigroup S with apartness is a left weak interior co-ideal of S.*

Proof. Let K be an interior co-ideal of a Γ-semigroup with apartness S and let $x, y, z \in S$ and $a, b \in \Gamma$ be arbitrary elements such that $xaybz \in K$. Then $y \in K$ because K is an interior co-ideal of S. Thus $y \in K \vee z \in K$ which means that K is a left weak interior co-ideal in S.

Theorem 10. *The family $\mathfrak{W}_\mathfrak{l}\mathfrak{Intc}(S)$ of all left weak interior co-ideals of a* Γ*-semiring with apartness S forms a complete lattice.*

Proof. Let $\{K_i\}_{i \in I}$ be a family of left weak interior co-ideals of a Γ-semiring with apartness S.

(a) (i) Let $u, v \in S$ be such that $u \in \bigcup_{i \in I}$. Then there exists an index $k \in I$ such that $u \in K_k$. Thus $u \neq v$ or $v \in K_k \subseteq \bigcup_{i \in I} K_i$ by strongly extensionality of the left weak interior co-ideal K_k in R. This means that the set $\bigcup_{i \in I} K_i$ is a strongly extensional subset in S.

(ii) Let $x, y \in S$ and $a \in \Gamma$ be such that $xay \in \bigcup_{i \in I} K_i$. Then there exists an index $k \in I$ such that $xay \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i \in I} K_i$ or $y \in K_k \subseteq \bigcup_{i \in I} K_i$ since L_k is a co-subsemigroup of S. This means that the set $\bigcup_{i \in I} K_i$ is a co-subsemigroup of S.

(iii) Let $x, y, z \in S$ and $a, b \in \Gamma$ be such that $xaybz \in \bigcup_{i \in I} K_i$. Then there exists an index $k \in I$ such that $xaybz \in K_k$. Thus $y \in K_k \subseteq \bigcup_{i \in I} K_i$ or $z \in K_k \subseteq \bigcup_{i \in I} K_i$. This shows that the set $\bigcup_{i \in I} K_i$ is a left weak interior co-ideal of S.

Based on (i), (ii) and (iii), we conclude that the set $\bigcup_{i\in I} K_i$ is a left weak interior co-ideal of S.

(b) Let X be the family of all left weak interior co-ideals of Γ-semiring with apartness S contained in $\bigcap_{i\in I} K_i$. Then $\cup X$ is the maximal interior co-ideal of S contained in $\bigcap_{i\in I} K_i$, according to (a).

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup X$, then $(\mathfrak{W}_l\mathfrak{Intc}(S), \sqcup, \sqcap)$ a is a complete lattice.

Corollary 11. *For any subset X of Γ-semigroup with apartness S there is the maximal left weak interior co-ideal contained in X.*

Proof. The proof of this Corollary is obtained directly from part (b) of the evidence in the previous theorem.

Corollary 12. *For any element $x \in S$ there is the maximal left weak interior co-ideal K_x in Γ-semigroup with apartness S such that $x \lhd K_x$.*

Proof. One should take $X = \{u \in S : u \neq x\}$ and apply the previous corollary.

Remark 1. *Without major difficulties, the previous claims concerning the left weak interior co-ideals can be transformed into the claims concerning the right weak interior co-ideals.*

3.3. Quasi-interior co-ideals

Quasi-interior ideals in a quasi-ordered semigroup are discussed in [14] by D. A. Romano. In this subsection, firstly, we will recall the determination of the notions of left, right, and two-sided quasi-interior ideals in Γ-semigroups.

- A non-empty subset B of a Γ-semigroup S is said to be left quasi-interior ideal of S, if B is a Γ-co-subsemigroup of S and holds $S\Gamma B\Gamma S\Gamma B \subseteq B$. Thus, the following formulas

$B \neq \emptyset$,

$(\forall x, y \in S)(\forall a \in \Gamma)((x \in B \wedge y \in B) \implies xay \in B)$,

$(\forall x, y, z, u \in S)(\forall a, b, c \in \Gamma)((y \in B \wedge u \in B) \implies xaybzcu \in B)$

are valid formulas in a Γ-semigroup S.

- A non-empty subset B of a Γ-semigroup S is said to be right quasi-interior ideal of S, if B is a Γ-co-subsemigroup of S and holds $B\Gamma S\Gamma S\Gamma B \subseteq B$. Thus, the following formulas

$B \neq \emptyset$,

$(\forall x, y \in S)(\forall a \in \Gamma)((x \in B \wedge y \in B) \implies xay \in B)$,

$(\forall x, y, z, u \in S)(\forall a, b, c \in \Gamma)((x \in B \wedge z \in B) \implies xaybzcu \in B)$

are valid formulas in a Γ-semigroup S.

- A non-empty subset B of a Γ-semigroup S is said to be quasi-interior ideal of S, if B is a Γ-subsemigroup of S and B is a left quasi-interior ideal and a right quasi-interior ideal of S.

In this subsection, we introduce the notion of left quasi-interior co-ideals as a generalization of interior co-ideals of Γ-semigroup with apartness and study its properties.

Definition 4. *Let S be a Γ-semigroup with apartness.*

(4) *A strongly extensional subset K of S is said to be left quasi-interior co-ideal of S if K is a co-subsemigroup of S and the following holds*

$$(\forall x, y, z, u \in S)(\forall a, b, c \in \Gamma)(xaybzcu \in K \implies (y \in K \vee u \in K)),$$

(5) *A strongly extensional subset K of S is said to be right quasi-interior co-ideal of S if K is a co-subsemigroup of S and the following holds*

$$(\forall x, y, z, uS)(\forall a, b, c \in \Gamma)(xaybzcu \in K \implies (x \in K \vee z \in K));$$

(6) *A strongly extensional subset K of S is said to be quasi-interior co-ideal of S if it is both a left quasi-interior co-ideal and a right quasi-interior co-ideal of S.*

Example 10. *Let $\mathbb{Q}$ be a field of rational numbers, $S := \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Q}\}$ be the semigroup of matrices over the filed $\mathbb{Q}$ and $\Gamma = S$. The ternary operation in S over $\mathbb{Q}$ is the standard multiplication of matrices. Then S is a Γ-semigroup. Then $K =: \{\begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} \mid d \in \mathbb{Q} \wedge d \neq 0\}$ is a right quasi-interior co-ideal of the Γ-semigroup S.*

Theorem 13. *Let S be a Γ-semigroup with apartness. If $K\,(\neq S)$ is a left quasi-interior co-ideal of S, then the set $K^{\lhd}$ is a left quasi-interior ideal of S.*

Proof. That the set $K^{\lhd}$ is inhabited follows from the condition $K \neq \emptyset$.

Let $x, y, t \in S$ and $a\Gamma$ be arbitrary element such that $x \lhd K$, $y \lhd K$ and $t \in K$. Then $xay \in K$ or $xay \neq t \in K$ by strongly extensionality of K in S. The first option would give $x \in K$ or $y \in K$ which is contrary to assumptions. Therefore, it must be $xay \neq t \in K$. This means $xay \lhd K$.

Let $x, y, z, u, t \in S$ and $a, b, c \in \Gamma$ be arbitrary elements such that $y \lhd K$ and $u \lhd K$ and $t \in K$. Then $xaybzcu \in K$ or $xaybzcu \neq t \in K$ by strongly extensionality of K in S. The first option $xaybzcu \in K$ would give $y \in K$ or $u \in K$ because K is a left quasi-interior co-ideal of S, which contradicts the hypotheses $y \lhd K$ and $u \lhd K$. So, it must be $xaybzcu \neq t \in K$. This means $xaybzcy \lhd K$.

Analogous to the previous, it can be proved:

Theorem 14. *Let S be a Γ-semigroup with apartness. If $K\,(\neq S)$ is a right quasi-interior co-ideal of S, then the set $K^{\lhd}$ is a right quasi-interior ideal of S.*

Theorem 15. *Every interior co-ideal of Γ-semigroup with apartness S is a left quasi-interior co-ideal of S.*

Proof. Let K ne an interior co-ideal of a Γ-semigroup with apartness S and let $x, y, z, u \in S$ and $a, b, c \in \Gamma$ be such that $xayb(zxu) = xaybzcu \in K$. Then $y \in K$ because $zcu \in S$ and K is an interior co-ideal of S. So, the set K is a left quasi-interior co-ideal of S.

Corollary 16. *Every co-ideal of a Γ-semigroup with apartness S is a left quasi-interior co-ideal of S.*

Proof. Let K be a co-ideal of a Γ-semigroup with apartness S. then K is an interior co-ideal of S by Theorem 1. Now, it remains to apply Theorem 15.

Theorem 17. *The family $\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intc}(S)$ of all left quasi-interior co-ideals of a Γ-semigroup with apartness S forms a complete lattice.*

Proof. Let $\{K_i\}_{i\in I}$ be a family of left quasi-interior co-ideals of a Γ-semigroup with apartness S.

(a) (i) Let $x, y \in S$ be such that $x \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $x \in K_k$. Thus $x \neq y$ or $y \in K_k \subseteq \bigcup_{i\in I} K_i$ because K_k is a strongly extensional subset of S. This means that the set $\bigcup_{i\in I} K_i$ is a strongly extensional subset of S.

(ii) Let $x, y \in S$ and $a \in \Gamma$ be arbitrary elements such that $xay \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $xay \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ or $y \in K_k \subseteq \bigcup_{i\in I} K_i$ since K_k is a co-subsemigroup of S. This means that the set $\bigcup_{i\in I} K_i$ is a co-subsemigroup of S.

(iii) Let $x, y, z, u \in S$ and $a, b, c \in \Gamma$ be arbitrary elements such that $xaybzcu \in \bigcup_{i\in I} K_i$. Then there exists an index $l \in I$ such that $xaybzcu \in K_k$. Thus $y \in K_k \subseteq \bigcup_{i\in I} K_i$ or $u \in K_k \subseteq \bigcup_{i\in I} K_i$ because K_k is a left quasi-interior co-ideal of S.

Based on what is found in (i), (ii) and (iii), we conclude that $\bigcup_{i\in I} K_i$ is a left quasi-interior co-ideal of S.

(b) Let X be the family of all left quasi-interior co-ideals contained in $\bigcap_{i\in I} K_i$. Then $\cup X$ is the maximal left quasi-interior co-ideal contained in X, according to (a) in this proof.

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup X$, then $(\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intr}(S), \sqcup, \sqcap)$ is a complete lattice.

Analogous to the previous, it can be proved:

Theorem 18. *The family* $\mathfrak{Q}_{\mathfrak{r}}\mathfrak{intc}(S)$ *of all right quasi-interior co-ideals of a* Γ*-semigroup with apartness* S *forms a complete lattice.*

4. Final comments and conclusion

Bishop's constructive framework (in the sense of the books [1, 2, 6, 7]) includes Intuitionist logic (see, for example, [20]). In addition to considering recognizable algebraic substructures in a given algebraic structure, intuitionistic logic enables the recognition and acceptance of another class of substructures in the observed algebraic structure whose elements are determined as constructive duals of the first class of substructures. Such substructures were first observed by Ruitenburg in his dissertation ([15]). Then, this class of substructures was observed by Romano ([9]) and Troelstra and van Dalen ([20], Chapter VIII). Apart from the above, since the principle of TND (the principle of excluding the third) is not a valid formula in Intuitionist logic, algebraic structures can be observed within Bishop's constructive orientation, as constructions created over the relational system $(S, =, \neq)$ where $\neq$ is apartness relation (for example, [3, 4, 11]). Thus, in this report, the duals of some classical types of ideals in Γ-semigroup with apartness such as interior, weak interior and quasi-interior ideals are presented. This paper is literally a continuation of the author's previously published articles [10, 12, 13] on Γ-semigroups with apartness.

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