Acta Universitatis Apulensis No. 68/2021
ISSN: 1582-5329 pp. 109-166
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2021.68.10

A CLASS OF STURM-LIOUVILLE BVPS WITH A NONINTEGRABLE WEIGHT

A. Benmezai, S. Mellal and J. Henderson

Abstract. Under eigenvalue criteria we prove existence of nodal solutions to the nonlinear boundary value problem

$$\begin{cases} -u'' + qu = \rho u f(t,u) \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases}$$

where ρ is a positive real parameter, $q \in C([0,1), \mathbb{R})$, $\int_0^1 q = +\infty$ and $f : [0,1] \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ is continuous. The cases where the nonlinearity $uf(t,u)$ is asymptotically linear, sublinear and superlinear are considered.

2010 *Mathematics Subject Classification*: 34B15, 34B40.

Keywords: Sturm-Liouville BVPs, Half-eigenvalue problem, Global bifurcation theory.

1. Introduction

Sturm-Liouville boundary value problems (bvp for short) have been the subject of hundreds of articles during the previous five decades, where existence and multiplicity of solutions have been investigated. Many of these articles concern existence of nodal solutions for second order differential equations subject to various boundary conditions; see, for example, [1], [4], [5], [7], [9], [10], [11] [12], [13], [14], [15], [16], [17], [19], [20], [21], [22] [23], [24], [25], [26], [27] and references therein.

Nodal solutions appear as eigenfunctions to the half eigenvalue problem

$$\begin{cases} -u'' + qu = \sigma m u + \alpha u^+ - \beta u^- \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{1}$$

where σ is a real parameter, $q, m, \alpha, \beta \in C([0,1], \mathbb{R})$ and $m > 0$ in $[0,1]$.

To the authors' knowledge, such a bvp has been studied for the first time in [4], where H. Berestycki introduced the concept of half-eigenvalue. He proved that the bvp (1) admits two increasing sequences of half-eigenvalues $\left(\sigma_k^+\right)_{k\geq 1}$ and $\left(\sigma_k^-\right)_{k\geq 1}$ such that $\vartheta_{k,\nu}$, the eigenfunction associated with σ_k^ν, admits exactly $(k-1)$ zeros in $(0,1)$, all are simple and $\nu\vartheta_{k,\nu}'(0) > 0$. The conditions $q, m, \alpha, \beta \in C\left([0,1], \mathbb{R}\right)$ and $m > 0$ in $[0,1]$ have been relaxed in [2] to $q, m, \alpha, \beta \in L^1\left([0,1], \mathbb{R}\right)$, $m \geq 0$ a.e. in $(0,1)$ $m > 0$ a.e. in a subinterval (ξ, η) of $[0,1]$. Notice that the concept of half-eigenvalue generalizes that of eigenvalue and for the role played by this notion, we refer the reader to [4], [6], [10], [24], [25], and [26].

In this article, we consider the case of the bvp (1) where $m, \alpha, \beta \in C\left([0,1], \mathbb{R}\right)$, $m \geq 0$ in $(0,1)$, $m(t_0) > 0$ for some $t_0 \in [0,1]$, and $q \in C\left([0,1), \mathbb{R}\right)$ with $\int_0^1 q(t)dt = +\infty$. Notice that the results obtained in [4] and in [2] do not cover such a situation. However, we prove in Section 3 that the Berysticki's result holds true for such a version of the bvp (1).

In Section 4, we investigate existence and multiplicity of nodal solutions to the bvp

$$\begin{cases} -u'' + qu = g(t,u) \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{2}$$

where $q \in C\left([0,1), \mathbb{R}\right)$ with $\int_0^1 q(t)dt = +\infty$ and $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous. The nonlinearity g is supposed to be sublinear, assymptotically linear and superlinear. This interest is mainly motivated by that in [19], [17] , [16] and [15] where is considerd the version of the bvp (1) with $q = 0$ and the nonlinearity g is separable variable; Namely

$$\begin{cases} -u''(t) = a(t)\, g(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

where $a : [0,1] \to [0,+\infty)$ is continuous and does not vanish identically and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Let $g_0 = \lim_{s\to 0} g(s)/s$, $g_\infty = \lim_{|s|\to\infty} g(s)/s$ and $(\mu_k)_{k\geq 1}$ be the sequence of eigenvalues of the bvp

$$\begin{cases} -u''(t) = \mu a(t)\, u(t), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Authors of the paper [19] under the assumptions that

(A) $a > 0$ in $[0,1]$,
(B) a is continuously differentiable,
(C) $g(-s) = -g(s)$ for all $s \in \mathbb{R}$,
(D) $g(s)s > 0$ for all $s \neq 0$,
(E) g is locally Lipschitzian,
(F) in the case where $g_0 = \infty$, g is nondecreasing and $g(s)/s$ is nonincreasing on $(0, s_0]$ for some $s_0 > 0$,

proved by means of a shooting method, that if for some integer k, $\lambda_k < g(s)/s < \lambda_{k+1}$ for all $s \neq 0$, then except the trivial function, the bvp (3) has no solution and if $g_0 < \lambda_k < g_\infty$ or $g_\infty < \lambda_k < g_0$, then the bvp (3) has a solution having exactly $k-1$ zeros in $(0,1)$, all are simple.

In [16], R. Ma and B. Thompson improved the existence result in [19]. Just under Hypotheses **(A)** and **(D),** they proved that if $0 < g_0 < \lambda_k < g_\infty < \infty$ or $0 < g_\infty < \lambda_k < g_0 < \infty$, then the bvp (3) has two solutions u_+ and u_-, each having exactly $k-1$ zeros in $(0,1)$, all are simple and for $\nu = +$ or $-$, $\nu u_\nu'(0) > 0$. In [17], where Hypothesis **(A)** is relaxed to:

(A') $a \geq 0$ in $[0,1]$ and does not vanish identically on any subinterval of $[0,1]$, they obtained the same result.

As it is mentioned in [16], we conclude from the above result that if Hypotheses **(A')** and **(D)** hold and if there are integers k, i such that $0 < g_0 < \lambda_k \leq \lambda_{k+i} < g_\infty < \infty$ or $0 < g_\infty < \lambda_k \leq \lambda_{k+i} < g_0 < \infty$, then for each $j \in \{0, 1, .., i\}$ the bvp (3) has two solutions $u_{+,j}$ and u_-, j, each having exactly $k+j-1$ zeros in $(0,1)$, all are simple and for $\nu = +$ or $-$, $\nu u_{\nu,j}'(0) > 0$.

In [15], the authors considered the cases where the nonlinearity f is superlinear and sublinear. They proved that if Hypotheses **(A)**, **(D)** hold and $g_0 = 0$, $g_\infty = \infty$ or Hypotheses **(A)**, **(D)**, **(F)** hold and $g_\infty = 0$, then for each $j \in \mathbb{N} = \{1, ...\}$ the bvp (3) has two solutions $u_{j,+}$ and $u_{j,-}$, each having exactly $j-1$ zeros in $(0,1)$, all are simple and for $\nu = +$ or $-$, $\nu u_{j,\nu}'(0) > 0$.

Main results of Section 4 concern nodal solutions to the bvp (2) in the cases where the nonlinearity g is respectively asymptotically linear, superlinear and sublinear. All are obtained by means of the global bifurcation theory due to P. H. Rabinowitz and they provide existence and multiplicity of nodal solutions with less conditions relative to that obtained in the above cited papers.

2. Preliminaries

2.1. General setting

For statements of main results in this paper needed to introduce some notations: in what follows, we let

$$
\begin{aligned}
&E = C([0,1],\mathbb{R}), \quad E^+ = \{m \in E : m \geq 0 \text{ in } [0,1]\},\\
&\Gamma^+ = \{m \in E^+ : m > 0 \text{ in a subinterval of } [0,1]\},\\
&\Gamma^{++} = \{m \in \Gamma^+ : m > 0 \text{ in } [0,1]\},\\
&Q = \left\{q \in C([0,1),\mathbb{R}) : \textstyle\int_0^1 q(s)ds = +\infty\right\},\\
&Q^+ = \{q \in Q : q(t) \geq 0 \text{ for all } t \in (0,1) \text{ and } \liminf_{t\to 1} q(t) > 0\},\\
&Q_\# = \left\{q \in Q : \textstyle\int_0^1 (1-s)q(s)ds < \infty\right\},\\
&W = \{u \in C([0,1),\mathbb{R}) : u(0) = \lim_{t\to 1} u(t) = 0\},\\
&C_b^1([0,1),\mathbb{R}) = \left\{u \in C^1([0,1),\mathbb{R}) : \sup_{t\in[0,1)} \left|u'(t)\right| < \infty\right\}\\
&W^1 = W \cap C_b^1([0,1),\mathbb{R}), \quad W^2 = W^1 \cap C^2([0,1),\mathbb{R}).
\end{aligned}
$$

The linear spaces W and W_1 are respectively equipped with the norms $\|\cdot\|$ and $\|\cdot\|_1$ defined by $\|u\| = \sup_{t\in[0,1]} |u(t)|$ and $\|u\|_1 = \sup_{t\in[0,1]} |u'(t)|$. Obviously, $(W, \|\cdot\|)$ and $(W^1, \|\cdot\|_1)$ are Banach spaces.

For an integer $k \geq 1$, S_k^+ denotes the set of all the functions u in W^1 having exactly $(k-1)$ zeros in $(0,1)$, all are simple and u is positive in a right neighbourhood of 0, $S_k^- = -S_k^+$ and $S_k^+ = S_k^+ \cup S_k^-$. For $u \in S_k$, $(z_j)_{j=0}^{j=k}$ with $0 = z_0 < z_1 < ... < z_k = 1$ and $u(z_j) = 0$ for $j = 1, ..., k-1$, is said to be the sequence of zeros of u.

Throughout this paper, for $q \in Q$ the operator $\mathcal{L}_q : C^2([0,1),\mathbb{R}) \to C([0,1),\mathbb{R})$ is defined by $\mathcal{L}_q u = -u'' + qu$.

For $\nu = +$ or $-$, let $I^\nu : W \to W$ be the operator defined for $u \in W$ by $I^\nu u(x) = \max(\nu u(x), 0) = u^\nu(x)$. We have for all $u \in W$

$$u = I^+u - I^-u \quad \text{and} \quad |u| = I^+u + I^-u.$$

This implies that, for all $u, v \in W$,

$$
\begin{aligned}
|I^+u - I^+v| &\leq \left(\tfrac{|u-v|}{2} + \tfrac{||u|-|v||}{2}\right) \leq |u-v|,\\
|I^-u - I^-v| &\leq \left(\tfrac{|u-v|}{2} + \tfrac{||u|-|v||}{2}\right) \leq |u-v|,
\end{aligned}
\tag{4}
$$

and the operators I^+, I^- are continuous.

2.2. The Green's function and fixed point formulation

In all what follows, we let for $q \in Q^{+}$, Ψ_q be the unique solution of the initial value problem

$$\begin{cases} \mathcal{L}_q u = 0, \\ u(0) = 0,\ u'(0) = 1. \end{cases}$$

Lemma 1. *For all $q \in Q^{+}$, the function Ψ_q has the following properties:*

i) $\Psi_q(t) > 0$, $\Psi_q'(t) > 0$ *and* $\Psi_q''(t) \geq 0$ *for all* $t \in (0,1]$.

ii) $\lim_{t\to 1} \Psi_q'(t) = +\infty$.

iii) *The function* Ψ_q/Ψ_q' *is bounded at* $t = 1$.

iv) $\lim_{t\to 0} \Psi_q(t) \int_t^1 \frac{ds}{\Psi_q^2(s)} = 1$.

v) $\lim_{t\to 1} \Psi_q(t) \int_t^1 \frac{ds}{\Psi_q^2(s)} = 0$.

vi) *If* $q \in Q_{\#}$ *then* $\Psi_q(1) = \lim_{t\to 1} \Psi_q(t) < \infty$.

Proof. Let $q \in Q^{+}$ and let $a \in (0,1)$ be such that $\alpha = \inf_{s\in(a,1)} q(s) > 0$.

i) We have to prove that $\Psi_q'(t) > 0$ for all $t \in [0,1)$. Suppose on the contrary that $\Psi_q'(t_0) = 0$ for some $t_0 \in (0,1)$. In this case and since $\Psi_q'(0) = 1$, there is $t_* \in (0,t_0]$ such that $\Psi_q'(t_*) = 0$ and $\Psi_q(t), \Psi_q'(t) > 0$ for all $t \in [0,t_*)$. Therefore, we have from $\Psi_q'' = q\Psi_q$ that Ψ_q' is nondecreasing on $[0,t_*)$ and this leads to the contradiction

$$1 = \Psi_q'(0) \leq \Psi_q'(t_*) = 0.$$

ii) We have for all $t \in (a,1)$

$$\begin{aligned} \Psi_q'(t) &= \left(\Psi_q'(a) + \int_a^t \Psi_q''(s)\,ds\right) \\ &= \left(\Psi_q'(a) + \int_a^t q(s)\Psi_q(s)\,ds\right) \\ &\geq \left(\Psi_q'(a) + \left(\inf_{s\in(a,1)} \Psi_q(s)\right)\int_a^t q(s)\,ds\right) \end{aligned}$$

leading to $\lim_{t\to 1} \Psi_q'(t) = +\infty$.

iii) We have for all $t \geq a$

$$\begin{aligned} \left(\Psi_q'(t)\right)^2 - \left(\Psi_q'(a)\right)^2 &= 2\int_a^t \Psi_q''(s)\Psi_q'(s)ds = 2\int_a^t q(s)\Psi_q(s)\Psi_q'(s)ds \\ &\geq \alpha\left((\Psi_q(t))^2 - (\Psi_q(a))^2\right), \end{aligned}$$

leading to

$$\left(\Psi_q(t)/\Psi_q'(t)\right)^2 \leq \frac{1}{\alpha} + \left(\Psi_q'(a)/\Psi_q'(t)\right)^2 \text{ for all } t \geq a.$$

Hence, we deduce from Assertion **ii)**, existence of $a_* \in (a,1)$ such that

$$\Psi_q(t)/\Psi_q'(t) \leq \sqrt{\frac{2}{\alpha}} \text{ for all } t \geq a_*.$$

iv) By means of L'Hopital's rule we obtain

$$\lim_{t\to 0} \Psi_q(t) \int_t^1 \frac{ds}{\Psi_q^2(s)} = \lim_{t\to 0} \frac{\int_t^1 \Psi_q^{-2} ds}{\left(\Psi_q(t)\right)^{-1}} = \lim_{t\to 0} \frac{1}{\Psi_q'(t)} = 1.$$

v) Again by means of L'Hopital's rule we obtain

$$\lim_{t\to 1} \Psi_q(t) \int_t^1 \frac{ds}{\Psi_q^2(s)} = \lim_{t\to 1} \frac{1}{\Psi_q'(t)} = 0.$$

vi) First, notice that if $q \in Q_\#$ then for all $t \in (a,1)$

$$\begin{aligned} \int_a^t \int_a^s q(\tau)\, d\tau ds &\leq \int_0^t \int_0^s q(\tau)\, d\tau ds \\ &= -(1-t) \int_0^t q(s)\, ds + \int_0^t (1-s) q(s)\, ds \\ &\leq 2 \int_0^1 (1-s) q(s)\, ds. \end{aligned}$$

Then, for all $s \in (a,1)$

$$\begin{aligned} \Psi_q'(s) &= \left(\Psi_q'(a) + \int_a^t \Psi_q''(s)\, ds\right) = \left(\Psi_q'(a) + \int_a^t q(\tau)\Psi_q(\tau)\, d\tau\right) \\ &\leq \left(\Psi_q'(a) + \Psi_q(s) \int_a^s q(\tau)\, d\tau\right) \end{aligned}$$

leading to

$$\frac{\Psi_q'(s)}{\Psi_q(s)} \leq \frac{\Psi_q'(a)}{\Psi_q(s)} + \int_a^s q(\tau)\, d\tau \leq \frac{\Psi_q'(a)}{\Psi_q(a)} + \int_a^s q(\tau)\, d\tau.$$

Integrating on (a,t), we obtain

$$\ln\left(\frac{\Psi_q(t)}{\Psi_q(a)}\right) \leq \frac{\Psi_q'(a)}{\Psi_q(a)} + \int_a^t \int_a^s q(\tau)\, d\tau ds \leq \frac{\Psi_q'(a)}{\Psi_q(a)} + 2\int_0^1 (1-s) q(s)\, ds,$$

leading to

$$\Psi_q(t) \leq \Psi_q(a) \exp\left(\frac{\Psi_q'(a)}{\Psi_q(a)} + 2\int_0^1 (1-s)q\,(s)\,ds\right).$$

As Ψ_q is increasing, we have $\Psi_q(1) = \lim_{t\to 1} \Psi_q(t) < +\infty$.

The proof of Lemma 1 is complete.

Because of Properties ii), iii), iv) and v) in Lemma 1, the function

$$\Phi_q(t) = \begin{cases} 1 & \text{if} \quad t = 0 \\ \Psi_q(t) \int_t^1 \dfrac{ds}{\Psi_q^2(s)} & \text{if} \quad t \in (0,1), \\ 0 & \text{if} \quad t = 1 \end{cases} \tag{5}$$

is well defined and it is the unique solution of the bvp

$$\begin{cases} \mathcal{L}_q u = 0 \text{ in } (0,1) \\ u(0) = 1,\ \lim_{t\to 1} u(t) = 0. \end{cases}$$

Lemma 2. *For all $q \in Q^+$, the function Φ_q has the following properties:*

a) $\Phi_q(t) > 0$, $\Phi_q'(t) < 0$ *and* $\Phi_q''(t) \geq 0$ *for all* $t \in (0,1)$,

b) *For all* $t \in [0,1]$, $\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = 1$,

c) *The function* Φ_q/Φ_q' *is bounded at* 1.

Proof. Let $q \in Q^+$ and $a \in (0,1)$ be such that $\alpha = \inf_{t\geq a} q(t) > 0$.

a) We have respectively from (5) and $\Phi_q'' = q\Phi_q$, that $\Phi_q(t) > 0$ and $\Phi_q''(t) \geq 0$ for all $t \in (0,1)$. Since the function Ψ_q' is increasing, we obtain from (5) that for all $t \in (0,1)$,

$$\Phi_q'(t) = \Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2} - \frac{1}{\Psi_q(t)} < \int_t^1 \frac{\Psi_q'}{\Psi_q^2}ds - \frac{1}{\Psi_q(t)} < -\frac{1}{\lim_{t\to 1}\Psi_q(t)} \leq 0.$$

b) We have from (5) that for all $t \in [0,1]$

$$\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = \Psi_q(t)\Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2} - \Psi_q(t)\left(\Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2} - \frac{1}{\Psi_q(t)}\right) = 1.$$

c) We have for $t \geq a$:

$$\begin{aligned} \left(-\Phi_q'(t)\right)^2 &= 2\int_t^1 \Phi_q''(s)\left(-\Phi_q'(s)\right)ds = \int_t^1 q(s)\Phi_q(s)\left(-\Phi_q'(s)\right)ds \\ &\geq \alpha\left(\Phi_q(t)\right)^2. \end{aligned}$$

This leads to

$$\left|\Phi_q(t)/\Phi_q'(t)\right|^2 = \left(\Phi_q(t)/-\Phi_q'(t)\right)^2 \le \frac{1}{\alpha} \text{ for all } t \ge T,$$

and so,

$$\sup_{t\ge T}\left|\Phi_q(t)/\Phi_q'(t)\right| \le \frac{1}{\sqrt{\alpha}}.$$

The proof of Lemma 2 is complete.

Set for $q \in Q^+$ and $0 \le \theta < \eta < 1$

$$\begin{aligned}
\Psi_{q,\theta}(t) &= \Phi_q(\theta)\Psi_q(t) - \Psi_q(\theta)\Phi_q(t),\\
\Phi_{q,\theta,\eta}(t) &= \frac{\Psi_q(\eta)\Phi_q(t) - \Phi_q(\eta)\Psi_q(t)}{\Psi_{q,\theta}(\eta)},\\
\Phi_{q,\theta}(t) &= \lim_{\eta\to 1}\Phi_{q,\theta,\eta}(t) = \frac{\Phi_q(t)}{\Phi_q(\theta)},\\
G_q(\theta,\eta,t,s) &= \begin{cases} 0 \text{ if } \min(t,s) \le \theta \\ \Phi_{q,\theta,\eta}(s)\Psi_{q,\theta}(t) \text{ if } \theta \le t \le s \le \eta \\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(s) \text{ if } \theta \le s \le t \le \eta \\ 0 \text{ if } \min(t,s) \ge \eta, \end{cases}\\
G_q(\theta,t,s) &= \lim_{\eta\to 1} G_q(\theta,\eta,t,s) = \begin{cases} 0 \text{ if } \min(t,s) \le \theta \\ \Phi_{q,\theta}(s)\Psi_{q,\theta}(t) \text{ if } \theta \le t \le s \\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(s) \text{ if } \theta \le s \le t. \end{cases}
\end{aligned}$$

$$\begin{aligned}
\Phi_{q,\theta}(t) &= \frac{\Phi_q(t)}{\Phi_q(\theta)},\ \Psi_{q,\theta}(t) = \Phi_q(\theta)\Psi_q(t) - \Psi_q(\theta)\Phi_q(t) \text{ and}\\
G_q(\theta,t,s) &= \begin{cases} 0 \text{ if } \min(t,s) \le \theta \\ \Phi_{q,\theta}(s)\Psi_{q,\theta}(t) \text{ if } \theta \le t \le s \\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(s) \text{ if } \theta \le s \le t. \end{cases}
\end{aligned}$$

We have then for all $q \in Q$ and all $\theta, \eta \in [0,1]$

$$\Phi_{q,\theta,\eta}\Psi_{q,\theta}' - \Phi_{q,\theta,\eta}'\Psi_{q,\theta} = \Phi_{q,\theta}\Psi_{q,\theta}' - \Phi_{q,\theta}'\Psi_{q,\theta} = 1 \text{ ou?} \tag{6}$$

and

$$G_q(\theta,t,s) = G_q(t,s) - \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(s)\Phi_q(t) \text{ for } t,s \ge \theta.$$

where

$$G_q(t,s) = G_q(0,t,s) = \begin{cases} \Phi_q(t)\Psi_q(s) \text{ if } 0 \le t \le s < 1 \\ \Phi_q(s)\Psi_q(t) \text{ if } 0 \le s \le t < 1. \end{cases} \tag{7}$$

Lemma 3. *We have for all $q \in Q^+$ and $\theta, \eta \in [0,1)$ with $\theta < \eta$:*

1. $G_q(\theta,\eta,t,s) \leq G_q(\theta,\eta,s,s)$ *for all* $t,s \in [\theta,\eta]$,
2. $G_q(\theta,t,s) \leq G_q(\theta,s,s)$ *for all* $t,s \in [\theta,1]$,
3. $G_q(\theta,\eta,t,s) \geq \rho_{\theta,\eta}(t)\, G_q(\theta,\eta,s,s)$ *for all* $t,s \in [\theta,\eta]$ *where* $\rho_{\theta,\eta}(t) = \min(t-\theta,\eta-t)/\Psi_{q,\theta}(\eta)$. *Moreover, if* $q \in Q_\#$ *then* $\Psi_q(1) = \lim_{t\to 1}\Psi_q(t) < \infty$ *and* $G_q(\theta,\eta,t,s) \geq \rho^*_{\theta,\eta}(t)\, G_q(\theta,\eta,s,s)$ *for all* $t,s \in [\theta,\eta]$ *where* $\rho^*_{\theta,\eta}(t) = \min(t-\theta,\eta-t)/\Psi_{q,\theta}(1)$.

Proof. Assertions 1 and 2 are obtained from the monotonicity of the functions $\Phi_{q,\theta,\eta}$, $\Phi_{q,\theta}$ and $\Psi_{q,\theta}$. We have

$$\begin{aligned}\frac{G_q(\theta,\eta,t,s)}{G_q(\theta,\eta,s,s)} &= \begin{cases}\frac{\Psi_{q,\theta}(t)}{\Psi_{q,\theta}(s)} \text{ if } \theta \leq t \leq s \leq \eta\\ \frac{\Phi_{q,\theta,\eta}(t)}{\Phi_{q,\theta,\eta}(s)} \text{ if } \theta \leq s \leq t \leq \eta\end{cases}\\ &\geq \begin{cases}\frac{\Psi_{q,\theta}(t)}{\Psi_{q,\theta}(\eta)} \text{ if } \theta \leq t \leq s \leq \eta\\ \Phi_{q,\theta,\eta}(t) \text{ if } \theta \leq s \leq t \leq \eta.\end{cases}\end{aligned} \tag{8}$$

Since

$$\Psi_{q,\theta}(t) = \int_\theta^t \Psi'_{q,\theta}(s)\,ds \geq \int_\theta^t \Psi'_{q,\theta}(\theta)\,ds = t-\theta$$

and

$$\Phi_{q,\theta,\eta}(t) = \int_t^\eta \left(-\Phi'_{q,\theta,\eta}(s)\right)ds \geq \int_t^\eta \left(-\Phi'_{q,\theta,\eta}(\eta)\right)ds = \frac{\eta-t}{\Psi_{q,\theta}(\eta)},$$

we obtain from (8),

$$\frac{G_q(\theta,\eta,t,s)}{G_q(\theta,\eta,s,s)} \geq \begin{cases}\frac{t-\theta}{\Psi_{q,\theta}(\eta)} \text{ if } \theta \leq t \leq s \leq \eta\\ \frac{\eta-t}{\Psi_{q,\theta}(\eta)} \text{ if } \theta \leq s \leq t \leq \eta.\end{cases} \geq \rho_{\theta,\eta}(t).$$

This ends the proof.

Lemma 4. *We have for all $q \in Q^+$*

i) $\overline{G}_q = \sup_{s,t\in[0,1]} G_q(t,s) = \sup_{0\leq t\leq 1} \Phi_q(t)\,\Psi_q(t) < \infty$,

ii) $\widetilde{G}_q = \sup_{\theta,t,s\in[0,1]} G_q(\theta,t,s) < \infty$.

Proof. Let $q \in Q^+$ and $T \in (0,1)$ be such that $\alpha = \inf_{t \geq T} q(t) > 0$.

i) Taking into consideration that Ψ_q is increasing, we obtain from (5), that for all $t, s \in [0,1]$

$$\begin{aligned} G_q(t,s) \leq \Phi_q(t)\Psi_q(t) &= \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right)\left(\Psi_q(t)\Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2(s)}\right) \\ &\leq \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right)\left(\Psi_q(t)\int_t^1 \frac{\Psi_q'(s)ds}{\Psi_q^2(s)}\right) \leq \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right). \end{aligned}$$

This together with **iii)** in Lemma 1 leads to

$$\overline{G}_q = \sup_{t,s\in[0,1]} G_q(t,s) \leq \sup_{t\in[0,1]} \Phi_q(t)\Psi_q(t) < \infty.$$

ii) Because of Φ_q is decreasing and Ψ_q is increasing we have for all $s, t \geq \theta$

$$\begin{aligned} 0 &\leq G_q(\theta,t,s) \leq \Phi_q(t)\Psi_q(t) + \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(t)\Phi_q(s) \\ &\leq \Phi_q(t)\Psi_q(t) + \Psi_q(\theta)\Phi_q(\theta) \\ &\leq 2\sup_{t\in[0,1]} \Phi_q(t)\Psi_q(t) < \infty. \end{aligned}$$

The proof of Lemma 4 is complete.

Lemma 5. *For all $q \in Q^+$, $\theta \in [0,1)$ and $h \in W$, $L_{q,\theta}h(t) = \int_0^1 G_q(\theta,t,s)h(s)ds$ is the unique solution in $(\theta,1)$ to the bvp:*

$$\begin{cases} \mathcal{L}_q u = h(t), & \theta < t < 1, \\ u(\theta) = \lim_{t\to 1} u(t) = 0 \end{cases}$$

and the operator $L_{q,\theta} : W \to W^1$ is continuous. Moreover, if $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function such that $F(0,0) = F(1,0) = 0$, then the operator $T_{q,\theta} : W \to W$ defined for $v \in W$ by

$$T_{q,\theta}u\,(t) = \int_0^1 G_q(\theta,t,s)F(s,v(s))ds$$

is completely continuous and $u \in W$ is a fixed point of $T_{q,\theta}$ if and only if u is a solution to the bvp

$$\begin{cases} \mathcal{L}_q v = F(t,v(t)), & \theta < t < 1, \\ v(\theta) = v(1) = 0. \end{cases}$$

Proof. Let $h \in W$ and set $H\left(t\right) = L_{q,\theta}h(t)$. We have

$$H(\theta) = \int_0^1 G(\theta,\theta,s)h(s)ds = 0$$

and differentiating twice in the relation

$$H(t) = \int_0^1 G(\theta,t,s)h(s)ds = \Phi_{q,\theta}\left(t\right)\int_\theta^t \Psi_{q,\theta}\left(s\right)h(s)ds + \Psi_{q,\theta}\left(t\right)\int_t^1 \Phi_{q,\theta}\left(s\right)h(s)ds$$

we obtain

$$H''(t) = q(t)H(t) + \left(\Phi'_{q,\theta}\left(t\right)\Psi_{q,\theta}\left(t\right) - \Phi_{q,\theta}\left(t\right)\Psi'_{q,\theta}\left(t\right)\right)h(t) \text{ for all } t \geq \theta.$$

This together with (6) lead to

$$\mathcal{L}_q H(t) = h(t) \text{ for all } t \geq \theta.$$

We have for all $t > \theta$:

$$H(t) = \Phi_q(t)\int_\theta^t \Psi_q(s)h(s)ds + \Psi_q(t)\int_t^1 \Phi_q(s)h(s)ds - \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(t)\int_\theta^1 \Phi_q(s)h(s)ds.$$

Let us prove that $\lim_{t\to 1} H(t) = 0$. Clearly, if $\int_\theta^1 \Psi_q(s)h(s)ds < \infty$ then $\lim_{t\to 1} \Phi_q(t)\int_\theta^t \Psi_q(s)h(s)ds = 0$ and if $\int_\theta^1 \Psi_q(s)h(s)ds = \infty$ then taking in consideration Assertions **d)** in Lemma 2, **i)** of Lemma 4 and $\lim_{t\to 1} h(t) = 0$, we obtain by means of the L'Hopital's rule

$$\begin{aligned} \lim_{t\to 1} \Phi_q(t)\int_\theta^t \Psi_q(s)h(s)ds &= \lim_{t\to 1} \frac{\int_\theta^t \Psi_q(s)h(s)ds}{(\Phi_q(t))^{-1}} \\ &= \lim_{t\to 1}\left(\frac{\Phi_q(t)}{-\Phi'_q(t)}\right)\left(\Phi_q(t)\Psi_q(t)\right)h(t) = 0. \end{aligned}$$

Similarly, if $\lim_{t\to 1}\Psi_q(t) < \infty$ then $\lim_{t\to 1}\Psi_q(t)\int_t^1 \Phi_q(s)h(s)ds = 0$ and if $\lim_{t\to 1}\Psi_q(t) = +\infty$ then taking in consideration **iii)** in Lemma 1, **i)** of Lemma 4 and $\lim_{t\to 1} h(t) = 0$, we obtain by means of the L'Hopital's rule

$$\begin{aligned} \lim_{t\to 1} \Psi_q(t)\int_t^1 \Phi_q(s)h(s)ds &= \lim_{t\to 1} \frac{\int_t^1 \Phi_q(s)h(s)ds}{(\Psi_q(t))^{-1}} \\ &= \lim_{t\to 1}(\frac{\Psi_q(t)}{\Psi'_q(t)})\Phi_q(t)\Psi_q(t)h(t) = 0. \end{aligned}$$

Now, for any $h \in W$, we have

$$\left\|L_{q,\theta}h\right\| = \sup_{t\in[0,1]}\left|L_{q,\theta}h(t)\right| = \sup_{t\in[0,1]}\left|\int_0^1 G(\theta,t,s)h(s)ds\right| \leq G_q\left\|h\right\|$$

and taking in consideration (6) we obtain

$$
\begin{aligned}
\left\|(L_{q,\theta}h)'\right\| &= \sup_{t\in(0,1)} \left|(L_{q,\theta}h)'(t)\right| \\
&= \sup_{t\in(0,1)} \left|\Phi'_{q,\theta}(t)\int_\theta^t \Psi_{q,\theta}(s)\,h(s)ds + \Psi'_{q,\theta}(t)\int_t^1 \Phi_{q,\theta}(s)\,h(s)ds\right| \\
&\le \sup_{t\in(0,1)} \left(-\Phi'_{q,\theta}(t)\int_\theta^t \Psi_{q,\theta}(s)\,|h(s)|\,ds + \Psi'_{q,\theta}(t)\int_t^1 \Phi_{q,\theta}(s)\,|h(s)|\,ds\right) \\
&\le \sup_{t\in(0,1)} \left(-\Phi'_{q,\theta}(t)\,\Psi_{q,\theta}(t)\int_\theta^t ds + \Psi'_{q,\theta}(t)\,\Phi_{q,\theta}(t)\int_t^1 ds\right)\|h\| \\
&\le \|h\|.
\end{aligned}
$$

The above estimates prove that the operator $L_{q,\theta}$ is well defined and is continuous.

Now, We proof that $T_{q,\theta}$ is completely continuous. Notice that $T_{q,\theta} = \daleth \circ L_{q,\theta} \circ \mathbb{F}$ where $\mathbb{F} : W \to W$ is defined by $\mathbb{F}u(t) = F(t, u(t))$ and $\daleth$ is the compact embeding of W^1 in W. Because that the mapping $\mathbb{F}$ is continuous and bounded, the operator $T_{q,\theta}$ is completely continuous.

At the end, if u is a fixed point of $T_{q,\theta}$ and $h = \mathbb{F}u$, then $u = L_{q,\theta}h$ and

$$
\begin{cases}
\mathcal{L}_q u(t) = h(t) = F(t, v(t)),\ \theta < t < 1, \\
u(\theta) = \lim_{t\to 1} u(t) = 0.
\end{cases}
$$

In the remainder of this paper, for $q \in Q^+$ and $m \in E$, we let $L_{q,m}, L^+_{q,m}, L^-_{q,m} : W \to W$ be the operators defined by

$$
\begin{aligned}
&L_{q,m}u(t) = \textstyle\int_0^1 G_q(t,s)m(s)u(s)ds, \\
&L^+_{q,m}u(t) = \left(L_{q,m} \circ I^+\right)u(t) = L_{q,m}u^+(t), \\
&L^-_{q,m}u(t) = \left(L_{q,m} \circ I^-\right)u(t) = L_{q,m}u^-(t).
\end{aligned}
$$

It follows from Lemma 5 that $L_{q,m}$ is compact and for $\nu = +$ or $-$, $L^\nu_{q,m}$ is completely continuous.

2.3. Comparison results

The following three lemmas will play important roles in in this article.

Lemma 6 ([2]). *Let j and k be two integers such that $j \ge k \ge 2$ and let $(\xi_l)_{l=0}^{l=k}$, $(\eta_l)_{l=0}^{l=j}$ be two families of real numbers such that*

$$
\begin{aligned}
&\xi_0 = \xi < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < \xi_k = \eta, \\
&\eta_0 = \xi < \eta_1 < \eta_2 < \cdots < \eta_{j-1} < \eta_j = \eta.
\end{aligned}
$$

If $\xi_1 < \eta_1$, then there exist two integers m and n having the same parity, $1 \leq m \leq k-1$ and $1 \leq n \leq j-1$ such that

$$\xi_m < \eta_n \leq \eta_{n+1} \leq \xi_{m+1}.$$

Lemma 7. *For $i = 1, 2$ let $\phi_i \in S_\rho^{k_i,\nu} \cap W_2$ having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k_i}$. If for some integers m, n with $m \leq k_1 - 1$ we have $n \leq k_2 - 1$ $z_m^1 \leq z_n^2 < z_{n+1}^2 \leq z_{m+1}^1$ and $\phi_1\phi_2 > 0$, then*

$$\int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 \begin{cases} > 0 \text{ if } z_m^1 < z_n^2 \text{ or } z_{n+1}^2 < z_{m+1}^1, \\ = 0 \text{ if } z_m^1 = z_n^2 < z_{n+1}^2 = z_{m+1}^1. \end{cases}$$

Proof. Let $Wr = \phi_1\phi_2' - \phi_2\phi_1'$ be the Wronksian of ϕ_1 and ϕ_2 and without loss of generality, suppose that $\phi_1, \phi_2 > 0$ in $\left(z_n^2, z_{n+1}^2\right)$. We have then $Wr(0) = \lim_{t\to 1} Wr(t) = 0$ and

$$\int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = Wr\left(z_n^2\right) - \lim_{t \to z_{n+1}^2} Wr\left(t\right).$$

Therefore, we distinguish the following cases:

i) $z_m^1 \leq z_n^2 < z_{n+1}^2 = z_{m+1}^1$: In this case we have

$$\phi_1\left(z_n^2\right) = \phi_2\left(z_m^1\right) = \phi_1\left(z_{n+1}^2\right) = \phi_1\left(z_{m+1}^2\right) = 0,$$

leading to

$$\begin{aligned} \int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 &= Wr\left(z_n^2\right) - \lim_{t \to z_{n+1}^2} Wr\left(t\right) \\ &= Wr\left(z_m^1\right) - \lim_{t \to z_{m+1}^2} Wr\left(t\right) = 0. \end{aligned}$$

ii) $z_m^1 \leq z_n^2 < z_{n+1}^2 < z_{m+1}^1$: In this case we have

$$z_{n+1}^2 < 1,\ \phi_1\left(z_{n+1}^2\right) > 0,\ \phi_2\left(z_n^2\right) = \phi_2\left(z_{n+1}^2\right) = 0,\ \phi_1\left(z_{n+1}^2\right) > 0 \text{ and } \phi_2'\left(z_{n+1}^2\right) < 0,$$

leading to

$$\begin{aligned} \int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 &= Wr\left(z_n^2\right) - Wr\left(z_{n+1}^2\right) \\ &\geq -Wr\left(z_{n+1}^2\right) = -\phi_1\left(z_{n+1}^2\right)\phi_2'\left(z_{n+1}^2\right) > 0. \end{aligned}$$

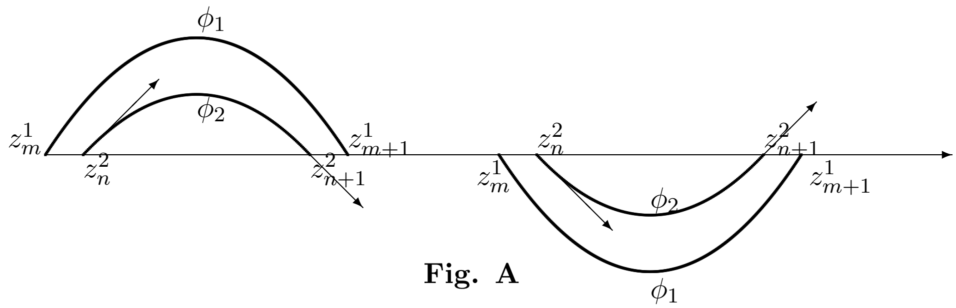

Fig. A

iii) $z_m^1 < z_n^2 < z_{n+1}^2 \leq z_{m+1}^1$: In this case we have $\phi_1\left(z_n^2\right) > 0,\ \phi_2'\left(z_n^2\right) >$ and

$$\lim_{t \to z_{n+1}^2} Wr\left(t\right) = \begin{cases} 0 & \text{if } z_{n+1}^2 = 1 \\ \phi_1\left(z_{n+1}^2\right)\phi_2'\left(z_{n+1}^2\right) & \text{if } z_{n+1}^2 < 1. \end{cases}$$

Thus, we obtain

$$\begin{aligned} \int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 &= Wr\left(z_n^2\right) - \lim_{t \to z_{n+1}^2} Wr\left(t\right) \\ &\geq Wr\left(z_n^2\right) = \phi_1\left(z_n^2\right)\phi_2'\left(z_n^2\right) > 0. \end{aligned}$$

This ends the proof.

We end this section with the following lemma which is an adapted version of the Sturmian comparison result.

Lemma 8. *Let $q \in Q$ and for $i = 1, 2$, $m_i \in \Gamma^+$ and $w_i \in C^2\left([0,1), \mathbb{R}\right)$ satisfying*

$$\mathcal{L}_q w_i = m_i w_i \ \text{in} \ (x_1, x_2)$$

and suppose that w_2 does not vanish identically, $m_1 \geq m_2$ and $m_1 > m_2$ in a subset of positive measure. If either

i) *$x_2 < 1$ and $w_2(x_1) = w_2(x_2) = 0$, or*

ii) *$x_2 = 1$ and $w_2(x_1) = \lim_{t\to 1} w_i(t) = 0$ for $i = 1, 2$*

then there exists $\tau \in (x_1, x_2)$ such that $W^1(\tau) = 0$.

Proof.

i) By the contrary suppose that $w_1 > 0$ in (x_1, x_2) and without loss of generality assume that $w_2 > 0$ in (x_1, x_2), then we have the contradiction:

$$0 \geq w_1(x_2)\, w_2'(x_2) - w_1(x_1)\, w_2'(x_1) =$$
$$\int_{x_1}^{x_2} w_2 \mathcal{L}_q w_1 - w_1 \mathcal{L}_q w_2 = \int_{x_1}^{x_2} (m_1 - m_2) w_1 w_2 > 0.$$

ii) By the contrary suppose that $w_1 > 0$ in $(x_1, 1)$ and without loss of generality assume that $w_2 > 0$ in $(x_1, 1)$, we have for $t > x_1$ that

$$(w_1(t)\, w_2'(t) - w_1(t)\, w_2'(t)) - w_1(x_1)\, w_2'(x_1) =$$
$$\int_{x_1}^{t} w_2 \mathcal{L}_q w_1 - w_1 \mathcal{L}_q w_2 = \int_{x_1}^{t} (m_1 - m_2) w_1 w_2 > 0.$$

Since from Lemma 5 $w_1, w_2 \in W^1$, we have

$$\lim_{t \to 1} \left(w_1(t)\, w_2'(t) - w_1'(t)\, w_2(t) \right) = 0 \tag{9}$$

and so, the contradiction

$$0 \geq -w_1(x_1)\, w_2'(x_1) = \int_{x_1}^{1} (m_1 - m_2) w_1 w_2 > 0.$$

The proof is complete.

2.4. The positive eigenvalue

The main result of this subsection concerns the existence of positive eigenvalues on the bounded interval $[\theta, 1]$.

Theorem 9. *For all $q \in Q$, $m \in \Gamma^{++}$ and $\theta \in [0, 1)$, the eigenvalue problem*

$$\begin{cases} \mathcal{L}_q u = \mu m u, & in\ (\theta, 1), \\ u(\theta) = \lim_{t \to 1} u(t) = 0, \end{cases} \tag{10}$$

admits a unique positive eigenvalue $\mu_1^+(q, m, \theta)$. Moreover for q, m fixed, the function $\theta \to \mu_1(\theta) := \mu_1(q, m, \theta)$ is continuous increasing and we have $\lim_{\theta \to 1} \mu_1(\theta) = +\infty$.

Proof. Let $q \in Q$, $m \in \Gamma^{++}$, $\theta \in [0, 1)$ and let ϖ be a positive constant such that $\widehat{q} = q + \varpi m > 0$ in $[0, 1)$. Consider the eigenvalue problem

$$\begin{cases} \mathcal{L}_{\widehat{q}} u = \mu m u, & \text{in } (\theta, 1) \\ u(\theta) = \lim_{t \to 1} u(t) = 0 \end{cases} \tag{11}$$

and notice that μ_0 is a positive eigenvalue of the bvp (11) if and only if $\mu_0 - \varpi$ is a positive eigenvalue of the bvp (10).

We have from Lemma 5 that μ is a positive eigenvalue of (11) if and only if μ^{-1} is a positive eigenvalue of the linear compact operator $L_{\widehat{q},\theta} : W \to W$ where

$$L_{\widehat{q},m,\theta}u(t) = \int_0^1 G_{\widehat{q}}(\theta, t, s)m(s)u(s)ds.$$

Let $u_\theta \in W$ be the function defined by

$$u_\theta(t) = \begin{cases} 0 & \text{if } t \notin \left[\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right], \\ (t - \frac{2\theta+1}{3})(\frac{2+\theta}{3} - t) & \text{if } t \in \left[\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right], \end{cases}$$

we have then $L_{\widehat{q},m,\theta}u_\theta(t) \geq 0 = u_\theta(t)$ for $t \in \left[0, \frac{2\theta+1}{3}\right] \cup \left[\frac{2+\theta}{3}, 1\right]$ and $L_{\widehat{q},m,\theta}u_\theta(t), u_\theta(t) > 0$ for $t \in \left(\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right)$. This shows that $L_{\widehat{q},m,\theta}u_\theta \geq c_\theta u_\theta$ where $c_\theta = \inf\left\{L_{\widehat{q},\theta}u_\theta(t)/u_\theta(t) : t \in \left(\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right)\right\} > 0$ and $r(L_{\widehat{q},m,\theta}) > 0$. We have from the Krein-Rutman theorem, that $r(L_{\widehat{q},\theta})$ is a positive eigenvalue of L_θ having a positive eigenvector ϕ_θ. Obviously, $\widehat{\mu}_1(\theta, \widehat{q}, m) = 1/r(L_{\widehat{q},m,\theta})$ is a positive eigenvalue of the eigenvalue problem (11) and $\mu_1(\theta, q, m) = \widehat{\mu}_1(\theta, \widehat{q}, m) - \varpi$ is a positive eigenvalue of the eigenvalue problem (10).

Now, let us prove uniqueness of the positive eigenvalue. Suppose that λ is a positive eigenvalue of the eigenvalue problem (10) having an eigenfunction ψ, we have then

$$0 = \int_\theta^1 \psi \mathcal{L}_{\widehat{q}}\phi_\theta + \phi_\theta \mathcal{L}_{\widehat{q}}\psi = (\mu_1(\theta, q, m) - \lambda) \int_\theta^1 m\phi_\theta\psi$$

leading to $\lambda = \mu_1(\theta, q, m)$.

Let now $\theta_1, \theta_2 \in (0, 1)$ be such that $\theta_1 < \theta_2$ and set for $i = 1, 2,\ \mu_i = \mu_1(\theta_i, q, m)$ with the corresponding eigenfunction ψ_i. We have

$$\begin{aligned} 0 > -\psi_2'(\theta_2)\psi_1(\theta_2) &= \textstyle\int_{\theta_2}^1 \psi_2 \mathcal{L}_{\widehat{q}}\psi_1 - \psi_1 \mathcal{L}_{\widehat{q}}\psi_2'' \\ &= (\mu_1 - \mu_2) \textstyle\int_{\theta_2}^1 m\psi_1\psi_2 \end{aligned}$$

leading to $\mu_1 < \mu_2$, proving that the function $\theta \to \mu_1(\cdot)$ is an increasing.

At this stage let us prove the continuity of the function $\theta \to \mu_1(\cdot)$. Let $[\gamma, \delta] \subset$

$[0,1]$ and $\theta_1, \theta_2 \in [\gamma, \delta]$ be such that $\theta_1 < \theta_2$. We have for all $u \in W$ with $\| u \| = 1$

$$\begin{aligned}
&\left| L_{\widehat{q},m,\theta_2} u(t) - L_{\widehat{q},m,\theta_1} u(t) \right| \\
= &\left| \int_{\theta_2}^{1} G_{\widehat{q}}(\theta_2, t, s)\, muds - \int_{\theta_1}^{1} G_{\widehat{q}}(\theta_1, t, s)\, muds \right| \\
= &\begin{cases} 0, & \text{if } t \le \theta_1 < \theta_2, \\ \left| \int_{\theta_1}^{1} G_{\widehat{q}}(\theta_1, t, s)\, muds \right|, & \text{if } \theta_1 < t \le \theta_2, \\ \\ \left| \int_{\theta_2}^{1} G_{\widehat{q}}(\theta_2, t, s)\, muds - \int_{\theta_1}^{1} G_{\widehat{q}}(\theta_1, t, s)\, muds \right| & \text{if } \theta_1 < \theta_2 < t. \end{cases}
\end{aligned}$$

Set

$$\chi = \|m\| \left[\left(\int_{\gamma}^{1} \phi_{\widehat{q}} ds \right) \frac{\phi_{\widehat{q}}(\gamma)}{\phi_{\widehat{q}}^2(\delta)} + \overline{G}_{\widehat{q}} + \Phi_{\widehat{q}}(\gamma)\, \Psi_{\widehat{q}}(\delta) \right]$$

then we have for $\theta_2 \ge t > \theta_1$

$$\begin{aligned}
&\left| \int_{\theta_1}^{1} G_{\widehat{q}}(\theta_1, t, s)\, muds \right| \le \|m\| \int_{\theta_1}^{1} G_{\widehat{q}}(\theta_1, t, s)\, ds \\
&= \|m\| \left(\int_{\theta_1}^{1} G_{\widehat{q}}(t, s)\, ds - \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)} \phi_{\widehat{q}}(t) \int_{\theta_1}^{1} \phi_{\widehat{q}} ds \right) \\
&= \|m\| \big(\int_{\theta_1}^{t} G_{\widehat{q}}(t, s)\, ds + \int_{t}^{1} G_{\widehat{q}}(t, s)\, ds \\
&- \tfrac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} \phi_{\widehat{q}}(t) \int_{\theta_1}^{t} \phi_{\widehat{q}} ds - \tfrac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} \phi_{\widehat{q}}(t) \int_{t}^{1} \phi_{\widehat{q}} ds \big) \\
&= \|m\| \big(\int_{\theta_1}^{t} G_{\widehat{q}}(t, s)\, ds - \tfrac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} \phi_{\widehat{q}}(t) \int_{\theta_1}^{t} \phi_{\widehat{q}} ds \big) + \psi_{\widehat{q}}(t) \int_{t}^{1} \phi_{\widehat{q}} ds - \tfrac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} \phi_{\widehat{q}}(t) \int_{t}^{1} \Psi_{\widehat{q}} ds \big) \\
&= \|m\| \big(\int_{\theta_1}^{t} G_{\widehat{q}}(t, s)\, ds - \tfrac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} \phi_{\widehat{q}}(t) \int_{\theta_1}^{t} \phi_{\widehat{q}} ds \big) + \int_{t}^{1} \phi_{\widehat{q}} ds \left(\tfrac{\psi_{\widehat{q}}(t)}{\phi_{\widehat{q}}(t)} - \tfrac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} \right) \phi_{\widehat{q}}(t) \big) \\
&\le \|m\| \left[\left(\int_{\gamma}^{1} \phi_{\widehat{q}} ds \right) \tfrac{\phi_{\widehat{q}}(\gamma)}{\phi_{\widehat{q}}^2(\delta)} + \overline{G}_{\widehat{q}} + \Phi_{\widehat{q}}(\gamma)\, \Psi_q(\delta) \right] |\theta_2 - \theta_1| \le \chi\, |\theta_2 - \theta_1|
\end{aligned}$$

and for $\theta_1 < \theta_2 < t$,

$$\begin{aligned}
&\left| \int_{\theta_2}^{1} G_{\widehat{q}}(\theta_2, t, s)\, muds - \int_{\theta_1}^{1} G_{\widehat{q}}(\theta_1, t, s)\, muds \right| \le \\
&\left| \int_{\theta_2}^{1} \left(G_{\widehat{q}}(\theta_2, t, s) - G_{\widehat{q}}(\theta_1, t, s) \right) muds \right| + \left| \int_{\theta_1}^{\theta_2} G_{\widehat{q}}(\theta_1, t, s)\, muds \right| \\
&= \left| \left(\int_{\theta_2}^{1} \phi_{\widehat{q}} muds \right) \left(\tfrac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} - \tfrac{\psi_{\widehat{q}}(\theta_2)}{\phi_{\widehat{q}}(\theta_2)} \right) \phi_{\widehat{q}}(t) \right| + \left| \int_{\theta_1}^{\theta_2} G_{\widehat{q}}(\theta_1, t, s)\, muds \right| \\
&\le \|m\| \left[\left(\int_{\gamma}^{1} \phi_{\widehat{q}} ds \right) \tfrac{\phi_{\widehat{q}}(\gamma)}{\phi_{\widehat{q}}^2(\delta)} + \overline{G}_{\widehat{q}} \right] |\theta_2 - \theta_1| \le \chi\, |\theta_2 - \theta_1| .
\end{aligned}$$

The above estimates show that the mapping $\theta \to L_{\widehat{q},m,\theta}$ is locally Lipschitzian and so, it is continuous. Let (θ_n) be a sequence converging to θ_* and let θ_-, θ_+ be such that $(\theta_n) \subset [\theta_-, \theta_+]$. Therefore we have for all $n \ge 1$,

$$0 < \mu_1(\theta_+) \le \mu_1(\theta_n) \le \mu_1(\theta_-)$$

and the sequence $(\mu_1(\theta_n, \widehat{q}, m))$ converges (up to a subsequence) to some $\mu_* > 0$. We conclude by Lemma 2.13 in [3] and by uniqueness of the positive eigenvalue that $\mu_* = \mu_1(\theta_*)$. Thus, the continuity of the mapping $\mu_1(\cdot)$ is proved.

It remains to prove that

$$\lim_{\theta \to 1} \mu_1^+(\theta) = \lim_{\theta \to 1} \frac{1}{r(L_{\widehat{q},m,\theta})} = +\infty.$$

We have for all $u \in W$ with $\|u\| = 1$

$$\begin{aligned} \left|L_{\widehat{q},m,\theta} u(t)\right| &\leq \int_\theta^1 G_{\widehat{q}}(\theta, t, s)\, m(s) ds \\ &\leq \int_\theta^1 G_{\widehat{q}}(t, s)\, m(s) ds + \frac{\Psi_{\widehat{q}}(\theta)}{\Phi_{\widehat{q}}(\theta)} \int_\theta^1 \Phi_{\widehat{q}}(t) \Phi_{\widehat{q}}(s) m(s) ds \\ &\leq \int_\theta^1 G_{\widehat{q}}(t, s)\, m(s) ds + \Psi_{\widehat{q}}(\theta) \int_\theta^1 \Phi_{\widehat{q}}(s) m(s) ds. \end{aligned}$$

Arguing as in the proof of Lemma 5, we obtain $\lim_{\theta \to 1} \Psi_{\widehat{q}}(\theta) \int_\theta^1 \Phi_{\widehat{q}}(s) m(s) ds = 0$ and because of $\int_\theta^1 G_{\widehat{q}}(t, s)\, m(s) ds \leq \overline{G}_{\widehat{q}} \int_\theta^1 m(s) ds$, we have $\lim_{\theta \to 1} \int_\theta^1 G_{\widehat{q}}(t, s)\, m(s) ds = 0$ uniformely on $[0, 1]$. Therefore, we have proved that $\lim_{\theta \to 1} r(L_{\widehat{q},m,\theta}) = \lim_{\theta \to +\infty} \left\|L_{\widehat{q},m,\theta}\right\| = 0$ and this ends the proof.

3. The half-eigenvalue problem

Consider for $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$ the bvp:

$$\begin{cases} \mathcal{L}_q u = \lambda m u + \alpha u^+ - \beta u^-, & \text{in } (0,1) \\ u(0) = \lim_{t \to 1} u(t) = 0, \end{cases} \tag{12}$$

where λ is a real parameter.

Because that the function $u \to \lambda m u + \alpha u^+ - \beta u^-$ is linear on the cones $\{u \in E : u \geq 0 \text{ in } [0, 1]\}$ and $\{u \in E : u \leq 0 \text{ in } [0, 1]\}$, the bvp (12) is said to be half-linear.

Definition 1. *We say that λ_0 is a half-eigenvalue of (12) if there exists a nontrivial solution (λ_0, u_0) of (12). In this situation, $\{(\lambda_0, tu_0),\ t > 0\}$ is a half-line of nontrivial solutions of (12) and μ_0 is said to be simple if all solutions (λ_0, u) of (12), with $uu_0 > 0$ in a right neighborhood of 0, are on this half-line. There may exist another half-line of solutions $\{(\lambda_0, tv_0),\ t > 0\}$, but then we say that λ_0 is simple, if $u_0 v_0 < 0$ in a right neighborhood of 0 and all solutions (λ_0, v) of (12) lie on these two half lines.*

The case of the bvp (12) where $q \in E$ has been considered by Berestycki in [4]. He has proved that the bvp (12) admits two increasing sequences of half-eigenvalues. So, the main goal of this section is to prove that the Berestycki result holds true for the case $q \in Q$. We begin with the following list of lemmas.

Proposition 1. *Let $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$. If (λ, ϕ) is a nontrivial solution to the bvp (12), then $\phi \in S_k^\nu$ for some integer $k \geq 1$ and $\nu = +$ or $-$.*

Proof. Let $\varepsilon > 0$ be small enough and let $A > 0$ be such that $\mu_1(q-\alpha, m+\varepsilon) > -A$. Consider the bvp

$$\begin{cases} \mathcal{L}_{q+Am}u = \lambda mu + \alpha u^+ - \beta u^- \text{ in } (0,1), \\ u\,(0) = \lim_{t\to 1} u\,(t) = 0, \end{cases} \tag{13}$$

and notice that λ is a half-eigenvalue of the bvp (13) if and only if $(\lambda - A)$ is a half-eigenvalue to the bvp (12). Thus, we have to prove that if (λ, ϕ) is a nontrivial solution to the bvp (13), then $\phi \in S_k^\nu$ for some integer $k \geq 1$ and $\nu = +$ or $-$. To this aim, let (λ, ϕ) is a nontrivial solution to the bvp (13), we claim first that all zeros of ϕ in $[0, 1)$ are simple. Indeed, noticing that the right hand-side in (13) is Lipschitzian, if $\phi\,(x_*) = \phi'\,(x_*) = 0$ for some $x_* \in [0, 1)$ then the standard existence and uniqueness result of a solution to an initial value problem leads to $\phi = 0$. This contradicts (λ, ϕ) is a nontrivial solution to the bvp (13).

Now, we claim that ϕ has a finite number of zeros. To the contrary, assume that ϕ has an infinite sequence of zeros, say (z_n) such that $\lim z_n = z_*$, we distinguish then the following two cases:

i. $z_* \in [0, 1)$, in this situation we have

$$\phi\,(z_*) = \lim \phi\,(z_n) = 0 \text{ and } \phi'\,(z_*) = \lim \frac{\phi\,(z_n) - \phi\,(z_*)}{z_n - z_*} = 0.$$

This contradicts the simplicity of zeros of ϕ in $[0, 1)$.

ii. $z_* = 1$, in this case ϕ satisfies for all $n \geq 1$

$$\begin{cases} \mathcal{L}_{q+Am}u = \lambda mu + \alpha u^+ - \beta u^- \text{ in } (0,1), \\ u\,(z_n) = \lim_{t\to 1} u\,(t) = 0. \end{cases}$$

Let for all $n \geq 1$ $\mu_n = \mu_1(q + Am - \alpha, m + \varepsilon, z_n)$ the positive eigenvalue given by Theorem 9 and let ψ_n the normalized positive eigenfunction associated with μ_n. Notice that

$$\mu_n = \mu_1(q+Am-\alpha, m+\varepsilon, z_n) \geq \mu_1(q+Am-\alpha, m+\varepsilon) = \mu_1(q-\alpha, m+\varepsilon) + A > 0.$$

We claim now that for all integers $n \geq 1$, $\lambda > \mu_n$. Indeed, let $l \geq n$ be such that $\phi > 0$ in (z_l, z_{l+1}), we obtain from Lemma 7 that

$$0 < \int_{z_l}^{z_{l+1}} -\psi_n \mathcal{L}_q \phi + \phi \mathcal{L}_q \psi_n + \mu_n \varepsilon \int_{z_l}^{z_{l+1}} \phi \psi_n = (\lambda - \mu_n) \int_{z_l}^{z_{l+1}} m \phi \psi_n$$

leading to $\lambda > \mu_n$.

Therefore, we obtain from Theorm 9 the contradiction

$$\lambda \geq \lim \mu_n = \lim \mu_1(q + Am - \alpha, m + \varepsilon, z_n) = +\infty.$$

This completes the proof of the lemma.

Proposition 2. *For $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$, $k \geq 1$ and $\nu = +$ or $-$ the bvp (12) admits at most one half eigenvalue having an eigenfunction in S_k^ν.*

Proof. Let (λ_1, ϕ_1) and (λ_2, ϕ_2) be two nontrivial solutions to the bvp (12) such that $\lambda_1 \neq \lambda_2$ and $\phi_1, \phi_2 \in S_k^\nu$ for some integer $k \geq 1$ and $\nu = +, -$, and denote for $i = 1, 2$ $\left(z_j^i\right)_{j=0}^{j=k}$ the sequence of zeros of ϕ_i. First, we claim that there exists j_0 such that $z_{j_0}^1 \neq z_{j_0}^2$; indeed, assume that $\phi_1\left(z_j^2\right) = 0$ for all $j \in \{1, ..., k-1\}$ and $\lambda_1 < \lambda_2$ and note that there exists $j_1 \in \{1, ..., k-1\}$ such that $meas\left(\{m > 0\} \cap \left(z_{j_1}^2, z_{j_1+1}^2\right)\right) > 0$ and $\phi_1 \phi_2 > 0$ in $\left(z_{j_1}^2, z_{j_1+1}^2\right)$. Applying Lemma 8, we conclude that there is $\tau \in \left(z_{j_1}^2, z_{j_1+1}^2\right)$ such that $\phi_1(\tau) = 0$ and this contradicts $\phi_1 \in S_k^\nu$.

Now, let $k_1 = \max\left\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\right\}$ and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$ and without loss of generality, assume that $\xi_1 = z_{k_1+1}^1 < \eta_1 = z_{k_1+1}^2$. We obtain from Lemma 6 that there exist two integers $m, n \geq 1$ having the same parity such that

$$\xi_m = z_{k_1+m}^1 < \eta_n = z_{k_1+n}^2 < \eta_{n+1} = z_{k_1+n+1}^2 \leq \xi_{m+1} = z_{k_1+m+1}^1$$

and we have from Lemma 7 that

$$0 < \int_{\xi_0}^{\xi_1} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = (\lambda_1 - \lambda_2) \int_{\xi_0}^{\xi_1} m \phi_1 \phi_2 \tag{14}$$

$$0 < \int_{\eta_n}^{\eta_{n+1}} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = (\lambda_2 - \lambda_1) \int_{\eta_n}^{\eta_{n+1}} m \phi_1 \phi_2. \tag{15}$$

Therefore, we obtain from (14) that $\lambda_1 > \lambda_2$, and from (15) the contradiction $\lambda_1 < \lambda_2$. This ends the proof.

Proposition 3. *Let $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$ and assume that $(\lambda_1, \phi_1), (\lambda_2, \phi_2)$ are two solutions of the bvp (12) such that $\phi_i \in S_\rho^{k_i,\nu}$ for $i = 1, 2$. If $k_2 > k_1$ then $\lambda_2 > \lambda_1$.*

Proof. By the way of contradiction assume that $\lambda_2 \leq \lambda_1$ and let for $i = 1, 2$, $\left(z_j^i\right)_{j=0}^{j=k}$ be the sequence of zeros of ϕ_i. Set $k_* = \max\left\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\right\}$ and consider $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ the families defined by $\xi_j = z_{k_*+j}^1$ and $\eta_j = z_{k_*+j}^2$. We distinguish then two cases.

i) $\xi_1 = z_{k_*+1}^1 > \eta_1 = z_{k_*+1}^2$. In this case we have from Lemma 7

$$0 < \int_{\eta_0}^{\eta_1} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = (\lambda_2 - \lambda_1) \int_{\eta_0}^{\eta_1} m\phi_1\phi_2$$

leading to the contradiction $\lambda_1 < \lambda_2$.

ii) $\xi_1 = z_{k_*+1}^1 < \eta_1 = z_{k_*+1}^2$. In this case, Lemma 6 guarantees existence of two integers m, n having the same parity such that

$$\xi_m = z_{k_1+m}^1 < \eta_n = z_{k_1+n}^2 < \eta_{n+1} = z_{k_1+n+1}^2 \leq \xi_{m+1} = z_{k_1+m+1}^1.$$

and we have from Lemma 7

$$0 < \int_{\eta_n}^{\eta_{n+1}} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = (\lambda_2 - \lambda_1) \int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2,$$

leading also to the contradiction $\lambda_1 < \lambda_2$.

This ends the proof.

Proposition 4. *Let $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$. If λ is a half-eigenvalue of the bvp (12), then λ is simple.*

Proof. Let λ be a half-eigenvalue of the bvp (12) having two eigenfunctions ϕ_1, ϕ_2 and without loss of generality, assume that $\phi_1, \phi_2 > 0$ in a right neighborhood of 0. Because of Proposition 3 we have that $\phi_1, \phi_2 \in S_k^+$ for some integer $k \geq 1$. For $i = 1, 2$, let $(z_j^i)_{j=0}^{j=k-1}$ be the sequence of zeros of ϕ_i. We have that $z_j^1 = z_j^2$ for all $j = 0, \ldots, k$. By induction, clearly $z_0^1 = z_0^2 = 0$ and if $z_j^1 = z_j^2$ then $z_{j+1}^1 = z_{j+1}^2$. Indeed, if for example $z_{j+1}^1 < z_{j+1}^2$, then Lemma 7 leads to the contradiction

$$0 < \int_{z_j^1}^{z_{j+1}^1} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = 0.$$

Because of the positive homogeneity of (12) and $\phi_1, \phi_2 \in S_k^+$, $\phi_1'(0) > 0$, $\phi_2'(0) > 0$ and $\psi_1 = (\phi_1'(0))^{-1}\phi_1$, $\psi_2 = (\phi_2'(0))^{-1}\phi_2$ are eigenfunctions associated with λ satisfying

$$\psi_1(0) = \psi_2(0) = 0 \text{ and } \psi_1'(0) = \psi_2'(0) = 1.$$

Therefore, $\psi = \psi_1 - \psi_2$ satisfies

$$\begin{cases} \mathcal{L}_q\psi = \mu m\psi + \alpha\psi^+ - \beta\psi^- \text{ in } (0, z_j^1), \\ \psi(0) = \psi'(0) = 0, \end{cases}$$

proving that $\psi_1 = \psi_2$ in $[0,1]$. This completes the proof.

In what follows and when for functions $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$ the half-eigenvalue of the bvp (12) associated with an eigenfunction in S_k^ν exists, this will be denoted by $\lambda_k^\nu(q, m, \alpha, \beta)$.

Proposition 5. *Let $q_1, q_2 \in Q$, $m \in \Gamma^+$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in E$ and assume that for some $k \geq 1$ and $\nu = \pm$, $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1)$, $\lambda_k^\nu(q_2, m, \alpha_1, \beta_1)$, $\lambda_k^\nu(q_1, m, \alpha_2, \beta_1)$ and $\lambda_k^\nu(q_1, m, \alpha_1, \beta_2)$ exist.*

1. *If $\alpha_1 \leq \alpha_2$ a.e. in $(0,1)$, then $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1) \geq \lambda_k^\nu(q_1, m, \alpha_2, \beta_1)$.*
2. *If $\beta_1 \leq \beta_2$ a.e. in $(0,1)$, then $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1) \geq \lambda_k^\nu(q_1, m, \alpha_1, \beta_2)$.*
3. *If $q_1 \leq q_2$ a.e. in $(0,1)$, then $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1) \leq \lambda_k^\nu(q_2, m, \alpha_1, \beta_1)$.*

Proof. We present the proof of Assertion 1. Assertion 2 is checked similarly and Assertion 3 is a consequence of Assertions 2 and 1. Suppose that $\alpha_1 \leq \alpha_2$ and for $i = 1, 2$, set $\lambda_i = \lambda_k^\nu(m, \alpha_i, \beta_1)$. Let ϕ_i be the eigenfunction associated with λ_i having a sequence of zeros $(z_j^i)_{j=0}^{j=k}$. We distinguish two cases:

i). $z_j^1 = z_j^2$ for all $j \in \{1, \dots, k-1\}$. Let $j_1 \in \{1, \dots, k-1\}$ be such that $\text{meas}(\{m > 0\} \cap (z_{j_1}^2, z_{j_1+1}^2)) > 0$, we have

$$\begin{aligned} 0 &= \int_{z_{j_1}^2}^{z_{j_1+1}^2} \phi_2\mathcal{L}_q\phi_1 - \phi_1\mathcal{L}_q\phi_2 = (\lambda_1 - \lambda_2)\int_{z_{j_1}^2}^{z_{j_1+1}^2} m\phi_1\phi_2 \\ &+ \int_{z_{j_1}^2}^{z_{j_1+1}^2} \left(\alpha_1\phi_1^+\phi_2 - \alpha_2\phi_2^+\phi_1\right) + \int_{z_{j_1}^2}^{z_{j_1+1}^2} \left(\beta_1\phi_1^-\phi_2 - \beta_1\phi_2^-\phi_1\right) \\ &= (\lambda_1 - \lambda_2)\int_{z_{j_1}^2}^{z_{j_1+1}^2} m\phi_1\phi_2 + \int_{z_{j_1}^2}^{z_{j_1+1}^2} \left(\alpha_1\phi_1^+\phi_2 - \alpha_2\phi_2^+\phi_1\right). \end{aligned} \tag{16}$$

Thus, from (16) in both the case $\phi_1, \phi_2 > 0$ in $(z_{j_1}^2, z_{j_1+1}^2)$ and the case $\phi_1, \phi_2 < 0$ in $(z_{j_1}^2, z_{j_1+1}^2)$, we obtain $\lambda_1 \geq \lambda_2$.

ii) $z^1_{j_0} \neq z^2_{j_0}$ for some j_0: In this case set $k_1 = \max\{l \leq k : z^1_j = z^2_j \text{ for all } j \leq l\}$. If $z^1_{k_1+1} < z^2_{k_1+1}$, then

$$0 < \int_{z^1_{k_1}}^{z^1_{k_1+1}} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = (\lambda_1 - \lambda_2) \int_{z^1_{k_1}}^{z^1_{k_1+1}} m\phi_1\phi_2 + \int_{z^1_{k_1}}^{z^1_{k_1+1}} (\alpha_1 - \alpha_2)\phi_1\phi_2$$

proving that $\mu_1 > \mu_2$ and if $z^2_{k_1+1} < z^1_{k_1+1}$ then considering the families $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ with $\xi_j = z^1_{k_1+j}$ and $\eta_j = z^2_{k_1+j}$, we obtain from Lemma 6 that there exist two integers $m, n \geq 1$ having the same parity such that

$$\xi_m = z^2_{k_1+m} < \eta_n = z^1_{k_1+n} < \eta_{n+1} = z^1_{k_1+n+1} \leq \xi_{m+1} = z^2_{k_1+m+1}.$$

Therefore, we obtain from Lemma 7

$$0 < \int_{\eta_n}^{\eta_{n+1}} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = (\lambda_1 - \lambda_2) \int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2 + \int_{\eta_n}^{\eta_{n+1}} (\alpha_1 - \alpha_2)\phi_1\phi_2,$$

leading to $\lambda_1 > \lambda_2$.

This completes the proof.

Proposition 6. *Let $q \in Q$, $m_1, m_2 \in \Gamma^+$ and $\alpha, \beta \in E$. Assume that $m_1 \leq m_2$ in $(0,1)$, $m_1 < m_2$ in a subset of positive measure and $\lambda^\nu_k(q, m_1, \alpha, \beta)$, $\lambda^\nu_k(q, m_2, \alpha, \beta)$ exist for some integer $k \geq 1$ and $\nu = +$ or $-$. If either $\lambda^\nu_k(q, m_1, \alpha, \beta) \geq 0$ or $\lambda^\nu_k(q, m_2, \alpha, \beta) \geq 0$, then $\lambda^\nu_k(q, m_1, \alpha, \beta) > \lambda^\nu_k(q, m_2, \alpha, \beta)$ and if either $\lambda^\nu_k(q, m_1, \alpha, \beta) \leq 0$ or $\lambda^\nu_k(q, m_2, \alpha, \beta) \leq 0$, then $\lambda^\nu_k(q, m_1, \alpha, \beta) < \lambda^\nu_k(q, m_2, \alpha, \beta)$.*

Proof. Assume that for $i = 1, 2$ $\lambda_i = \lambda^\nu_k(m_1, \alpha, \beta)$ exists and has an eigenfunction ϕ_i having a sequence of zeros $(z^i_j)_{j=0}^{j=k}$. First, we claim that there exists j_0 such that $z^1_{j_0} \neq z^2_{j_0}$. Indeed, if $\phi_1(z^2_j) = 0$ for all $j \in \{1, \ldots, k-1\}$ and $j_1 \in \{1, \ldots, k-1\}$ is such that $\text{meas}(\{m_2 > m_1\} \cap (z^2_{j_1}, z^2_{j_1+1})) > 0$, then taking in account that $\phi_1\phi_2 > 0$ in $(z^2_{j_1}, z^2_{j_1+1})$, we obtain by means of Lemma 8 in the case $\lambda_1 \leq \lambda_2$ (the other caes is checked similarly) that there exists $\tau \in (z^2_{j_1}, z^2_{j_1+1})$ such that $\phi_1(\tau) = 0$. Obviously, this contradicts $\phi_1 \in S^\nu_k$.

Now, let $k_1 = \max\{l \leq k : z^1_j = z^2_j \text{ for all } j \leq l\}$, and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z^1_{k_1+j}$ and $\eta_j = z^2_{k_1+j}$. Assume that $\lambda_1 \geq 0$ or $\lambda_2 \geq 0$, we distinguish then two cases.

i. $\xi_1 = z^1_{k_1+1} < \eta_1 = z^2_{k_1+1}$: In this case we have from Lemma 7

$$\begin{aligned}
0 < \int_{\xi_0}^{\xi_1} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 &= \int_{\xi_0}^{\xi_1} (\lambda_1 m_1 - \lambda_2 m_2)\phi_1\phi_2 \\
&= (\lambda_1 - \lambda_2) \int_{\xi_0}^{\xi_1} m_1 \phi_1 \phi_2 + \lambda_2 \int_{\xi_0}^{\xi_1} (m_1 - m_2)\phi_1\phi_2 \\
&= (\lambda_1 - \lambda_2) \int_{\xi_0}^{\xi_1} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\xi_0}^{\xi_1} (m_1 - m_2)\phi_1\phi_2
\end{aligned}$$

and this proves that in both the cases $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, we have $\lambda_1 > \lambda_2$.

ii. $\xi_1 = z^1_{k_1+1} > \eta_1 = z^2_{k_1+1}$: In this case Lemma 6 guarantees existence of two integers m, n having the same parity such that

$$\eta_n = z^2_{k_1+n} < \xi_m = z^1_{k_1+m} < \xi_{m+1} = z^1_{k_1+m+1} \leq \eta_{n+1} = z^2_{k_1+n+1}.$$

As above, we have from Lemma 7

$$\begin{aligned}
0 < \int_{\xi_m}^{\xi_{m+1}} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 &= \int_{\xi_m}^{\xi_{m+1}} (\lambda_2 m_2 - \lambda_1 m_1)\phi_1\phi_2 \\
&= (\lambda_2 - \lambda_1) \int_{\xi_m}^{\xi_{m+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\xi_m}^{\xi_{m+1}} (m_2 - m_1)\phi_1\phi_2 \\
&= (\lambda_1 - \lambda_2) \int_{\xi_m}^{\xi_{m+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\xi_m}^{\xi_{m+1}} (m_1 - m_2)\phi_1\phi_2
\end{aligned}$$

and this proves that in both the cases $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, we have $\lambda_1 > \lambda_2$.

Assume that $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$, we distinguish then two cases.

iii. $\xi_1 = z^1_{k_1+1} > \eta_1 = z^2_{k_1+1}$: In this case we have from Lemma 7

$$\begin{aligned}
0 > \int_{\eta_0}^{\eta_1} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 &= \int_{\eta_0}^{\eta_1} (\lambda_1 m_1 - \lambda_2 m_2)\phi_1\phi_2 \\
&= (\lambda_1 - \lambda_2) \int_{\eta_0}^{\eta_1} m_1 \phi_1 \phi_2 + \lambda_2 \int_{\eta_0}^{\eta_1} (m_1 - m_2)\phi_1\phi_2 \\
&= (\lambda_1 - \lambda_2) \int_{\eta_0}^{\eta_1} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\eta_0}^{\eta_1} (m_1 - m_2)\phi_1\phi_2
\end{aligned}$$

and this proves that in both the cases $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$, we have $\lambda_1 < \lambda_2$.

iv. $\xi_1 = z^1_{k_1+1} < \eta_1 = z^2_{k_1+1}$: In this case Lemma 6 guarantees existence of two integers m, n having the same parity such that

$$\xi_m = z^1_{k_1+m} < \eta_n = z^2_{k_1+n} < \eta_{n+1} = z^2_{k_1+n+1} \leq \xi_{m+1} = z^1_{k_1+m+1}.$$

As above, we have from Lemma 7

$$\begin{aligned} 0 &> \int_{\eta_n}^{\eta_{n+1}} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = \int_{\eta_n}^{\eta_{n+1}} (\lambda_2 m_2 - \lambda_1 m_1)\phi_1\phi_2 \\ &= (\lambda_2 - \lambda_1) \int_{\eta_n}^{\eta_{n+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\eta_n}^{\eta_{n+1}} (m_2 - m_1)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{\eta_n}^{\eta_{n+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\eta_n}^{\eta_{n+1}} (m_1 - m_2)\phi_1\phi_2 \end{aligned}$$

and this proves that in both the cases $\lambda_1 \le 0$ and $\lambda_2 \le 0$, we have $\lambda_1 < \lambda_2$. The proof is complete.

Lemma 10. *Let (ϕ_n) be a sequence in S_k^ν converging in W^1 to some $\phi \in S_l^\kappa$, then $l \le k$ and $\kappa = \nu$.*

Proof. On the contrary suppose that $\phi \in S_l^\nu$ for some $l > k$ and let $(z_j)_{j=0}^{j=l}$ be the sequence of zeros of ϕ. Let $\delta > 0$ be small enough that there exists an integer $n_* \ge 1$ such that $\phi\phi_{n_*} > 0$ in the intervals $[\delta, z_1 - \delta]$ and $[z_j + \delta, z_{j+1} - \delta]$ for $j = 1, ..., l-2$.

Also, for each integer $j \in \{1, ..., l-1\}$ there exists $n_j \ge n_*$ such that the function ϕ_{n_j} has exactly one zero in $[z_j + \delta, z_{j+1} - \delta]$. Otherwise if there is a subsequence (ϕ_{n_i}) such that for all $i \ge 1$, ϕ_{n_i} has at least two zeros, then we can choose $x_{n_i}^1$ and $x_{n_i}^2$ in $[z_j + \delta, z_{j+1} - \delta]$ such that

$$\phi'_{n_i}\left(x_{n_i}^1\right) \le 0 \le \phi'_{n_i}\left(x_{n_i}^2\right).$$

Let

$$\begin{array}{ll} x_{\inf}^1 = \liminf x_{n_i}^1 & x_{\sup}^1 = \limsup x_{n_i}^1 \\ x_{\inf}^2 = \liminf x_{n_i}^2 & x_{\sup}^2 = \liminf x_{n_i}^2. \end{array}$$

Hence, we have since $\phi = \lim \phi_n$ in W^1,

$$\phi\left(x_{\inf}^1\right) = \phi\left(x_{\inf}^2\right) = \phi\left(x_{\sup}^1\right) = \phi\left(x_{\sup}^2\right) = 0$$

leading to $\lim x_{n_i}^1 = \lim x_{n_i}^2 = z_j$ then to

$$\phi'(z_j) = \lim \phi'_{n_l}\left(x_{n_i}^1\right) = \lim \phi'_{n_l}\left(x_{n_i}^2\right) = 0,$$

contradicting the simplicity of z_j.

Now, we claim that there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $\phi\phi_n > 0$ in $(0, \delta)$. Indeed, if there a subsequence (ϕ_{n_i}) such that for all $i \ge 1$, ϕ_{n_i} has at least a zero x_{n_i} with $\nu\phi'_{n_i}(x_{n_i}) < 0$, then we obtain as above for $x_- = \liminf x_{n_i}$ and $x_+ = \limsup x_{n_i}$ $\phi(x_-) = \phi(x_+) = 0$ and $x_- = x_+ = 0$. Therefore, we have

$$0 < \nu\phi'(0) = \lim \nu\phi'_{n_i}(x_{n_i}) \le 0,$$

contradicting the simplicity of the zero $z_0 = 0$. The proof of the lemma is complete.

Proposition 7. *Let $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$ and let (m_n) be a sequence of functions in Γ^+ such that $\lim m_n = m$ in E. If for some integer $k \geq 1$ and $\nu = +$ or $-$, $\lambda_k^\nu (q, m_n, \alpha, \beta)$ exits for all $n \geq 1$ with $\lim_{n \to +\infty} \lambda_k^\nu (q, m_n, \alpha, \beta) = \lambda \in \mathbb{R}$, then $\lambda = \lambda_k^\nu (q, m, \alpha, \beta)$.*

Proof. Let for all integers $n \geq 1$ $\phi_n \in S_k^\nu$ be the normalized eigenfunction associated with $\lambda_{k,n}^\nu = \lambda_k^\nu (q, m_n, \alpha, \beta) = \lambda_k^\nu (q^+, m_n, \alpha + q^-, \beta + q^-)$. Therefore, we have for all integers $n \geq 1$

$$\phi_n (t) = \lambda_{k,n}^\nu L_{q^+, m_n} \phi_n (t) + L^+_{q^+, \alpha + q^-} \phi_n (t) - L^-_{q^+, \beta + q^-} \phi_n (t).$$

Since all the operators in the above equation are compact and (ϕ_n) is bounded, up to a subsequence, (ϕ_n) converges to some ϕ with $\|\phi\| = 1$ and

$$\phi (t) = \lambda_k^\nu L_{q^+, m} \phi (t) + L^+_{q^+, \alpha + q^-} \phi (t) - L^-_{q^+, \beta + q^-} \phi (t).$$

This proves that λ_k^ν is a half-eigenvalue of the bvp (12).

We have from Lemma 10 that $\phi \in S_l^\nu$ with $l \leq k$. Let us prove that $l = k$. We claim that there is an integer $n_+ \geq 1$ such that $\phi\phi_{n_+} > 0$ in $(z_{l-1} + \delta, 1)$. Indeed, if there a subsequence (ϕ_{n_i}) such that for all $i \geq 1$, ϕ_{n_i} has at a zero $x_{n_i} \in (z_{l-1} + \delta, 1)$ and ϕ_{n_i} does not vanish in $(x_{n_i}, 1)$ then

$$\lambda_{k,n}^\nu = \mu_1(q + Am - \omega, m + \varepsilon, x_{n_i}) \geq \mu_1(q + Am - \omega, m + \varepsilon, x_{n_i})$$

where

$$\omega = \begin{cases} \alpha \text{ if } \phi_{n_i} > 0 \text{ in } (x_{n_i}, 1) \\ \beta \text{ if } \phi_{n_i} < 0 \text{ in } (x_{n_i}, 1) \end{cases}$$

and $\omega = \max(|\alpha|, |\beta|)$.

Passing to the limit, we obtain the contradiction

$$+\infty > \lambda_k^\nu \geq \lim \mu_1(q + Am - \omega, m + \varepsilon, x_{n_i}) = +\infty.$$

From all the above, we obtain for all $n \geq \max\{n^*, n_+, n_1, ...n_{l-1}\}$ ϕ_{n_i} belongs to S_l^ν, and $l = k$. The proof is complete.

Lemma 11. *([2])Let $q \in Q$, $m \in \Gamma^{++}$ and $\alpha, \beta \in E$. For all $\theta \in (0, 1)$ the bvp*

$$\begin{cases} \mathcal{L}_q u = \lambda m u + \alpha u^+ - \beta u^-, \ \text{in } (0, \theta), \\ u(0) = u(\theta) = 0, \end{cases}$$

admits two increasing sequence of simple half eigenvalues $\left(\lambda_k^+(q, m, \alpha, \beta, \theta)\right)_{k \geq 1}$ and $\left(\lambda_k^-(q, m, \alpha, \beta, \theta)\right)_{k \geq 1}$ such that for all integers $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-line of solutions lies on $\{\lambda_k^\nu(q, m, \alpha, \beta, \theta)\} \times S_k^\nu,$. Moreover, for all integers $k \geq 1$ and $\nu = +$ or $-$, the function $\theta \to \lambda_k^\nu (\theta) := \lambda_k^\nu (\theta, q, m, \alpha, \beta, \theta)$ is continuous decreasing and $\lim_{\theta \to 0} \lambda_k^\nu(\theta) = +\infty$.

Lemma 12. *For all functions $q \in Q$, $m \in \Gamma^{++}$ and $\alpha, \beta \in E$, the bvp (12) admits two increasing sequences of half-eigenvalues $\left(\lambda_k^+(q, m, \alpha, \beta)\right)_{k \geq 1}$ and $\left(\lambda_k^-(q, m, \alpha, \beta)\right)_{k \geq 1}$ such that for all integers $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-line of solutions lies on $\{\mu_k^\nu(m, \alpha, \beta)\} \times S_k^\nu$.*

Proof. Let $q \in, Q$, $m \in \Gamma^{++}$ and $\alpha, \beta \in E$. Clearly for $k = 1$, we have $\lambda_k^+(q, m, \alpha, \beta) = \mu_1^+(q - \alpha, m, 0)$ and $\lambda_k^-(q, m, \alpha, \beta) = \mu_1^+(q - \beta, m, 0)$ that existence is guaranteed by Theorem 9. Fix $k \geq 2$, $\nu = +$ or $-$ and set $\omega_1 = \alpha$ and $\omega_2 = \beta$. Let for $\theta \in (0, 1)$ $\lambda_{k-1}^\nu(\theta) = \lambda_{k-1}^\nu(q, , m, \alpha, \beta, \theta)$ and for $i = 1, 2$ $\mu_i(\theta) = \mu_l^\nu(q - \omega_i, m, \theta)$ given respectively by Lemma 11 and Theorem 9. Because that the function $\lambda_{k-1}^\nu(\cdot)$ is decreasing, the functions $\mu_i(\cdot)$ are increasing and

$$\lim_{\theta \to 0} \lambda_{k-1}^\nu(\theta) = \lim_{\theta \to 1} \mu_i(\theta) = +\infty,$$

the equation $\lambda_{k-1}^\nu(\theta) = \mu_i(\theta)$ admits a unique solution $\theta_{k,i} \in (0, 1)$.

Let for $\theta \in (0, 1)$, ψ_θ be the eigenfunction associated with $\lambda_{k-1}^\nu(\theta)$ and for $i = 1, 2$ $\phi_{\theta,i}$ be the eigenfunction associated with $\mu_i(\theta)$. We distinguish the following cases:

a) $\psi_\theta'(\theta) > 0$ for all $\theta \in (0, 1)$. In this case $\lambda_k^\nu = \lambda_{k-1}^\nu(\theta_{k,1}) = \mu_i(\theta_{k,1})$ is the half-eigenvalue having as an eigenfunction the function $\psi_k \in S_k^\nu$ defined by

$$\psi_k(t) = \begin{cases} \psi_{\theta_{k,1}}(t) & \text{for } t \in [0, \theta_{k,1}], \\ \phi_{\theta_{k,1},1}(t) \left(\psi_{\theta_{k,1}}(\theta_{k,1}) / \phi'_{\theta_{k,1},1}(\theta_{k,1})\right) & \text{for } t \in [\theta_{k,1}, 1]. \end{cases}$$

b) $\psi_\theta'(\theta) < 0$ for all $\theta \in (0, 1)$. In this case $\lambda_k^\nu = \lambda_{k-1}^\nu(\theta_{k,2}) = \mu_i(\theta_{k,2})$ is the half-eigenvalue having as an eigenfunction the function $\psi_k \in S_k^\nu$ defined by

$$\psi_k(t) = \begin{cases} \psi_{\theta_{k,2}}(t) & \text{for } t \in [0, \theta_{k,2}], \\ \phi_{\theta_{k,2}}(t) \left(\psi_{\theta_{k,2}}(\theta_{k,2}) / \phi'_{\theta_{k,2}}(\theta_{k,2})\right) & \text{for } t \in [\theta_{k,2}, 1]. \end{cases}$$

This ends the proof.

Lemma 13. *Let $q \in Q$, $m \in \Gamma^{++}$ and set for all $k \geq 1$*

$$\mu_k(q, m) = \lambda_k^+(q, m, 0, 0) = \lambda_k^-(q, m, 0, 0).$$

Then for any interval $[\gamma, \delta] \subset (0, 1)$, $\mu_k(q, m) < \mu_k(q, m, [\gamma, \delta])$ where $(\mu_k(q, m, [\gamma, \delta]))$ is the sequence of eigenvalues of the bvp

$$\begin{cases} \mathcal{L}_q u = \mu m u, \ \text{in } (\gamma, \delta), \\ u(\gamma) = u(\delta) = 0. \end{cases}$$

Proof. Fix $k \geq 1$ and set $\mu_1 = \mu_k(q,m)$ and $\mu_2 = \mu_k(q,m,[\gamma,\delta])$. Let for $i = 1,2,\ \phi_i$ be an eigenfunction associated with μ_i, having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k}$, and without loss of generality, suppose that $\phi_1\phi_2 > 0$ in a right neighborhood of γ. We distinguish two cases.

i) $\phi_2 > 0$ in (γ,δ) (i.e. $k = 1$): In this case we obtain by Lemma 7

$$0 < \int_\gamma^\delta \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = (\mu_2 - \mu_1) \int_\gamma^\delta m\phi_1\phi_2$$

leading to $\mu_2 > \mu_1$.

ii) $\phi_2(t_0) = 0$ for some $t_0 \in (\gamma,\delta)$: In this case consider the family $(\xi_j)_{j=0}^{j=k_0}$ defined by $\xi_0 = \gamma$, $\xi_{k_0} = \delta$ and $\phi_1(\xi_j) = 0$ for $j \in \{1,...,k_0-1\}$ and note that $k_0 \leq k$. Thus, we have from Lemma 6 that there exist two integers m,n having the same parity, such that $\xi_m < z_n^2 < z_{n+1}^2 \leq \xi_{m+1}$. Therefore, we have $\phi_1,\phi_2 > 0$ in $\left(z_n^2, z_{n+1}^2\right)$ and we obtain by Lemma 7

$$\begin{aligned} 0 &< \int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 \\ &= (\mu_2 - \mu_1) \int_{z_n^2}^{z_{n+1}^2} m\phi_1\phi_2 \end{aligned}$$

leading to $\mu_2 > \mu_1$.

This ends the proof.

Theorem 14. *For all $q \in Q$, $m \in \Gamma^+$ and $\alpha,\beta \in E$ the bvp (12) admits two increasing sequences of simple half-eigenvalues $\left(\lambda_k^+(q,m,\alpha,\beta)\right)_{k\geq 1}$ and $\left(\lambda_k^-(q,m,\alpha,\beta)\right)_{k\geq 1}$ such that for all integers $k \geq 1$, the corresponding half-line of solutions lies on $\{\mu_k^\nu(m,\alpha,\beta)\} \times S_k^\nu$, $\nu = +,-$ with $\lim_{k\to\infty} \mu_k^\nu(q,m,\alpha,\beta) = +\infty$, aside from these solutions and the trivial one, there are no other solutions of (12). Furthermore, for $k \geq 1$ and $\nu = +$ or $-$, the half-eigenvalue $\lambda_k^\nu(\cdot,\cdot,\cdot,\cdot)$ has the following properties:*

1. *Let $q \in Q$, $m \in \Gamma^+$ and $\alpha_1,\alpha_2,\beta \in E$. If $\alpha_1 \leq \alpha_2$ in $(0,1)$, then $\lambda_k^\nu(q,m,\alpha_1,\beta) \geq \lambda_k^\nu(q,m,\alpha_2,\beta)$.*
2. *Let $q \in Q$, $m \in \Gamma^+$ and $\alpha,\beta_1,\beta_2 \in E$. If $\beta_1 \leq \beta_2$ in $(0,1)$, then $\lambda_k^\nu(q,m,\alpha,\beta_1) \geq \lambda_k^\nu(q,m,\alpha,\beta_2)$.*
3. *Let $q_1,q_2 \in Q$, $m \in \Gamma^+$ and $\alpha,\beta \in E$. If $q_1 \leq q_2$ in $(0,1)$, then $\lambda_k^\nu(q_1,m,\alpha,\beta) \leq \lambda_k^\nu(q_2,m,\alpha,\beta)$.*

4. *Let $m_1, m_2 \in \Gamma^+$, $\alpha, \beta \in E$, with $m_1 \leq m_2$ in $(0,1)$ and $m_1 < m_2$ in a subset of positive measure. If $\lambda_k^\nu(m_1, \alpha, \beta) \geq 0$ or $\lambda_k^\nu(m_2, \alpha, \beta) \geq 0$, then $\lambda_k^\nu(q, m_1, \alpha, \beta) > \lambda_k^\nu(q, m_2, \alpha, \beta)$ and if $\lambda_k^\nu(q, m_1, \alpha, \beta) \leq 0$ or $\lambda_k^\nu(q, m_2, \alpha, \beta) \leq 0$, then $\lambda_k^\nu(q, m_1, \alpha, \beta) < \lambda_k^\nu(q, m_2, \alpha, \beta)$.*

5. *If $m \in \Gamma^+$ and $(m_n) \subset \Gamma^+$ are such that $\lim m_n = m$ in E, then $\lim_{n\to\infty} \lambda_k^\nu(q, m_n, \alpha, \beta) = \lambda_k^\nu(q, m, \alpha, \beta)$ for all $\alpha, \beta \in E$.*

Proof. Let $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$ and (ϵ_n) be a decreasing sequence of real numbers converging to 0 and let $A > 0$ be such that $\min(\mu_1(q-\alpha, m+\epsilon_1), \mu_1(q-\beta, m+\epsilon_1)) > -A$. Consider the BVP

$$\begin{cases} \mathcal{L}_{q+Am}u = \lambda m u + \alpha u^+ - \beta u^- \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{17}$$

and notice that λ is a half-eigenvalue of the (17) if and only if $(\lambda - A)$ is a half-eigenvalue of the bvp (12). Let for k and ν fixed, $\lambda_{k,n}^\nu = \lambda_k^\nu(q + Am, m + \epsilon_n, \alpha, \beta)$ and let $[\gamma, \delta] \subset (\xi, \eta)$ be such that $m > 0$ a.e. in (γ, δ).

First, because of

$$\begin{aligned} \lambda_{k,1}^\nu = \lambda_k^\nu(q + Am, m + \epsilon_1, \alpha, \beta) &\geq \lambda_1^\nu(q, m + \epsilon_1, \alpha, \beta) + A \\ &\geq \min(\mu_1(q-\alpha, m+\epsilon_1), \mu_1(q-\beta, m+\epsilon_1)) + A > 0, \end{aligned}$$

we have by Proposition 6 that for all $n \in \mathbb{N}$, $\lambda_{k,n+1}^\nu \geq \lambda_{k,n}^\nu \geq \lambda_{k,1}^\nu > 0$.

Set $\widetilde{q} = q + Am + (|\alpha| + |\beta|)$, Proposition 5, Lemma 13 and Proposition 6 lead to

$$0 < \lambda_{k,n}^\nu \leq \mu_k(\widetilde{q}, m + \epsilon_n) \leq \mu_k(\widetilde{q}, m + \epsilon_n, [\gamma, \delta]) \leq \mu_k(\widetilde{q}, m, [\gamma, \delta])$$

proving that $\lim \lambda_{k,n}^\nu = \lambda_k^\nu \in \mathbb{R}$. Thus, we conclude from Proposition 11 that $\lambda_k^\nu = \lambda_k^\nu(q + Am, m, \alpha, \beta)$.

Now, we need to prove that $\lim_{k\to\infty} \lambda_k^\nu(q + Am, m, \alpha, \beta) = +\infty$. To this aim set $\omega = |\alpha| + |\beta|$ and let $B > 0$ be such that $\overline{q} = q + Am - \omega + B(m + \epsilon_1) > 0$ in $[0, 1)$. We have then from Propositions 5 and 6:

$$\begin{aligned} \lambda_k^\nu(q + Am, m, \alpha, \beta) &\geq \lambda_k^\nu(q + Am, m + \epsilon_1, \alpha, \beta) \\ &\geq \lambda_k^\nu(q + Am, m + \epsilon_1, \omega, \omega) \\ &\geq \lambda_k^\nu(q + Am, m + \epsilon_1, \omega, \omega) \\ &= \mu_k(q + Am - \omega, m + \epsilon_1) \\ &= \mu_k(\overline{q}, m + \epsilon_1) - B. \end{aligned}$$

Because that $(\mu_k(\overline{q}, m+\epsilon_1))$ is the sequence of characteristic-values of the positive compact operator $L_{\overline{q},m+\epsilon_1} : W \to W$ defined for $u \in W$ by

$$L_{\overline{q},m+\epsilon_1}u(t) = \int_0^1 G_{\overline{q}}(t,s)\,(m\,(s)+\epsilon_1)\,u(s)ds,$$

we have that $\lim_k \mu_k(\overline{q}, m+\epsilon_1) = +\infty$, proving that $\lim_k \lambda_k^\nu(q+Am, m, \alpha, \beta) = +\infty$.

At the end, Assertions 1, 2, 3, 4 and 5 follow from Propositions 1-7.

For the particular case of the bvp (12) where $\alpha = \beta = 0$, namely for the bvp

$$\begin{cases} \mathcal{L}_q u = \mu m u, & \text{in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{18}$$

we obtain from Theorem the following corollary.

Corollary 15. *For all pairs (q,m) in $Q \times \Gamma^+$, the set of eigenvalues of the bvp (18) consists in an unbounded increasing sequence of simple eigenvalues $(\mu_k(q,m))_{k\geq 1}$ such that eigenfunctions associated with $\mu_k(q,m)$ belong to S_k. Moreover, the mapping $\mu_k(\cdot,\cdot)$ has the following properties:*

1. *Let $q \in Q$, $m_1, m_2 \in \Gamma^+$ with $m_1 \leq m_2$ in $(0,1)$ and $m_1 < m_2$ in a subset of positive measure. If $\mu_k(q,m_1) \geq 0$ or $\mu_k(q,m_2) \geq 0$, then $\mu_k(q,m_1) > \mu_k(q,m_2)$ and if $\mu_k(q,m_1) \leq 0$ or $\mu_k(q,m_2) \leq 0$, then $\mu_k(q,m_1) < \mu_k(q,m_2)$.*

2. *If $m \in \Gamma^+$ and $(m_n) \subset \Gamma^+$ are such that $\lim m_n = m$ in E, then $\lim_{n\to\infty} \mu_k(q,m_n) = \mu_k(q,m)$.*

3. *Let $q_1, q_2 \in Q$ and $m \in \Gamma^+$. If $q_1 \leq q_2$, then $\mu_k(q_1,m) \leq \mu_k(q_2,m)$ for all $k \geq 1$.*

The following proposition is a consequence of Assertion 2 in Corollary 15 and it will be used in the following section.

Proposition 8. *Let $q \in Q$ and $m \in \Gamma^+$ be such that $\mu_k(q,m) = 1$ for some integer $k \geq 1$. Then there exists $\varepsilon_0 > 0$ such that for all $p \in \Gamma^+$ with $\|p-m\| \leq \varepsilon_0$, $\mu_l(q,p) = 1$ implies $l = k$.*

Proof. Let $\epsilon_0 > 0$ be such that $\epsilon_0 < \min(\mu_{k+1}(q,m) - \mu_k(q,m), \mu_k(q,m) - \mu_{k-1}(q,m))$. Because of the continuity of the functions $\mu_{k-1}(q,m)$, $\mu_{k+1}(q,m)$, there exists $\varepsilon_0 > 0$ such that for all $p \in \Gamma^+$, $\|p-m\| \leq \varepsilon_0$ implies

$$\mu_{k-1}(q,m) - \epsilon_0 \leq \mu_{k-1}(q,p) \leq \mu_{k-1}(q,m) + \epsilon_0 \tag{19}$$

and

$$\mu_{k+1}(q,m)-\epsilon_0 \leq \mu_{k+1}(q,p) \leq \mu_{k+1}(q,m)+\epsilon_0. \tag{20}$$

Let $p \in \Gamma^+$ with $\|p-m\| \leq \varepsilon_0$ and suppose that $\mu_l(q,p)=1$ for some integer $l \geq 1$. If $l<k$, we have then from (19) the contradiction

$$1=\mu_l(q,p) \leq \mu_{k-1}(q,p) \leq \mu_{k-1}(q,m)+\epsilon_0 < \mu_k(q,m),$$

and if $l>k$, we have then from (20) the contradiction

$$1=\mu_l(q,p) \geq \mu_{k+1}(q,p) \geq \mu_{k+1}(q,m)-\epsilon_0 > \mu_k(q,m)=1.$$

This shows that $l=k$ and the lemma is proved.

4. Nodal solutions to the nonlinear bvp

4.1. Main results

In all this section, ρ is a positive real parameter, q is a function in Q, m, α and β are functions in E and $f:[0,1]\times(\mathbb{R}\setminus\{0\})\to\mathbb{R}$ is continuous function. Main results of this section concern existence of nodal solutions to the bvp

$$\begin{cases} \mathcal{L}_q u=\rho u f(t,u) \text{ in } (0,1) \\ u(0)=\lim_{t\to 1} u(t)=0, \end{cases} \tag{21}$$

where the function f is assumed to satisfy one of the following Hypotheses (22), (23) and (24).

$$\begin{cases} \lim_{u\to 0} f(t,u)=m(t), \\ \lim_{u\to -\infty} f(t,u)=\beta(t) \text{ and} \\ \lim_{u\to +\infty} f(t,u)=\alpha(t) \text{ in } E. \end{cases} \tag{22}$$

$$\begin{cases} \lim_{u\to 0} f(t,u)=m(t) \text{ in } E \text{ and} \\ \lim_{|u|\to +\infty} \left(\inf_{t\in[0,1]} f(t,u)\right)=+\infty. \end{cases} \tag{23}$$

$$\begin{cases} \lim_{u\to 0} u f(t,u)=0, \\ \lim_{u\to 0} \left(\inf_{t\in[0,1]} f(t,u)\right)=+\infty, \\ \lim_{u\to -\infty} f(t,u)=\beta(t) \text{ and} \\ \lim_{u\to +\infty} f(t,u)=\alpha(t) \text{ in } E. \end{cases} \tag{24}$$

Remark 1. *Notice that if the nonlinearity f satisfies one of the Hypotheses (22), (24) and (23), then there is $\omega_0 \in \Gamma^{++}$ such that $f(t,u)+\omega_0(t)>0$ for all $t\in[0,1]$ and $u\in\mathbb{R}$.*

The statements of the main results of this section and their proofs require introducing some notations. In all this section we let:

$$\widetilde{q} = q^+ + \rho\left(m^- + 2\omega_0\right), \quad \widetilde{m} = \rho\left(m^+ + 2\omega_0\right) + q^-, \quad \widetilde{f}(t,u) = \rho\left(f(t,u) - m\right),$$

$$\widetilde{\alpha} = \rho\left(\alpha - m\right), \quad \widetilde{\beta} = \rho\left(\beta - m\right), \quad \varphi = \inf\left(\alpha, \beta\right) \quad \text{and} \quad \psi = \sup\left(\alpha^+, \beta^+\right),$$

where ω_0 is as in Remark 1.

Since in all this section the weight q is fixed in Q, we let for all $\chi \in \Gamma^+$ and all $k \geq 1$, $\mu_k\left(\chi\right) = \mu_k\left(q, \chi\right)$. In particular we let for all $k \geq 1$ and $\nu = +$ or $-$,

$$\widetilde{\mu}_k = \mu_k\left(\widetilde{q}, \widetilde{m}\right), \quad \widetilde{\lambda}_k^\nu = \lambda_k^\nu\left(\widetilde{q}, \widetilde{m}, \widetilde{\alpha}, \widetilde{\beta}\right).$$

The operators $T_0,\ T_\infty : W \to W$ are defined as follows

$$\begin{aligned} T_0 u(t) &= \textstyle\int_0^1 G_{\widetilde{q}}(t,s) u(s) \widetilde{f}(s, u(s)) ds, \\ T_\infty u(t) &= T_0 u(t) - L^+_{\widetilde{q},\widetilde{\alpha}} u(t) + L^-_{\widetilde{q},\widetilde{\beta}} u(t) \\ &= \textstyle\int_0^1 G_{\widetilde{q}}(t,s) u(s) f^*(s, u(s)) ds, \end{aligned}$$

where $f^*(s,u) = u\widetilde{f}(s,u) - \widetilde{\alpha} u^+ + \widetilde{\beta} u^-$. We have from Lemma 5 that $T_0,\ T_\infty$ are completely continuous.

The following Theorems 16, 18 and 17 are the main results of this section. They provide respectively existence and multiplicity results for the cases where the nonlinearity f is asymptoticaly linear, sublinear and superlinear.

Theorem 16. *Assume that Hypothesis (22) holds true.*

1. *Let i, j be two integers such that $i \geq j \geq 1$. The bvp (21) admits in each of $S_j^+, \ldots, S_i^+, S_j^-, , \ldots, S_i^-$ a solution if one of the following Hypothesis (25), (26), (27) and (28) holds true.*

$$\varphi, m^+ \in \Gamma^+ \text{ and } \mu_i(\varphi) < \rho < \mu_j(m^+), \tag{25}$$

$$\begin{cases} \varphi \in \Gamma^+, m^+ = 0,\ \mu_i(\varphi) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+ \end{cases} \tag{26}$$

$$\psi, m \in \Gamma^+ \text{ and } \mu_i < \rho < \mu_j(\psi), \tag{27}$$

$$\begin{cases} m \in \Gamma^+, \psi = 0,\ \mu_i(m) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{28}$$

2. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (21) admits in each of $S^+_{2j}, \ldots, S^+_i, S^-_{2j-1}, \ldots, S^-_i$ a solution if one of the following Hypothesis (29) and (30) holds true.*

$$m, \beta^+ \in \Gamma^+ \text{ and } \mu_i(m) < \rho < \mu_j(\beta^+), \tag{29}$$

$$\begin{cases} m \in \Gamma^+, \beta^+ = 0, \ \mu_i(m) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{30}$$

3. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (21) admits in each of $S^+_{2j-1}, \ldots, S^+_i, S^-_{2j}, \ldots, S^-_i$ a solution if one of the following Hypothesis (31) and (32) holds true.*

$$m, \alpha^+ \in \Gamma^+ \text{ and } \mu_i(m) < \rho < \mu_j(\alpha^+), \tag{31}$$

$$\begin{cases} m \in \Gamma^+, \alpha^+ = 0, \ \mu_i(m) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{32}$$

Theorem 17. *Assume that Hypothesis (23) holds true and let $j \geq 1$. The bvp (21) admits for all $k \geq j$ a solution in S^+_k and in S^-_k if one of the following Hypotheses (33) and (34) holds true.*

$$m^+ \in \Gamma^+ \text{ and } \mu_j(m^+) > \rho, \tag{33}$$

$$m^+ = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{34}$$

Theorem 18. *Assume that Hypothesis (24) holds true, $q \in Q_\#$ and let $j \geq 1$.*

1. *The bvp (21) admits for all $k \geq j$ a solution in S^+_k and in S^-_k if one of the following Hypotheses (35) and (36) holds true.*

$$\psi \in \Gamma^+ \text{ and } \mu_j(\psi) > \rho, \tag{35}$$

$$\psi = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{36}$$

2. *The bvp (21) admits a solution in S^+_k for all $k \geq 2j$ and a solution in S^-_k for all $k \geq 2j-1$ if one of the following Hypotheses (37) and (38) holds true.*

$$\beta^+ \in \Gamma^+ \text{ and } \mu_j(\beta^+) > \rho, \tag{37}$$

$$\beta^+ = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{38}$$

3. *The bvp (21) admits a solution in S^+_k for all $k \geq 2j-1$ and a solution in S^-_k for all $k \geq 2j$ if one of the following Hypotheses (39) and (40) holds true.*

$$\alpha^+ \in \Gamma^+ \text{ and } \mu_j(\alpha^+) > \rho, \tag{39}$$

$$\alpha^+ = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{40}$$

4.2. Related Lemmas

In this subsection we prove some intermediate results.

Lemma 19.

1. *If $m \in \Gamma^+$ and $\mu_l(m) < \rho$ for some $l \geq 1$, then $\widetilde{\mu}_k < 1$ for all $k \leq l$.*
2. *If $m^+ \in \Gamma^+$ and $\mu_l(m^+) > \rho$ for some $l \geq 1$, then $\widetilde{\mu}_k > 1$ for all $k \geq l$.*
3. *If $m = -m^-$ and $\mu_l(\chi_0) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\mu}_k > 1$ for all $k \geq l$.*

Proof. If $m^+ \in \Gamma^+$, we have then

$$\begin{aligned} \widetilde{\mu}_k &= \mu_k\left(q^+ + \rho\left(m^- + 2\omega_0\right), \rho\left(m^+ + 2\rho\omega_0\right) + q^-\right) \\ &= \mu_k\left(q^+ + 2\rho\omega_0 + \rho m^- - \widetilde{\mu}_k\left(2\rho\omega_0 + q^-\right), \rho m^+\right) \\ &= \mu_k\left(q + (1-\widetilde{\mu}_l)\left(2\rho\omega_0 + q^-\right), \rho m^+\right) \\ &= \left(\mu_k\left(q + (1-\widetilde{\mu}_l)\left(2\rho\omega_0 + q^-\right) + \rho m^-, m^+\right)/\rho\right). \end{aligned} \tag{41}$$

Suppose that $m = m^+ \in \Gamma^+$, $\mu_l(m) < \rho$ for some $l \geq 1$ and $\widetilde{\mu}_k \geq 1$ for some $k \leq l$. We obtain from (41) and Assertion 3 in Proposition 5 the contradiction

$$1 \leq \widetilde{\mu}_k = \left(\mu_k\left(q + (1-\widetilde{\mu}_k)\left(2\rho\omega_0 + q^-\right), m\right)/\rho\right) \leq \left(\mu_k(m)/\rho\right) \leq (\mu_l(m)/\rho) < 1.$$

This proves Assertion 1.

Similarly, suppose that $m^+ \in \Gamma^+$, $\mu_l(m) < \rho$ for some $l \geq 1$ and $\widetilde{\mu}_k \leq 1$ for some $k \geq l$. We obtain from (41) and Assertion 3 in Proposition 5 the contradiction

$$1 \geq \widetilde{\mu}_k = \left(\mu_k\left(q + (1-\widetilde{\mu}_k)\left(2\rho\omega_0 + q^-\right) + \rho m^-, m\right)/\rho\right) \leq \left(\mu_k(m)/\rho\right) \geq (\mu_l(m)/\rho) > 1.$$

This proves Assertion 2.

Suppose that $m = -m^-$ (i.e. $m^+ = 0$), $\mu_l(q, \chi_0) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$ and $\widetilde{\mu}_k \leq 1$ for some $k \geq l$. We read from

$$\begin{aligned} \widetilde{\mu}_k &= \mu_k\left(q^+ + \rho\left(m^- + 2\omega_0\right), \rho\left(m^+ + 2\omega_0\right) + q^-\right) \\ &= \mu_k\left(q^+ + \rho\left(m + 2\omega_0\right), 2\rho\omega_0 + q^-\right) \end{aligned}$$

that

$$\mu_k\left(q + (1-\widetilde{\mu}_k)\left(2\rho\omega_0 + q^-\right), \chi\right) = 0 \text{ for all } \chi \in \Gamma^+.$$

Therefore, Assertion 3 in Proposition 5 leads to the contradiction

$$0 = \mu_k\left(q + (1-\widetilde{\mu}_k)\left(2\rho\omega_0 + q^-\right), \chi_0\right) \geq \mu_k(\chi_0) \geq \mu_l(\chi_0) > 0.$$

This Proves Assertion 3 and ends the proof.

Lemma 20. *For all integers $l \geq 1$ and $\nu = +$ or $-$:*

1. *If $\varphi \in \Gamma^+$ and $\mu_l(\varphi) < \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_k^\nu < 1$ for all $k \leq l$.*
2. *If $\psi \in \Gamma^+$ and $\mu_l(\psi) > \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_l^\nu > 1$ for all $k \geq l$.*
3. *If $\psi = 0$ and $\mu_l(\chi_0) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\mu}_k > 1$ for all $k \geq l$.*

Proof. To prove Assertion 1, we have to show that $\widetilde{\lambda}_l^\nu > 1$. By the way of contradiction, suppose that $\mu_l(\varphi) < \rho$ and $\widetilde{\lambda}_l^\nu \geq 1$ and let u,$v \in S_l^\nu$ be the eigenfunctions associated respectively with $\mu_l(\rho\varphi) = (\mu_l(\varphi)/\rho)$ and $\widetilde{\lambda}_l^\nu$. Notice that

$$\begin{cases} \mathcal{L}_q u = \mu_l(\rho\varphi)\rho\varphi u \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q v = \left(\widetilde{\lambda}_l^+ - 1\right)(\rho m + 2\rho\omega_0 + q^-)v + \rho\alpha v^+ - \rho\beta v^- \text{ in } (0,1), \\ v(0) = \lim_{t\to 1} v(t) = 0, \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$ and $(y_j)_{j=0}^{j=l}$ be respectively the sequences of zeros of u and v. We distinguish then the following two cases:

i) $x_1 \leq y_1$: in this case we have the contradiction:

$$\begin{aligned} 0 &\leq \int_{x_0}^{x_1} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\leq \int_{x_0}^{x_1} \mu_l(\rho\varphi)\rho\varphi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_0}^{x_1} \left(\mu_l(\rho\varphi)\varphi - \alpha\right)\rho u^+ v^+ + \left(\mu_l(\rho\varphi)\varphi - \beta\right)\rho u^- v^- < 0. \end{aligned}$$

ii) $y_1 < x_1$: in this case Lemma 6 guarantees existence of two integers m, n having the same parity such that $y_m < x_n < x_{n+1} \leq y_{m+1}$ and Lemma 7 leads to the contradiction:

$$\begin{aligned} 0 &< \int_{x_n}^{x_{n+1}} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\leq \int_{x_n}^{x_{n+1}} \mu_l(\rho\varphi)\rho\varphi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_n}^{x_{n+1}} \left(\mu_l(\rho\varphi)\varphi - \alpha\right)\rho u^+ v^+ + \left(\mu_l(\rho\varphi)\varphi - \beta\right)\rho u^- v^- < 0. \end{aligned}$$

We prove Assertion 2 by the same way. Suppose that $\mu_l(\psi) > \rho$ and $\widetilde{\lambda}_l^\nu \leq 1$ and let $u,v \in S_l^\nu$ be the eigenfunctions associated respectively with $\mu_l(\rho\psi) = \mu_l(\psi)/\rho$ and $\widetilde{\lambda}_l^\nu$. We have that

$$\begin{cases} \mathcal{L}_q u = \mu_l(\rho\psi)\rho\psi u \ \text{ in } (0,1), \\ u\,(0) = \lim_{t\to 1} u(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q v = \left(\widetilde{\lambda}_l^\nu - 1\right)(\rho m + 2\rho\omega_0 + q^-)v + \rho\alpha v^+ - \rho\beta v^- \text{ in } (0,1), \\ v\,(0) = \lim_{t\to 1} v(t) = 0, \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$ and $(y_j)_{j=0}^{j=l}$ be respectively the sequences of zeros of u and v. We distinguish then the following two cases:

a) $x_1 \leq y_1$: in this case we have the contradiction:

$$\begin{aligned} 0 &\leq \int_{x_0}^{x_1} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\leq \int_{x_0}^{x_1} \mu_l(\rho\psi)\rho\psi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_0}^{x_1} \left(\mu_l(\rho\psi)\psi - \alpha\right)\rho u^+ v^+ + \left(\mu_l(\rho\varphi)\varphi - \beta\right)\rho u^- v^- < 0. \end{aligned}$$

b) $y_1 < x_1$: in this case Lemma 6 guarantees existence of two integers m, n having the same parity such that $y_m < x_n < x_{n+1} \leq y_{m+1}$ and Lemma 7 leads to the contradiction:

$$\begin{aligned} 0 &< \int_{x_n}^{x_{n+1}} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\leq \int_{x_n}^{x_{n+1}} \mu_l(\rho\psi)\rho\psi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_n}^{x_{n+1}} \left(\mu_l(\rho\psi)\psi - \alpha\right)\rho u^+ v^+ + \left(\mu_l(\rho\varphi)\varphi - \beta\right)\rho u^- v^- < 0. \end{aligned}$$

We have for all $k \geq 1$ and $\nu = +$ or $-$,

$$\begin{aligned} \widetilde{\lambda}_k^\nu &= \lambda_k^\nu\left(q^+ + \rho(m^- + 2\omega_0) + q^-, \rho(m^+ + 2\omega_0) + q^-, \rho\left(\alpha - m\right), \rho\left(\beta - m\right)\right) \\ &= \lambda_k^\nu\left(q^+ + \rho(m^+ + 2\omega_0) + q^-, \rho(m^+ + 2\omega_0) + q^-, \rho\alpha, \rho\beta\right). \end{aligned}$$

This can be read that for all $\chi \in \Gamma^+$

$$0 = \lambda_k^\nu\left(q + \left(1 - \widetilde{\lambda}_k^\nu\right)\left(\rho(m^+ + 2\omega_0) + q^-\right), \chi, \rho\alpha, \rho\beta\right).$$

Therefore, if $\psi = 0$, $\mu_l(\chi_0) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$ and $\widetilde{\lambda}_k^\nu \leq 1$ for some $k \geq l$, Proposition 5 leads to the contradiction

$$\begin{aligned} 0 &= \lambda_k^\nu \left(q + \left(1 - \widetilde{\lambda}_k^\nu\right)\left(\rho(m^+ + 2\omega_0) + q^-\right), \chi_0, \rho\alpha, \rho\beta\right) \\ &\geq \lambda_k^\nu (q, \chi_0, 0, 0) = \mu_k(\chi_0) \geq \mu_l(\chi_0) > 0. \end{aligned}$$

The proof is complete.

Lemma 21. *1. If $\alpha^+ \in \Gamma^+$ and $\mu_l(\alpha^+) > \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l - 1$and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l$.*

2. If $\alpha^+ = 0$ and $\mu_l(\chi_0) > \rho$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l - 1$and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l$.

3. If $\beta^+ \in \Gamma^+$ and $\mu_l(\beta^+) > \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l$ and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l - 1$.

4. If $\beta^+ = 0$ and $\mu_l(\chi_0) > \rho$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l - 1$and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l$.

Proof. To be brief, we present the proof of Assertions 1 and 2, the other assertions are obtained similarly. Suppose that $\alpha^+ \in \Gamma^+$ and $\mu_l(\alpha^+) > \rho$ and let ϕ, ϑ, ψ be respectively the eigenfunctions associated respectively with $\mu_l(\alpha)$, $\widetilde{\lambda}_{2l-1}^+$ and $\widetilde{\lambda}_{2l}^-$. Thus ϕ, ϑ, ψ satisfy

$$\begin{cases} \mathcal{L}_q\phi = \mu_l(\rho\alpha)\rho\alpha\phi \ \text{ in } (0,1), \\ \phi(0) = \lim_{t\to 1}\phi(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q\vartheta = \left(\widetilde{\lambda}_{2l-1}^+ - 1\right)(\rho m + 2\rho\omega_0 + q^-)\vartheta + \rho\alpha\vartheta^+ - \rho\beta\vartheta^-, \text{ in } (0,1), \\ \vartheta(0) = \lim_{t\to 1}\vartheta(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q\psi = \left(\widetilde{\lambda}_{2l}^- - 1\right)(\rho m + 2\rho\omega_0 + q^-)\psi \ + \rho\alpha\psi^+ - \rho\beta\psi^-, \text{ in } (0,1), \\ \psi(0) = \lim_{t\to 1}\psi(t) = 0. \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$, $(y_j)_{j=0}^{j=2l-1}$ and $(y_j)_{j=0}^{j=2l}$ be respectively the sequences of zeros of ϕ, ϑ and ψ. Thus, if $\widetilde{\lambda}_{2l-1}^+ \leq 1$ then

$$\left(\widetilde{\lambda}_{2l-1}^+ - 1\right)(\rho m + 2\rho\omega_0 + q^-) + \rho\alpha < \rho\alpha < \frac{\mu_l(\alpha)}{\rho}\rho\alpha = \mu_l(\rho\alpha)\rho\alpha$$

and we obtain from Lemma 8 that in each interval (y_{2j}, y_{2j+1}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$.

Similarly, if $\widetilde{\lambda}_{2l}^{-} \leq 1$ then

$$\left(\widetilde{\lambda}_{2l}^{-} - 1\right)\left(\rho m + 2\rho\omega_0 + q^{-}\right) + \rho\alpha < \mu_l\left(\rho\alpha\right)\rho\alpha$$

and we obtain from Lemma 8 that in each interval (y_{2j+1}, y_{2j+2}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$.

Suppose that $\alpha^{+} = 0$, $\mu_l(\chi_0) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^{+}$ and let ϕ, ϑ, ψ be respectively the eigenfunctions associated respectively with $\mu_l(\chi_0)$, $\widetilde{\lambda}_{2l-1}^{+}$ and $\widetilde{\lambda}_{2l}^{-}$. Thus ϕ, ϑ, ψ satisfy

$$\begin{cases} \mathcal{L}_q\phi = \mu_l(\chi_0)\chi_0\phi \ \text{ in } (0,1), \\ \phi\left(0\right) = \lim_{t\to 1}\phi(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q\vartheta = \left(\widetilde{\lambda}_{2l-1}^{+} - 1\right)(\rho m^{+} + 2\rho\omega_0 + q^{-})\vartheta + \rho\alpha\vartheta^{+} - \rho\beta\vartheta^{-}, \text{ in } (0,1), \\ \vartheta\left(0\right) = \lim_{t\to 1}\vartheta(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q\psi = \left(\widetilde{\lambda}_{2l}^{-} - 1\right)(\rho m + 2\rho\omega_0 + q^{-})\psi \ + \rho\alpha\psi^{+} - \rho\beta\psi^{-}, \text{ in } (0,1), \\ \psi\left(0\right) = \lim_{t\to 1}\psi(t) = 0. \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$, $(y_j)_{j=0}^{j=2l-1}$ and $(y_j)_{j=0}^{j=2l}$ be respectively the sequences of zeros of ϕ, ϑ and ψ. Thus, if $\widetilde{\lambda}_{2l-1}^{+} \leq 1$ then

$$\left(\widetilde{\lambda}_{2l-1}^{+} - 1\right)\left(\rho m + 2\rho\omega_0 + q^{-}\right) + \rho\alpha < \rho\alpha < \frac{\mu_l\left(\alpha\right)}{\rho}\rho\alpha = \mu_l\left(\rho\alpha\right)\rho\alpha$$

and we obtain from Lemma 8 that in each interval (y_{2j}, y_{2j+1}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$.

Similarly, if $\widetilde{\lambda}_{2l}^{-} \leq 1$ then

$$\left(\widetilde{\lambda}_{2l}^{-} - 1\right)\left(\rho m + 2\rho\omega_0 + q^{-}\right) + \rho\alpha < \mu_l\left(\rho\alpha\right)\rho\alpha$$

and we obtain from Lemma 8 that in each interval (y_{2j+1}, y_{2j+2}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$.

Lemma 22. *Let (m_n) be a sequence in Γ^{+} such that $\lim_{n\to+\infty}(\inf_{t\in[0,1]} m_n\left(t\right)) = +\infty$. Then for all $q \in Q$ and $k \geq 1$, $\lim_{n\to+\infty}\mu_k\left(m_n\right) = 0$.*

Proof. For arbitrary $A > 0$, there is $n_A \geq 1$ such that $m_n \geq A$ for all $n \geq n_A$. Thus, we obtain by means of Assertion 1 in Corollary 15 that for all $k \geq 1$ and $n \geq n_A$,

$$\left|\mu_k\left(m_n\right)\right| \leq \left|\mu_k\left(A\right)\right| = \left(\left|\mu_k\left(1\right)\right|/A\right),$$

proving that $\lim_{n\to+\infty}\mu_k\left(m_n\right) = 0$.

Lemma 23. *Assume that $q \in Q_{\#}$ and let u be a nontrivial solution to the bvp (21), then either $u \in S_k^\nu$ for some $k \geq 1$ and $\nu = +, -$ or u has an infinite monotone sequence of simple zeros.*

Proof. We distinguish two cases:

i) u has a finite number of zeros $(z_j)_{j=0}^{j=l}$, in this case we have for all j, $0 \leq j \leq l-1$,

$$|u(t)| \geq \rho^*_{z_j, z_{j+1}}(t) \sup_{t \in [z_j, z_{j+1}]} |u(t)| \text{ in } [z_j, z_{j+1}]$$

leading to

$$\begin{aligned} \left| \frac{u(t)}{t - z_j} \right| &\geq \sup_{t \in [z_j, z_{j+1}]} |u(t)| / \Psi_q(1) \text{ for } t \text{ near } z_j \text{ and} \\ \left| \frac{u(t)}{t - z_{j+1}} \right| &\geq \sup_{t \in [z_j, z_{j+1}]} |u(t)| / \Psi_q(1). \end{aligned}$$

Passing to the limits we obtain that $|u'(z_j)| > 0$ and $|u'(z_{j+1})| > 0$. This proves that all zeros of u are simple and $u \in S_l^\nu$ for some $\nu = +$ or $-$.

ii) u has an infinite number of zeros, in this case there is $z_* \in [0, 1]$ such that $u(z_*) = u'(z_*) = 0$. We claim that there is a monotone sequence of simple zeros (t_n) such that $\lim t_n = z_*$. Indeed, if this fails then there is an interval $[a, b] \subsetneq [0, 1]$ such that $u = 0$ in $[a, b]$ and $z_* \in [a, b]$. Set then

$$\begin{aligned} t_+ &= \sup \{t \geq b : u(s) = 0 \text{ for all } s \in [b, t]\}, \\ t_- &= \inf \{t \leq a : u(s) = 0 \text{ for all } s \in [t, a]\}. \end{aligned}$$

Since u is a nontrivial solution, we have $t_- > 0$ or $t_+ < 1$. Without loss of generality, suppose that $t_+ < 1$ and $u > 0$ in (t_+, t_*) where $t_* = \sup \{t > t_+ : u(t) > 0\}$. In one hand, we have

$$u'(t_+) = \lim_{t \overset{<}{\to} t_+} \frac{u(t)}{t - t_+} = 0.$$

In the other, we obtain from Lemma 3 the contradiction

$$u'(t_+) = \lim_{t \overset{>}{\to} t_+} \frac{u(t)}{t - t_+} \geq \left(\sup_{t \in [t_+, t_*]} u(t) / \Psi_q(1) \right) > 0.$$

This proves that there is a monotone sequence of zeros (t_n) of u and the simplicity of t_n is obtained again by means of Lemma 3. This achieves the proof.

The following lemma is a adapted version of Corduneanu compactness criterion:

Lemma 24. *A nonempty bounded subset Ω is relatively compact in W if*

(a) Ω is locally equicontinuous on $[0,1)$, that is, equicontinuous on every compact interval of $[0,1)$ and

(b) Ω is equiconvergent at 1, that is, given $\epsilon > 0$, there corresponds $T(\epsilon) \in (0,1)$ such that $|x(t)| < \epsilon$ for any $t \geq T(\epsilon)$ and $x \in \Omega$.

4.3. Proofs of Theorems 16 and 17

4.3.1. An associated bifurcation bvp

Consider the bvp

$$\begin{cases} \mathcal{L}_{\widetilde{q}} u = \mu \widetilde{m} u + u \widetilde{f}(t,u) \text{ in } (0,1), \\ u(0) = \lim_{t \to 1} u(t) = 0, \end{cases} \tag{42}$$

where μ is real parameter.

By a solution to the bvp (42), we mean a pair $(\mu, u) \in \mathbb{R} \times W^2$ satisfying the differential equation in the bvp (42). Notice that $u \in W^2$ is a solution to the bvp (21) if and only if $(1, u)$ is a solution to the bvp (42). For this reason, we will study the bifurcation diagram of the bvp (42) and by means of Rabinowitz's global bifurcation theory, we will prove that the set of solutions to the bvp (42) consists in an infinity of unbounded components, each branching from a point on the line $\mathbb{R} \times \{0\}$ joining a point on $\overline{\mathbb{R}} \times \{\infty\}$. Obviously, each component having the starting point and the arrival point oppositely located relatively to 1, carries a solution of the bvp (21) and main results of this section will be proved once we compute the number of such components.

Lemma 25. *From each $\widetilde{\mu}_l$ bifurcate two unbounded components of nontrivial solutions to the bvp (42) ζ_l^+ and ζ_l^-, such that $\zeta_l^\nu \subset \mathbb{R} \times S_l^\nu$.*

Proof. It follows from Lemma 5 that solutions to the bvp (42) are those satisfying tohe fixed point equation

$$u = \mu L_{\widetilde{q},\widetilde{m}} u + T_0(u). \tag{43}$$

In order to use the global bifurcation theory, let us prove that all characteristic values of $L_{\widetilde{q},\widetilde{m}}$ are of algebraic multiplicity one. To this aim let $u \in N\left((I - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}})^2\right)$ and set $v = u - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}} u$, then $v \in N(I - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}}) = \mathbb{R}\phi_k$ and $u - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}} u = \eta \phi_k$ for some $\eta \in \mathbb{R}$. In another way, v satisfies the bvp

$$\begin{cases} -v'' + \widetilde{q} v = \widetilde{\mu}_k \widetilde{m} v - \eta \widetilde{m} \phi_k, \text{ in } (0,1) \\ u(0) = \lim_{t \to 1} u(t) = 0. \end{cases}$$

Multiplying the differential equation in the above bvp by ϕ_k and integrating on $(0,1)$ we obtain

$$\eta\widetilde{\mu}_k \int_0^1 \widetilde{m}\phi_k^2 dt = 0.$$

leading to $\eta = 0$ and $u = \widetilde{\mu}_k L_{\widetilde{m}} u \in \mathbb{R}\phi_k$.

Now, we need to prove that $T_0(u) = \circ(\|u\|)$ near 0. Indeed, let $(u_n) \subset W$ with $\lim \|u_n\| = 0$, we have

$$\frac{|T_0 u_n(t)|}{\|u_n\|} \leq \int_0^1 G_{\widetilde{q}}(t,s) \left|\widetilde{f}(s,u_n(s))\right| ds \ \leq \overline{G}_{\widetilde{q}} \int_0^1 \left|\widetilde{f}(s,u_n(s))\right| ds$$

We have from Hypothesis (22) that $\widetilde{f}(s,u_n(s)) \to 0$ as $n \to +\infty$ for all $s \in (0,1)$. Thus, we conclude by the Dominated convergence Theorem that $T_0(u) = \circ(\|u\|)$ near 0.

Let l_k be the projection of W on $\mathbb{R}\phi_k$, $\widetilde{W} = \{u \in W : l_k u = 0\}$ and for $\xi > 0$, $\eta \in (0,1)$ and $\nu = +$ or $-$,

$$K^{\nu}_{\xi,\eta} = \{(\mu,u) \in \mathbb{R} \times W : |\mu - \widetilde{\mu}_k| < \xi \text{ and } \nu l_k u > \eta \|u\|\}.$$

Since Lemma 5 guarantees that the operators $L_{\widetilde{m}}$ and T_0 are respectively compact and completely continuous, we have from Theorem 1.40 and Theorem 1.25 in [21], that from $(\widetilde{\mu}_k, 0)$ bifurcate two components ζ_k^+ and ζ_k^- of nontrivial solutions to Equation (43) such that there is $\varsigma_0 > 0$, $\zeta_k^\nu \cap B(0,\varsigma) \subset K^\nu_{\xi,\eta}$ for all $\varsigma < \varsigma_0$ and if $u = \alpha\phi_k + w \in \zeta_k^\nu$ then $|\mu - \widetilde{\mu}_k| = \circ(1)$, $w = \circ(|\alpha|)$ for α near 0.

We claim that there is $\delta > 0$ such that $\zeta_k^\nu \cap B(0,\varsigma) \subset \mathbb{R} \times S_k^\nu$; for all $\varsigma < \delta$. Indeed, let $(\mu_n, u_n)_{n\geq 1} \subset \zeta_k^\nu$ be such that $\lim(\mu_n, u_n) = (\widetilde{\mu}_k, 0)$, we have from Hypothesis (22) that $f(s,u_n(s)) \to m$, that is $\lim \mu_n f(s,u_n(s)) = \mu_k m(s)$ and Lemma 8 guarantees that there is $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. Moreover, if $u_n = \alpha_n\phi_k + w_n$ then $\lim \frac{u_n}{\alpha_n} = \phi_k$ in E proving that $\nu u_n(t) > 0$ for t in a right neighborhood of 0 and $\nu u_n'(0) > 0$ (otherwise, if $u_n'(0)$ then the existence and uniqueness result for ODEs leads to $u_n = 0$).

Also, if $(\mu_*, u_*) \in \zeta_k^\nu$ then for all sequence $(\mu_n, u_n)_{n\geq 1} \subset \zeta_k^\nu$ be such that $\lim(\mu_n, u_n) = (\mu_*, u_*)$, we have $\lim \mu_n f(s,u_n(s)) = \mu_* f(s,u_*(s))$ in E and Lemma 8 guarantees existence of $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. This shows that $\zeta_k^\nu \subset \mathbb{R} \times S_k^\nu$ and ζ_k^ν is unbounded in $\mathbb{R} \times W$. The lemma is proved.

4.3.2. Proof of Theorem 16

Step 1. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the projection of the component ζ_l^ν on the real axis is bounded. Since the nonlinearity f satisfies

Hypothesis (22), there is $\gamma \in \Gamma^{++}$ be such that

$$-\gamma(t) \leq f(t,u) \leq \gamma(t) \text{ for all } t \in [0,1] \text{ and } u \in \mathbb{R}.$$

Let for $\kappa = +$ or $-$, $\psi_{k,\kappa} \in S_k^\nu$ be the eigenfunction associated with $\mu_{k,\kappa} = \mu_k(\widetilde{q} - \rho(m+\kappa\gamma), \widetilde{m})$ and $(\mu, u) \in \zeta_k^\kappa$. It follows from Lemma 6 and Lemma 7 that there exist two intervals (ξ_1, η_1) and (ξ_2, η_2) where $u\psi_{k,\kappa} \geq 0$ and such that

$$\begin{aligned} 0 \ &\geq \ \int_{\xi_1}^{\eta_1} \psi_{k,+}\mathcal{L}_{\widetilde{q}}u - u\mathcal{L}_{\widetilde{q}}\psi_{k,+} = \int_{\xi_1}^{\eta_1} (\mu - \mu_{k,+})\,\widetilde{m}\psi_{k,+}u + (f(s,u)+\gamma)\,u\psi_{k,+} \\ &\geq \ (\mu - \mu_{k,+})\int_{\xi_1}^{\eta_1} \widetilde{m}\psi_{k,+}uds \end{aligned}$$

$$\begin{aligned} 0 \ &\leq \ \int_{\xi_2}^{\eta_2} \psi_{k,-}\mathcal{L}_{\widetilde{q}}u - u\mathcal{L}_{\widetilde{q}}\psi_{k,-} = \int_{\xi_2}^{\eta_2} ((\mu - \mu_{k,-})\,\widetilde{m}u\psi_{k,-} + (f(s,u)-\gamma)\,u\psi_{k,-})\,ds \\ &\leq \ (\mu - \mu_{k,-})\int_{\xi_2}^{\eta_2} \widetilde{m}u\psi_{k,-}ds. \end{aligned}$$

The above inequalities lead to $\mu_{k,+} \leq \mu \leq \mu_{k,-}$.

Step 2. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the component ζ_l^νrejoins the point $(\widetilde{\lambda}_l^\nu, \infty)$. Notice that (43) is equivalent to

$$u = \mu L_{\widetilde{m}}u + L_{\widetilde{\alpha}-\widetilde{m}}I^+u - L_{\widetilde{\beta}-\widetilde{m}}I^-u + T_\infty u. \tag{44}$$

We proove that $K(u_n) = \circ(\|u_n\|)$ near ∞. Indeed; from lemma (4) in (**i**) we have

$$(|T_\infty u_n(t)| / \|u_n\|) \leq \int_0^1 P_n(s)ds,$$

where

$$P_n(s) = \overline{G}_{\widetilde{q}}\left| f(s,u_n(s))\frac{u_n(s)}{\|u_n\|} - \widetilde{\alpha}(s)\frac{u_n^+(s)}{\|u_n\|} + \widetilde{\beta}(s)\frac{u_n^-(s)}{\|u_n\|}\right|.$$

Therefore, we have to prove that $\int_0^1 P_n(s)ds \to 0$ as $n \to \infty$.

We distinguish the following three cases:

i) $\lim u_n(s) = +\infty$: In this case, from (22) we obtain

$$P_n(s) \leq \overline{G}_{\widetilde{q}}|\widetilde{f}(s,u_n(s)) - \widetilde{\alpha}(s)| \to 0 \quad \text{as } n \to +\infty$$

ii) $\lim u_n(s) = -\infty$: in this case, from (22) we obtain

$$P_n(s) \leq \overline{G}_{\widetilde{q}}|\widetilde{f}(s,u_n(s)) - \widetilde{\beta}(s)| \to 0 \quad \text{as } n \to +\infty$$

iii) $\lim u_n(s) \neq \pm\infty$: in this case there may exist subsequences $(u_{n_k^1}(s))$ and $(u_{n_k^2}(s))$ such that $(u_{n_k^1}(s))$ is bounded and $\lim u_{n_k^2}(s) = \pm\infty$. Arguing as in the above two cases we obtain that $\lim P_{n_k^2}(s) = 0$ and we have from (22)

$$P_{n_k^1}(s) \leq \overline{G}_{\widetilde{q}} \left(\widetilde{f}(t, u(t)) + \widetilde{\alpha}(s) + \widetilde{\beta}(s) \right) \left(|u_{n_k^1}(s)| / \|u_{n_k^1}\| \right) \to 0 \quad \text{as } k \to +\infty,$$

proving that $T_\infty(u_n) = \circ(\|u_n\|)$ at ∞.

Now, let (μ_n, u_n) be sequence in ζ_k^ν with $\lim_{n\to+\infty} \|u_n\| = +\infty$ then $v_n = (u_n/\|u_n\|)$ satisfies

$$v_n = \mu_n L_{\widetilde{q},\widetilde{m}} v_n + L^+_{\widetilde{q},\widetilde{\alpha}} v_n - L^-_{\widetilde{q},\widetilde{\beta}} v_n + (T_\infty(u_n)/\|u_n\|) \tag{45}$$

with $T_\infty(u_n) = o(\|u_n\|)$ at ∞. By the compactness of the operators $L_{\widetilde{m}}, L_{\widetilde{\alpha}-\widetilde{m}}, L_{\widetilde{\beta}-\widetilde{m}}$, we obtain from (45) existence of $v_+, v_- \in W$ such that for $\kappa = +$ or $-$, $\|v_\kappa\| = 1$ and

$$v_\kappa = \mu_\kappa L_{\widetilde{q},\widetilde{m}} v_\kappa + L^+_{\widetilde{q},\widetilde{\alpha}} v_\kappa - L^-_{\widetilde{q},\widetilde{\beta}} v_\kappa$$

where $\mu_+ = \limsup \mu_n$ and $\mu_- = \liminf \mu_n$. We have from Lemma 10 that for $\kappa = +$ or $-$, $v_\kappa \in S_l^\nu$ with $l \leq k$. We claim that there is an integer $n_+ \geq 1$ such that $v_\kappa v_n > 0$ in $(z_{l-1} + \delta, 1)$. Indeed, if there a subsequence (v_{n_i}) such that for all $i \geq 1$, v_{n_i} has at a zero $x_{n_i} \in (z_{l-1} + \delta, 1)$ and v_{n_i} does not vanish in $(x_{n_i}, 1)$ then

$$\mu_n = \mu_1(\widetilde{q} - \widetilde{f}(s, u_n), \widetilde{m}, x_{n_i}) \geq \mu_1(q - \widetilde{\gamma}, \widetilde{m}, x_{n_i}).$$

Passing to the limit, we obtain from Theorem 9 the contradiction

$$+\infty > \mu_\kappa \geq \lim \mu_1(q - \widetilde{\gamma}, \widetilde{m}, x_{n_i}) = +\infty.$$

From all the above, we obtain that for all $n \geq n_+$, v_{n_i} belongs to S_l^ν and $l = k$.

Step 3. Notice that $u \in W^1 \cap C^2([0,1), \mathbb{R})$ is a solution to the bvp (21) if and only if $(1, u)$ is a solution to the bvp (42). This means that any component ζ_k^ν having the starting point $(\widetilde{\mu}_k, 0)$ and the arrival point $(\widetilde{\lambda}_k^\nu, \infty)$, oppositely located relatively to 1, carries a solution of the bvp (21). Therefore, we have to compute in each of the cases stated in Theorem 16 the number of such components. To be brief, we present only the proofs of Assertions 1 and 3.

Suppose that there is two integers i and j such that $i \geq j \geq 1$ and $\max(\mu_i(\alpha), \mu_i(\beta)) < \rho < \mu_j(m)$. We have then from Assertion 1 in Lemma 19 and Assertion 1 in Lemma 21 that $\widetilde{\mu}_j > 1$ and $\widetilde{\lambda}_i^\nu < 1$. Therefore, for all integers $l \in \{j, ..i\}$ and $\nu = +$ or $-$, the component ζ_l^ν crosses the hyperplane $\{1\} \times W$.

Now, Suppose that there is two integers i and j such that $i \geq j \geq 1$, with $i \geq 2(j-1)$ and $\mu_i(m) < \rho < \mu_j(\beta)$. We have then from Assertion 1 in Lemma 19 and Assertion 2 in Lemma 20 that $\widetilde{\mu}_i < 1$, $\widetilde{\lambda}^-_{2j-1} > 1$ and $\widetilde{\lambda}^+_{2j} > 1$. Therefore, for all integers $l \in \{2j-1, ..., i\}$, the component ζ^-_l crosses the hyperplane $\{1\} \times W$ and for all integers $l \in \{2j, ..i\}$, the component ζ^+_l crosses the hyperplane $\{1\} \times W$.

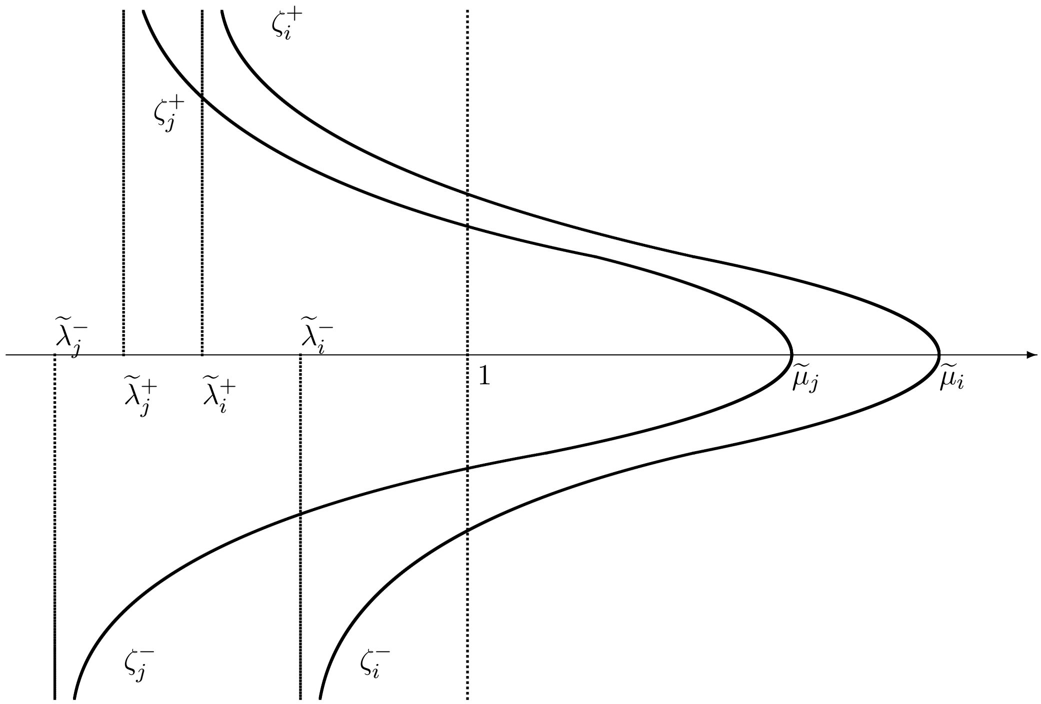

Fig. B: $\mu_i(\varphi) < \rho < \mu_j(m^+)$

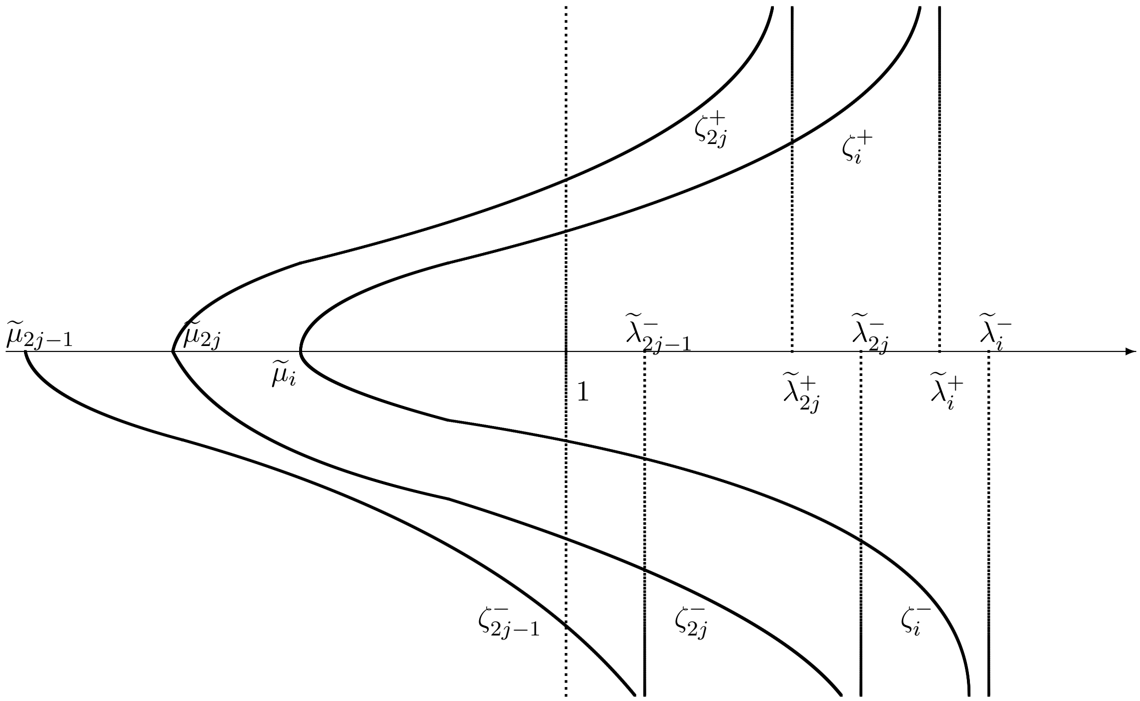

Fig. C: $\mu_i(m) < \rho < \mu_j(\alpha^+)$

4.3.3. Proof of Theorem 17

Step 1. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the projection of the component ζ_l^ν on the real axis is upper bounded . Since the nonlinearity f satisfies Hypothesis (23), there is $\gamma \in \Gamma^{++}$ be such that

$$f(t,u) \geq -\gamma(t) \text{ for all } t \in [0,1] \text{ and } u \in \mathbb{R}.$$

Because the nonlinearity f satisfies Hypothesis (23) there is $\gamma \in \Gamma^{++}$ such that

$$f(t,u) \geq -\gamma(t) \text{ for all } t \in [0,1] \text{ and } u \in \mathbb{R}.$$

Fix k and ν and let us prove first that if $(\mu,u) \in \zeta_k^\nu$ then $\mu \leq \mu_{k,-} = \mu_k(\widetilde{q} - \rho(m - \gamma), \widetilde{m})$. To this aim, let $\psi_k \in S_k^\nu$ be the eigenfunction associated with $\mu_{k,-}$, it follows from Lemma 6 and Lemma 7 that there exists an interval (ξ,η) where $u\psi_k \geq 0$ and we have

$$\begin{aligned} 0 &\leq \int_\xi^\eta \psi_k \mathcal{L}_{\widetilde{q}} u - u \mathcal{L}_{\widetilde{q}} \psi_k = \int_\xi^\eta \left((\mu - \mu_{k,-}) \widetilde{m} u \psi_k + (f(s,u) - \gamma) u \psi_k \right) ds \\ &\leq (\mu - \mu_{k,-}) \int_\xi^\eta \widetilde{m} u \psi_k ds \end{aligned}$$

leading to $\mu \leq \mu_{k,-}$.

Step 2. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the component ζ_l^νrejoins the point $(-\infty, \infty)$. Thus, we have to prove that for all $\mu < \mu_{k,-}$, there is a positive constant M_k^ν such that

$$\zeta_k^\nu \cap ([\mu, \mu_{k,-}] \times W) \subset [\mu, \mu_{k,-}] \times \overline{B}(0, M_k^\nu).$$

On the contrary, suppose that this fails and there is a sequence $(\mu_n, u_n)_{n \geq 1}$ in $\zeta_k^\nu \cap ([\mu, \mu_{k,-}] \times W)$ such that $\lim_{l \to \infty} \|u_n\| = +\infty$. That is for all $n \geq 1$

$$\begin{cases} \mathcal{L}_{\widetilde{q}} u_n = u_n \left(\mu_n + \widetilde{f}(t, u_n) \right) \text{ in } (0,1) \\ u_n(0) = \lim_{t \to 1} u_n(t) = 0, \end{cases}$$

from which we read that for all $n \geq 1$

$$\mu_k(\widetilde{q}, w_n) = 1 \tag{46}$$

where $w_n(t) = \mu_n + \widetilde{f}(t, u_n(t))$.

Let $\left(z_j^n \right)_{j=0}^{j=k}$ be the sequence of zeros of u_n, $I_j^n = \left[z_{j-1}^n, z_j^n \right]$, $\rho_j^n = \sup_{t \in I_j^n} |u_n(t)| = \left| u_n \left(y_j^n \right) \right|$ with $y_j^n \in I_j^n$. Because $\lim_{n \to \infty} \|u_n\| = +\infty$, there is j_n such that $\lim \rho_{j_n}^n = +\infty$. We claim that there is $a_* \in (0,1)$ such that if (n_s) is a sequence of integers such that $\lim_{s \to \infty} \rho_{j_{n_s}}^{n_s} = +\infty$ then $y_{j_{n,l_s}}^{n,l_s} \in (0, a_*)$. Indeed, if for any sequence (l_s) of integers such that $\lim_{s \to \infty} \rho_{j_{n_s}}^{n_s} = +\infty$ we have $\lim_{s \to \infty} y_{j_{n_s}}^{n_s} = 1$, then (u_{n_s}) is bounded on any interval $[0, a] \subset [0, 1)$. Therefore, from the equation

$$u_n(t) = \int_0^1 G_{\widetilde{q}}(t,s) \, u_n(s) \left(\mu_n + \widetilde{f}(s, u_n(s)) \right) ds$$

we conclude that (u_{n_s}) converges uniformly to $u \in W$ in all intervals $[0, a] \subset [0, 1)$ and

$$u(t) = \int_0^1 G_{\widetilde{q}}(t,s) \, u(s) \widetilde{f}(s, u_n(s)) \, ds.$$

Since for all $t \in [0,1)$

$$|u_n(t) - u(t)| \leq \int_0^1 G_{\widetilde{q}}(s,s) \left| u_n(s)\widetilde{f}(s, u_n(s)) - u(s)\widetilde{f}(s, u(s)) \right| ds,$$

we obtain by means of the Lebesgue dominated convergence theorem that $u_n \to u$ in W, leading to the contradiction $\|u\| = \lim_{n\to\infty} \|u_n\| = +\infty$.

Set $q_* = \sup_{t\in[0,a_*]} q(t)$ and let $A_* > 0$ be such that $f(t,u) > q_*$ for all $t \in [0, a_*]$ and $|u| > A_*$. We prove now, that if $I_j^n \subset [0,a_*]$ then $\lim \rho_j^n = +\infty$. On the contrary suppose that $\lim \rho_{j_n-1}^n \neq +\infty$ and $\lim \rho_{j_n+1}^n \neq +\infty$, that is u_n is bounded in $I_{j_n-1}^n \cup I_{j_n+1}^n$ and let ϖ be such that $\max\left(\rho_{j_n-1}^n, \rho_{j_n+1}^n\right) \leq \varpi$. Let $\alpha_{j_n}^n \in \left(z_{j_n-1}^n, y_{j_n}^n\right)$ and $\beta_{j_n}^n \in \left(y_{j_n}^n, z_{j_n}^n\right)$ be such that $\left|u_n\left(\alpha_{j_n}^n\right)\right| = \left|u_n\left(\beta_{j_n}^n\right)\right| = A_*$. Thus, we have

$$-u_n''(t)\, u_n(t) = u_n^2(t)\left(f(t,u_n(t)) - q(t)\right) \geq u_n^2(t)\left(q_* - q(t)\right) \geq 0 \text{ in } \left(\alpha_{j_n}^n, \beta_{j_n}^n\right)$$

leading to $\left|u_n'\left(\alpha_{j_n}^n\right)\right| = \sup_{t\in\left(\alpha_{j_n}^n, y_{j_n}^n\right)} |u_n'(t)|$ and $\left|u_n'\left(\beta_{j_n}^n\right)\right| = \sup_{t\in\left(y_{j_n}^n, \beta_{j_n}^n\right)} |u_n'(t)|$.

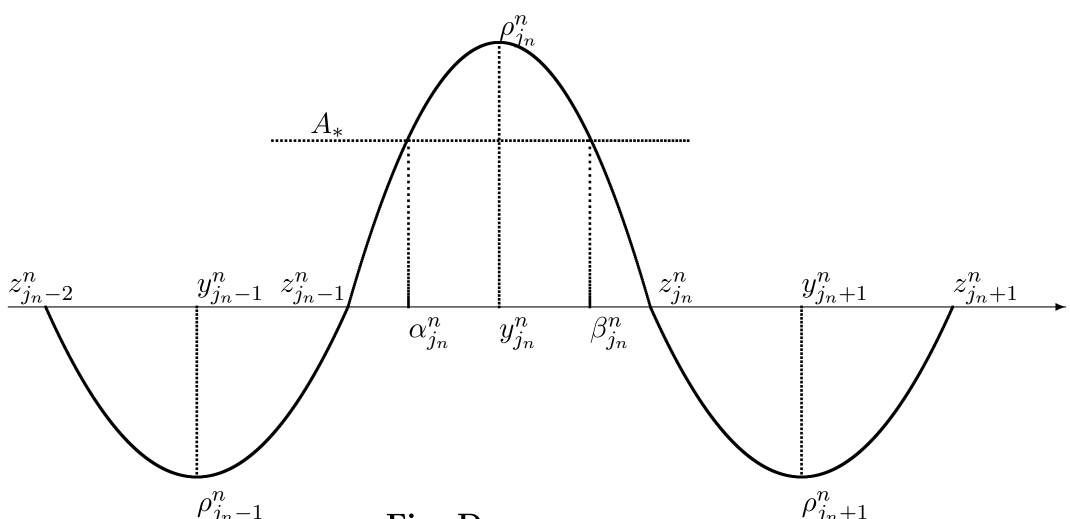

Fig. D

On the one hand, we have

$$\begin{aligned}\lim\left|u_n'\left(\alpha_{j_n}^n\right)\right| &= \lim\left(\sup_{t\in\left(\alpha_{j_n}^n,y_{j_n}^n\right)}\left|u_n'\left(t\right)\right|\right)=\lim\left|u_n'\left(\beta_{j_n}^n\right)\right| \\ &= \lim\left(\sup_{t\in\left(y_{j_n}^n,\beta_{j_n}^n\right)}\left|u_n'\left(t\right)\right|\right)=+\infty.\end{aligned}$$

Indeed, if for instance u_n' is bounded by a constant A in $\left(\alpha_{j_n}^n, y_{j_n}^n\right)$ then

$$\rho_{j_n}^n \le A_* + \int_{\alpha_{j_n}^n}^{y_{j_n}^n}\left|u_n'(s)\right| ds \le A_* + A,$$

contradicting $\lim \rho_{j_n}^n = +\infty$.

On the other hand, we have the contradiction

$$\left|u_n'\left(\alpha_{j_n}^n\right)\right| = \left|\int_{y_{j_n-1}^n}^{\alpha_{j_n}^n} u_n(s)\left(f\left(s,u_n(s)\right)-q(s)\right)ds\right| \le \max\left(\varpi, A_T\right)\left(q_*+\theta\right)<\infty$$

$$\left|u_n'\left(\beta_{j_n}^n\right)\right| = \left|\int_{\beta_{j_n}^n}^{y_{j_n+1}^n} u_n(s)\left(f\left(s,u_n(s)\right)-q(s)\right)ds\right| \le \max\left(\varpi, A_T\right)\left(q_*+\theta\right)<\infty$$

where $\theta = \sup\left\{\left|f\left(s,u\right)\right| : s\in[0,1] \text{ and } u\in\left[-\max\left(\varpi,A_T\right),\max\left(\varpi,A_T\right)\right]\right\}$. This shows that all bumps of u_n contained in $[0,a_*]$ are unbounded.

At this stage, for all $n\ge 1$ there is an interval $I_{j_n}^n = \left[z_{j_n-1}^n, z_{j_n}^n\right] \subset [0,a_*]$ such that $z_{j_n}^n - z_{j_n-1}^n \ge \frac{a_*}{k}$ and Lemma 3 leads to $\left|u_n(t)\right| \ge \frac{\rho_{j_n}^n}{4}$ for all $t\in\left[\gamma_{j_n}^n, \delta_{j_n}^n\right]$ where

$$\gamma_{j_n}^n = z_{j_n-1}^n + \frac{z_{j_n}^n - z_{j_n-1}^n}{4} \text{ and } \delta_{j_n}^n = z_{j_n}^n - \frac{z_{j_n}^n - z_{j_n-1}^n}{4}.$$

Set $\gamma_0 = \sup\gamma_{j_n}^n$ and $\delta_0 = \inf\gamma_{j_n}^n$ and notice that $\delta_0-\gamma_0 = \inf\left(\delta_{j_n}^n - \gamma_{j_n}^n\right) \ge \frac{T}{2k}$. Because of

$$u_n(t) = \int_{z_{j_n-1}^n}^{z_{j_n}^n} G\left(z_{j_n-1}^n, z_{j_n}^n, t, s\right) u(s)_n \widetilde{f}\left(s,u_n(s)\right)ds,$$

we obtain from Lemma 3 that

$$\left|u_n(t)\right| \ge \frac{\min\left(t-z_{j_n-1}^n, z_{j_n}^n - t\right)}{\Psi_{q,\theta}\left(z_{j_n}^n\right)}\rho_{j_n}^n \ge \frac{\min\left(t-z_{j_n-1}^n, z_{j_n}^n - t\right)}{\Psi_{q,\theta}\left(T\right)}\rho_{j_n}^n \to +\infty$$

for all $t \in [\gamma_0, \delta_0]$. Thus, we obtain from Lemma 13 and (46) that

$$\mu_k(\widetilde{q}, w_n, [\gamma_0, \delta_0]) > \mu_k(\widetilde{q}, w_n) = 1. \tag{47}$$

Let $A > \mu_k(\widetilde{q}, 1, [\gamma_0, \delta_0])$, there is $n_A \geq 1$ such that $w_n(t) = \mu_n + \widetilde{f}(t, u_n(t)) \geq A$ for all $n \geq n_A$ and $t \in [\gamma_0, \delta_0]$. Hence, we obtain by Assertion 1 in Corollary 15 the contradiction

$$1 < \mu_k(\widetilde{q}, w_n, [\gamma_0, \delta_0]) \leq \mu_k(\widetilde{q}, A, [\gamma_0, \delta_0]) = \frac{\mu_k(\widetilde{q}, 1, [\gamma_0, \delta_0])}{A} < 1.$$

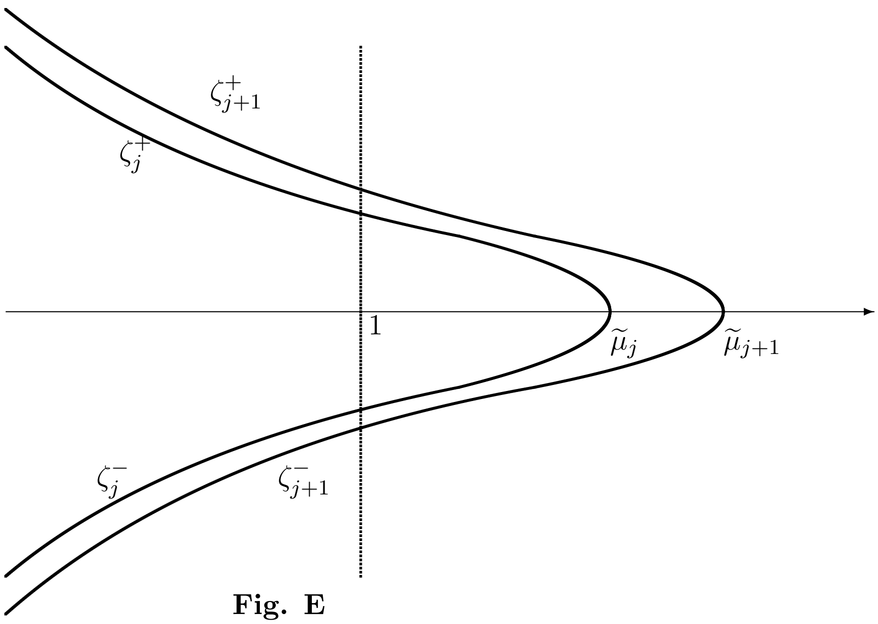

Fig. E

Step 3. At this stage, we have only to compute components that cross the hyperplane $\mu = 1$. Assume that Hypothesis (33) holds, then we have from Assertion 2 in Lemma 19 that $\widetilde{\mu}_k > 1$ for all $k \geq j$. Since for all $k \geq 1$ and $\nu = \pm$ the component ζ_k^ν reachs $(-\infty, \infty)$, ζ_k^ν crosses the hyperplane $\mu = 1$ for all $k \geq j$. Thus, the bvp (21) admits for all $k \geq j$ a solution in S_k^+ and in S_k^-. The case where $m^+ = 0$ and $\mu_j(\chi_0) > 0$ for some $\chi_0 \in \Gamma^+$ is obtained by means of Assertion 3 in Lemma 19.

The proof of Theorem 17 is complete.

4.4. Proof of Theorem 18

Set for $n \geq 1$

$$f_n(t,u) = \begin{cases} f(t,u) \text{ if } |u| \geq \frac{1}{n}, \\ f(t,n) \text{ if } |u| < \frac{1}{n} \end{cases}$$

and consider the bvp

$$\begin{cases} -u'' + qu = \rho u f_n(t,u) \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0. \end{cases} \tag{48}$$

We have then

$$\lim_{u\to+\infty} f_n(t,u) = \alpha(t), \ \lim_{u\to-\infty} f_n(t,u) = \beta(t) \text{ and } \lim_{u\to 0} f_n(t,u) = f(t,\frac{1}{n}) \text{ in } E.$$

To be brief, we present the proof of Assertion 1, the other Assertions are checked similarly. Because of $\lim_{n\to\infty}\left(\inf_{t\in[0,1]} \widetilde{f}(t,\frac{1}{n})\right) = +\infty$, for all $l \geq 1$ there exists $n_l \geq 1$ such that for all $n \geq n_l$, $\mu_l(\widetilde{q}, \widetilde{f}(t,\frac{1}{n})) < \rho$.

Fix $k \geq j$ and $\nu = +$ or $-$. For all $n \geq n_k$ Assertion 3 in Theorem 16 guarantees existence of $u_n \in S_k^\nu$ solution to the bvp (48).

Let ω_0 be that in Remark 1,

$$\overline{q} = q^+ + 2\rho\omega_0 \quad , \quad \overline{f}_n(t,u) = \rho\left(f_n(t,u) + 2\omega_0\right) + q^-$$

and observe that v is a solution to (48) if and only if v is a solution to the bvp

$$\begin{cases} -u'' + \overline{q}u = \rho u \overline{f}_n(t,u) \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0. \end{cases} \tag{49}$$

We claim that there is a positive constant m_k^ν such that $\|u_n\| \geq m_k^\nu$. To the contrary, suppose that (u_n) admits a subsequence (u_s) such that $\lim u_s = 0$ in E and let $A > \mu_k(\overline{q}, 1)$. There is $\gamma_A > 0$ such that for all $u \in \mathbb{R}$, $|u| < \gamma_A$ implies

$\inf_{t\in[0,1]} \overline{f}_n (t, u_s) > A$ and there is s_A such that $\|u_s\| < \gamma_A$ for all $s > s_A$. Thus, for all $s \geq \sup(1/\gamma_A, s_A)$, $\inf_{t\in[0,1]} \overline{f}_n (t, u_n(t)) > A$ and this leads to the contradiction

$$1 = \mu_k(\overline{q}, \overline{f}_n (t, u_n(t))) < \mu_k(\overline{q}, A) = \frac{\mu_k(\overline{q}, 1)}{A} < 1.$$

We prove now that there is positive constant M_k^ν such that $\|u_n\| \leq M_k^\nu$. To the contrary, suppose that there is a subsequence (u_r) of (u_n) such that $\lim \|u_r\| = \infty$. Arguing as in Step 2 in the proof of Theorem 17, we obtain that $v_r = u_r/\|u_r\|$ converges, up to a subsequence, to $v \in S_k^\nu$ satisfying

$$\begin{cases} \mathcal{L}_q v = \rho\alpha v^+ - \rho\beta v^- \text{ in } (0,1) \\ v(0) = \lim_{t\to 1} v(t) = 0. \end{cases}$$

Let $\phi \in S_k^\nu$ be the eigenfunction associated with $\mu_k(\rho\psi)$, that is ϕ satisfies

$$\begin{cases} \mathcal{L}_q \phi = \mu_k(\rho\psi)\, \rho\psi\phi \text{ in } (0,1) \\ \phi(0) = \lim_{t\to 1} \phi(t) = 0. \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$ and $(y_j)_{j=0}^{j=l}$ be respectively the sequences of zeros of v and ϕ. We distinguish then the following two cases:

i) $x_1 \leq y_1$: in this case we have the contradiction:

$$\begin{aligned} 0 &\leq \int_{x_0}^{x_1} v\mathcal{L}_q\phi - \phi\mathcal{L}_q v \\ &\leq \int_{x_0}^{x_1} \mu_k(\rho\psi)\rho\psi\phi v - \left(\rho\alpha v^+ - \rho\beta v^-\right)\phi \\ &= \int_{x_0}^{x_1} \left(\mu_k(\rho\psi)\rho\psi - \alpha\right)\rho\phi^+ v^+ + \left(\mu_k(\rho\psi)\rho\psi - \beta\right)\rho\phi^- v^- < 0. \end{aligned}$$

ii) $y_1 < x_1$: in this case Lemma 6 guarantees existence of two integers m, n having the same parity such that $y_m < x_n < x_{n+1} \leq y_{m+1}$ and Lemma 7 leads to the contradiction:

$$\begin{aligned} 0 &< \int_{x_n}^{x_{n+1}} v\mathcal{L}_q\phi - \phi\mathcal{L}_q v \\ &\leq \int_{x_n}^{x_{n+1}} \mu_k(\rho\psi)\rho\psi\phi v - \left(\rho\alpha v^+ - \rho\beta v^-\right)\phi \\ &= \int_{x_n}^{x_{n+1}} \left(\mu_k(\rho\psi)\rho\psi - \alpha\right)\rho\phi^+ v^+ + \left(\mu_k(\rho\psi)\rho\psi - \beta\right)\rho\phi^- v^- < 0. \end{aligned}$$

At this stage by means of Theorem 24 we prove that the sequence (u_n) is relatively compact. Let $[0,a] \subset [0,1)$, $t_1, t_2 \in [0,a]$ be such that $t_1 < t_2$ and $C_k^\nu = \sup\left\{\left|u\overline{f}(t,u)\right| : t \in [0,1] \text{ and } u \in [-M_k^\nu, M_k^\nu]\right\}$. We have

$$\begin{aligned}
&|u_n(t_2) - u_n(t_1)| \le C_k^\nu \int_0^1 |G_{\overline{q}}(t_2,s) - G_{\overline{q}}(t_1,s)|\, ds \le C_k^\nu (|\Phi_{\overline{q}}(t_2) - \Phi_{\overline{q}}(t_2)| \int_0^{t_1} \Psi_{\overline{q}}(s)\, ds \\
&+ \int_{t_1}^{t_2} |\Phi_{\overline{q}}(t_2)\Psi_{\overline{q}}(s) - \Phi_{\overline{q}}(s)\Psi_{\overline{q}}(t_1)|\, ds + |\Psi_{\overline{q}}(t_2) - \Psi_{\overline{q}}(t_1)| \int_{t_2}^1 \Phi_{\overline{q}}(s)\, ds) \\
&\le C_k^\nu \left(\left|\Phi'_{\widetilde{q}}(0)\right| \int_0^a \Psi_{\overline{q}}(s)\, ds + 2\Psi_{\overline{q}}(a) + \Psi'_{\overline{q}}(a) \int_0^1 \Phi_{\overline{q}}(s)\, ds +\right) |t_2 - t_1|.
\end{aligned}$$

This proves that (u_n) is equicontinuous on any interval $[0,a]$ contained in $[0,1)$.

By the mean value theorem, for all $n \ge 1$ and all $t \in [0,1)$ there is $t_n \in (t,1)$ such that

$$\left|\frac{u_n(t)}{1-t}\right| = \left|u_n'(t_n)\right| = \left|\int_0^1 \frac{\partial G_{\overline{q}}}{\partial t}(t_n,s) u_n(s) \overline{f}_n(s,u_n(s))\, ds\right| \le C_k^\nu.$$

This proves that the sequence (u_n) is equiconvergent at $t_0 = 1$.

Therefore, $\lim u_n = u$ (up to a subsequence) and $u(t) = \int_0^1 G_{\widetilde{q}}(t,s)\overline{f}(s,u(s))\, ds$ proving that u is a solution to the bvp (21). Furthermore, combining Lemma 23 with Lemma 10 we see that $u \in S_k^\nu$. This ends the proof.

4.5. Separable variable case

Consider the case of the bvp (21) where the nonlinearity f is a separable variables function, namely the case where the bvp (21) takes the form

$$\begin{cases} \mathcal{L}_q u = \rho\varkappa u h(u), \ t \in (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{50}$$

where $\varkappa \in \Gamma^+$ and $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{u\to 0} h(u) = h_0, \quad \lim_{u\to +\infty} h(u) = h_+, \quad \lim_{u\to -\infty} h(u) = h_-. \tag{51}$$

We obtain from Theorems 16, 17 and 18 the following corollary:

Corollary 26. *Assume that (51) holds.*

1. *Let i, j be two integers such that $i \ge j \ge 1$. The bvp (50) admits in each of $S_j^+, \dots, S_i^+, S_j^-, \dots, S_i^-$ a solution if one of the following Hypotheses (52), (53), (54) and (55) holds true.*

$$\begin{cases} h_0, h_+, h_- \in (0,+\infty) \ \textit{and} \\ (\mu_j(q,\varkappa)/\min(h_+,h_-)) < \rho < (\mu_i(q,\varkappa)/h_0), \end{cases} \tag{52}$$

$$\begin{cases} h_0 \leq 0,\ \ h_+, h_- \in (0, +\infty)\,,\ \ (\mu_i(q, \varkappa)/\min(h_+, h_-)) < \rho \\ \text{and } \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{53}$$

$$\begin{cases} h_0, h_+, h_- \in (0, +\infty) \ \text{ and} \\ (\mu_i(q, \varkappa)/h_0) < \rho < (\mu_j(q, \varkappa)/\max(h_+, h_-))\,, \end{cases} \tag{54}$$

$$\begin{cases} h_0 \leq 0,\ \ h_+, h_- \in (0, +\infty)\,,\ \ (\mu_j(q, \varkappa)/\max(h_+, h_-)) > \rho \\ \text{and } \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{55}$$

2. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (50) admits in each of $S^+_{2j}, \ldots, S^+_i, S^-_{2j-1}, \ldots, S^-_i$ a solution if one of the following Hypotheses (56), (57) holds true.*

$$\begin{cases} h_0, h_- \in (0, +\infty) \ \text{ and} \\ (\mu_i(q, \varkappa)/h_0) < \rho < (\mu_j(q, \varkappa)/h_-)\,, \end{cases} \tag{56}$$

$$\begin{cases} h_0 > 0,\ \ h_- \leq 0,\ \ (\mu_i(q, \varkappa)/h_0) < \rho \\ \text{and } \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{57}$$

3. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (50) admits in each of $S^+_{2j-1}, \ldots, S^+_i, S^-_{2j}, \ldots, S^-_i$ a solution if one of the following Hypotheses (58), (59) holds true.*

$$\begin{cases} h_0, h_+ \in (0, +\infty) \ \text{ and} \\ (\mu_i(q, \varkappa)/h_0) < \rho < (\mu_j(q, \varkappa)/h_+)\,, \end{cases} \tag{58}$$

$$\begin{cases} h_0 > 0,\ \ h_+ \leq 0,\ \ (\mu_i(q, \varkappa)/h_0) < \rho \\ \text{and } \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{59}$$

4. *The bvp (50) admits for all $k \geq j$ a solution in each of S^+_k and S^-_k if one of the following Hypotheses (60), (61), (62) and (63) holds true.*

$$\begin{cases} h_0 > 0,\ \ h_- = h_+ = +\infty \ \text{ and} \\ (\mu_j(q, \varkappa)/h_0) > \rho, \end{cases} \tag{60}$$

$$\begin{cases} h_0 \leq 0,\ \ h_- = h_+ = +\infty \ \text{ and} \\ \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{61}$$

$$\begin{cases} h_-, h_+ \in (0, +\infty)\,,\ \ h_0 = +\infty \ \text{ and} \\ (\mu_j(q, \varkappa)/\max(h_-, h_+)) > \rho, \end{cases} \tag{62}$$

$$\begin{cases} h_-, h_+ \leq 0,\ \ h_0 = +\infty \ \text{ and} \\ \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{63}$$

5. *The bvp (50) admits a solution in S_k^+ for all $k \geq 2j$ and a solution in S_k^- for all $k \geq 2j-1$, if one of the following Hypotheses (64), (65) holds true.*

$$\begin{cases} h_- > 0, \ h_0 = +\infty \text{ and} \\ (\mu_j(q, \varkappa)/h_-) > \rho, \end{cases} \tag{64}$$

$$\begin{cases} h_- \leq 0, \ h_0 = +\infty \text{ and} \\ \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{65}$$

6. *The bvp (50) admits a solution in S_k^+ for all $k \geq 2j-1$ and a solution in S_k^- for all $k \geq 2j$, if one of the following Hypotheses (66), (67) holds true.*

$$\begin{cases} h_+ > 0, \ h_0 = +\infty \text{ and} \\ (\mu_j(q, \varkappa)/h_+) > \rho, \end{cases} \tag{66}$$

$$\begin{cases} h_+ \leq 0, \ h_0 = +\infty \text{ and} \\ \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{67}$$

4.6. Comments

1. Under one of the Hypotheses (22), (23) and (24), the set of solutions to the bvp (21) is contained in $\cup_{k\geq 1, \nu=\pm} S_k^\nu$. Indeed, we have seen above that u is a solution to the bvp (21) if and only if u satisfies

$$\begin{cases} \mathcal{L}_{\widetilde{q}} u = u\widetilde{f}(t,u) \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{68}$$

where $\widetilde{q} = q + \omega_1$, $\widetilde{f}(t,u) = f(t,u) + \omega_1$ and $\omega_1 \in \Gamma^{++}$ is that in Remark 1. We read from (68) that u is a solution to bvp

$$\begin{cases} \mathcal{L}_{\widetilde{q}} v = v\widetilde{f}(t,u) \text{ in } (0,1) \\ v(0) = \lim_{t\to 1} v(t) = 0, \end{cases}$$

that is $\mu_l\left(\widetilde{q}, \widetilde{f}(t,u)\right) = 1$ for some $l \geq 1$ and the associated eigenfunction $u \in S_l^\nu$.

2. Let u be a solution to the bvp (21), according to the above comment, there is $k \geq 1$ such that $u \in S_k$. Let $(z_j)_{j=0}^{j=k}$ be the sequence and $t_q \in (0,1)$ be such that $q(t) > 0$ for all $t \geq t_q$. Set $t^* = \max(t_q, z_{k-1})$ and let $y_j \in (z_{k-1}, 1)$ be such that $u'(y_j) = 0$. We have then for all $t \geq t^*$

$$-u'(t) + \int_{y_j}^t q(s)u(s)ds = \int_{y_j}^t u(s)f(s,u(s))ds \tag{69}$$

leading to

$$\left|\int_{y_j}^{t} q(s)u(s)ds\right| = \left|u'(t)\right| + \int_{y_j}^{t} \left|u(s)f(s,u(s))\right| ds < \infty.$$

We deduce from the above inequality for both the cases $u > 0$ in $(z_{k-1}, 1)$ and $u < 0$ in $(z_{k-1}, 1)$ that

$$\int_{y_j}^{1} q(s)u(s)ds = \lim_{t \to \to 1} \int_{y_j}^{t} q(s)u(s)ds < \infty.$$

This proves that if u is a solution to the bvp (21) then $\int_0^1 q(s)u(s)ds$ converges. Therefore, we obtain from (69) that

$$\begin{aligned} \lim_{t\to 1} u'(t) &= \lim_{t\to 1} \left(\int_{y_j}^{t} q(s)u(s)ds - \int_{y_j}^{t} u(s)f(s,u(s))ds \right) \\ &= \int_{y_j}^{1} q(s)u(s)ds - \int_{y_j}^{1} u(s)f(s,u(s))ds. \end{aligned}$$

3. Let $q \in Q$, notice that if for some $m \in \Gamma^+$ and $l \geq 1$ $\mu_l(q,m) = 0$, then $\mu_l(q,\chi) = 0$ for all $\chi \in \Gamma^+$. Therefore, if $\mu_l(q,m) > 0$ (resp. < 0) for some $m \in \Gamma^+$ and $l \geq 1$ then $\mu_l(q,\chi) > 0$ (resp. < 0) for all $\chi \in \Gamma^+$. Indeed, if $\mu_l(q,\chi_0) > 0$ and $\mu_l(q,\chi_1) < 0$ for some $\chi_0, \chi_1 \in \Gamma^+$ and $l \geq 1$, then the continuity of the mapping

$$\mu_l(q,\cdot) : \{(1-r)\,\chi_0 + r\chi_1 : r \in [0,1]\} \to \mathbb{R}$$

leads to the existence of $r_0 \in (0,1)$ such that $\mu_l(q,(1-r_0)\,\chi_0 + r_0\chi_1) = 0$, then to the contradiction $\mu_l(q,\chi) = 0$ for all $\chi \in \Gamma^+$.

4. Let $q \in Q^+$ and $\chi_0 \in \Gamma^+$. The operator L_{q,χ_0} is then positive and we have for all $l \geq 1$

$$\mu_l(q,\chi_0) \geq \mu_1(q,\chi_0) = \frac{1}{r(L_{q,\chi_0})} > 0.$$

Therefore, $q \in Q^+$ is a particular situation where Assertion 3 in Lemmas 19 and 20 and Assertions 2 and 4 in Lemma 21 are satisfied.

Acknowledgements. The authors are thankful to the anonymous referee for his careful reading of the manuscript, which led to a substantial improvement of the original manuscript. .

References

[1] A. Benmezaï, W. Esserhane and J. Henderson, *Nodal solution for singular second order boundary value problems*, Electron. J. Differential Equations **2014** (2014), No. 156, 1-39.

[2] A. Benmezaï W. Esserhane, J. Henderson, *Sturm-Liouville BVPs with caratheodory nonlinearities,* Electron. J. Differential Equations, Vol. 2016 (2016), No. 298, pp. 1-49.

[3] A. Benmezaï, *Fixed point theorems in cones under local conditions,* Fixed Point Theory, **18** (2017), No. 1, 107-126.

[4] H. Berestycki, *On some non-linear Sturm-Liouville boundary value problems*, J. Differential Equations **26** (1977), 375-390.

[5] H. Berestycki, *Le nombre de solution de certains problèmes semi-linéaires élliptiques*, J. Functional Analysis 40 (1981), 1-29.

[6] P. A. Binding and B. P. Rynne, *Half-eigenvalues of periodic Sturm-Liouville problems*, J. Differential Equations **206** (2004), No. 2, 280-305.

[7] Y. Cui, J. Sun and Y. Zou, *Global bifurcation and multiple results for Sturm-Liouville boundary value problems*, J. Comput. Appl. Math. **235** (2011), 2185-2192.

[8] G. Dai, R. Ma and J. Xu, *Global bifurcation and nodal solutions of N-dimensional p-Lapalcian in unit ball,* Appl. Anal. **92** (2013), No. 7, 1345-1356.

[9] F. Genoud, *Bifurcation from infinity for an asymptotically linear boundary value problem on the half line*, Nonlinear Anal. **74** (2011), 4533-4543.

[10] F. Genoud and B. P. Rynne, *Half eigenvalues and the Fucik spectrum of multipoint boundary value problems*, J. Differential Equations **252**, No. 9 (2012), 5076-5095.

[11] F. Genoud and B. P. Rynne, Second order multi-point problems with variable coefficients, Nonlinear Anal. **74**, No. 18 (2011), 7269-7284.

[12] R. Ma and X. Han, Existence of nodal solutions of nonlinear eigenvalue problems with indefinite weight function, Nonlinear Anal. **71** (2009), 2119-2125.

[13] R. Ma and D. O'Regan, Nodal solutions for second order m-point boundary value problems with nonlinearities across several eigenvalues, Nonlinear Anal. **64** (2006), 1562-1577.

[14] R. Ma and B. Thompson, *A note on bifurcation from an interval*, Nonlinear Anal. **62** (2005), 743-749.

[15] R. Ma and B. Thompson, *Multiplicity results for second-order two point boundary value problems with superlinear or sublinear nonlinearities*, J. Math. Anal. Appl. **303** (2005), 726-735.

[16] R. Ma and B. Thompson, *Multiplicity results for second-order two point boundary value problems with nonlinearities across several eigenvalues*, Appl. Math. Lett. **18** (2005), 587-595.

[17] R. Ma and B. Thompson, *Nodal solutions for a nonlinear eigenvalue problems,* Nonlinear Anal. **59** (2004), 707-718.

[18] R. Ma and G. Dai, *Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity*, J. Funct. Anal. 265 (2013), 1443-1459.

[19] Y. Naïto and S. Tanaka, *On the existence of multiple solutions of the boundary value problem for nonlinear second order differential equations*, Nonlinear Anal. **56** (2004), 919-935.

[20] P. H. Rabinowitz, *Nonlinear Sturm-Liouville boundary value problems for second order ordinary differential equations,* Comm. Pure Appl. Math. **23** (1970), 939-962.

[21] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems,* J. Functional Anal. **7** (1971), 487-513.

[22] P. H. Rabinowitz, *On bifurcation from infinity,* J. Differential Equations **14** (1973), 462-475.

[23] B. P. Rynne, *Linear second order problems with Sturm-Liouville-type multipoint boundary conditions*, Electron. J. Differential Equations **2012** (2012), No. 146, 21 pp.

[24] B. P. Rynne, *Second order, three point, boundary value problem with jumping nonlinearities*, Nonlinear Anal. **68** (2008), 3294-3306.

[25] B. P. Rynne, *Spectral properties and nodal solutions for second-order m-point boundary value problems,* Nonlinear Anal. **67**, No. 12 (2007), 3318-3327.

[26] B. P. Rynne, *Half-eigenvalues of self adjoint, 2mth order differential operators and semilinear problems with jumping nonlinearities*, Differential Integral Equations **14** (2001), No. 9, 1129-1152.

[27] B. P. Rynne, *The Fucik spectrum of general Sturm-Liouville problems*, J. Differential Equations **161** (2000), 87-109.

Abdelhamid Benmezai
Higher National School of Mathematics,
Algiers, Algeria
email: *aehbenmezai@gmail.com*

Salima Mellal
Faculty of Mathematics, USTHB,
Algiers, Algeria
email: *salimamellal@yahoo.fr*

Johnny Henderson
Department of Mathematics,
Baylor University, Waco,
Texas 76798-7328,USA
email: *johnny_henderson@baylor.edu*