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SOLVABILITY AND STABILITY OF NEUTRAL CAPUTO-HADAMARD FRACTIONAL PANTOGRAPH-TYPE DIFFERENTIAL EQUATIONS

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Abstract. In this article, we discuss the existence, uniqueness and Ulam-type stability of solutions for Caputo-Hadamard fractional pantograph-type differential equations. The existence and uniqueness of solutions is establish by using Banach's fixed point theorem, while the existence of solutions is obtained from Krasnoselskii's fixed point theorem. We also present and study different types of Ulam stability. Finally, we give an illustrative example.

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1. Introduction

Pantograph equations arise in many applications such as number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics. Many scholars have carried out important studies on the theory of pantograph equations, for more details see [3, 12, 13, 14, 21]. Recently, fractional pantograph-type differential equations involving different fractional derivatives have been studied by different mathematicians, reader can refer to, [1, 5, 9, 17, 24, 25] and the references therein. On the other hand, existence and uniqueness of solutions to boundary value problems for pantograph fractional differential equations has attracted the attention of many researchers, see for example, [1, 2, 4, 6, 7]. Recently, Ulam stability for differential equations and fractional-type differential equations have been attracted by several authors [8, 10, 11, 16, 20, 22]. Moreover, Ulam stability of pantograph differential equations with fractional derivative has been studied by many scholars, see [9, 10, 13, 23, 24, 25] and the reference therein. In this work, we discuss the existence, uniqueness and Ulam-type stability of solutions for following neutral Caputo-Hadamard fractional pantograph-type differential equations:

$$\begin{cases} {}_{H}^{C}D^{\alpha}\left[{}_{H}^{C}D^{\beta}x\left(t\right)-\lambda g\left(t,x\left(\mu t\right)\right)\right]=f\left(t,x\left(t\right),x\left(\eta t\right)\right), \\ x\left(1\right)=\theta, \quad x\left(T\right)=\vartheta,\ \theta,\vartheta\in\mathbb{R}, \\ 1\leq t\leq T,\lambda>0,\ 0<\mu,\eta<1,0<\alpha,\beta\leq 1, \end{cases} \tag{1}$$

where ${}_{H}^{C}D^{\alpha}$ and ${}_{H}^{C}D^{\beta}$ are the Caputo-Hadamard type fractional derivatives, $g : J\times\mathbb{R}\to\mathbb{R}, f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

The rest of this work is organized as follows. In Section 2, we recall some definitions and lemma which are used throughout the paper. In Section 3, we discuss the existence and uniqueness of solutions for fractional boundary value problem (1). In section 4, we define and study the different types of Ulam stability for the fractional problem (1). In the last section, we give an illustrative example.

2. Preliminaries

In this section, we give notations, definitions and preliminary facts that will be used in the remainder of this work.

Definition 1. *[18] The Hadamard fractional integral of order ρ for a continuous function $h : [a,+\infty)\to\mathbb{R}$ is defined as*

$$_{H}I^{\rho}h\left(t\right)=\tfrac{1}{\Gamma(\rho)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{\rho-1}\frac{h\left(s\right)}{s}ds,\rho>0, \tag{2}$$

where $\log\left(.\right)=\log_{e}\left(.\right)$*, provided that the integral exist.*

Definition 2. *[15] For at least $n-$times differentiable function $h : [a,\infty)\to\mathbb{R}$ the Caputo-Hadamard fractional derivative of order ρ is defined as*

$${}_{H}^{C}D^{\rho}h(t)=\tfrac{1}{\Gamma(n-\rho)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{n-\rho-1}\delta^{n}\frac{h(s)}{s}ds, \tag{3}$$

where $n-1<\rho<n, n=[\rho]+1, \delta=t\frac{d}{dt}, [\rho]$ denotes the integer part of ρ and $\log\left(.\right)=\log_{e}\left(.\right)$.

Lemma 1. *[15] Let $x \in C_{\delta}^{n}([a,b],\mathbb{R})$. Then*

$$_{H}I^{\rho}\left({}_{H}^{C}D^{\rho}x\right)(t)=x(t)-\sum_{i=0}^{n-1}c_{i}(\log t)^{i},c_{i}\in\mathbb{R}, \tag{4}$$

where $C_{\delta}^{n}([a,b],\mathbb{R})=\left\{h:[a,b]\rightarrow\mathbb{R}:\delta^{n-1}h\in C([a,b],\mathbb{R})\right\}.$

We prove the following auxiliary lemma.

Lemma 2. *For a given $h(t)\in C(J,\mathbb{R})$, the solution of the fractional differential equation*

$${}_{H}^{C}D^{\alpha}\left[{}_{H}^{C}D^{\beta}x(t)-\lambda g(t,x(\mu t))\right]=h(t),t\in J,\lambda>0,0<\alpha,\beta\leq 1, \tag{5}$$

subject to the condition

$$x(1)=\theta,\quad x(T)=\vartheta. \tag{6}$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}h(s)\frac{ds}{s} \\ &+\frac{\lambda}{\Gamma(\beta)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}g(s,x(\mu s))\frac{ds}{s}+H_{x}(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} H_{x}(t) &= \theta+\frac{(\log t)^{\beta}}{(\log T)^{\beta}}\left[\frac{1}{\Gamma(\alpha+\beta)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}h(s)\frac{ds}{s}\right. \\ &\left.+\frac{\lambda}{\Gamma(\beta)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\beta-1}g(s,x(\mu s))\frac{ds}{s}+\theta-\vartheta\right]. \end{aligned} \tag{8}$$

Proof. Using Lemma 2, we obtain

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}h(s)\frac{ds}{s} \\ &+\frac{\lambda}{\Gamma(\beta)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}g(s,x(\mu s))\frac{ds}{s}+\frac{c_{0}}{\Gamma(\beta+1)}(\log t)^{\beta}+c_{1}. \end{aligned} \tag{9}$$

where $c_0, c_1 \in \mathbb{R}$.

From (6), we get $c_1 = \theta$ and

$$\begin{aligned} c_0 &= \frac{\Gamma(\beta+1)}{(\log T)^{\beta}} \left[\vartheta - \frac{1}{\Gamma(\alpha+\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha+\beta-1} h(s) \frac{ds}{s} \right. \\ & \left. - \frac{\lambda}{\Gamma(\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\beta-1} g(s, x(\mu s)) \frac{ds}{s} - \theta \right]. \end{aligned}$$

Substituting the value of c_0 and c_1 in (9) yields the solution (7).This completes the proof.

3. Existence and uniqueness results

We denote by $X = C(J, \mathbb{R})$ the Banach space of all continuous functions from J to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in J\}$.

In view of Lemma 2, we define an operator $O : X \to X$ as

$$\begin{aligned} Ox(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha+\beta-1} f(s, x(s), x(\eta s)) \frac{ds}{s} \\ & + \frac{\lambda}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} g(s, x(\mu s)) \frac{ds}{s} + H_x(t). \end{aligned} \tag{10}$$

The first result is concerned with the existence and uniqueness of solutions for the fractional problem (1) and is based on Banach's fixed point theorem.

Theorem 3. *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that:*

(H_1) : *There exists a constant $\omega > 0$ such that*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq \omega(|u_1 - v_1| + |u_2 - v_2|), t \in J, u_i, v_i \in \mathbb{R}, i = 1, 2.$$

(H_2) : *There exists a constant $\varpi > 0$ such that*

$$|g(t, u) - g(t, v)| \leq \varpi |u - v|, \ u, v \in C(J, \mathbb{R}).$$

If the inequality

$$2 \frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \omega + \frac{|\lambda| (\log T)^{\beta}}{\Gamma(\beta+1)} \varpi < \frac{1}{2}, \tag{11}$$

is valid, then problem (1) has a unique solution on J.

Proof. Let us fix $L=\sup_{t\in[t,T]}|f(t,0,0)|<\infty$ and $M=\sup_{t\in[t,T]}|g(t,0)|<\infty$ and define

$$r\geq\frac{2\left(\frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}L+\frac{|\lambda|(\log T)^{\beta}}{\Gamma(\beta+1)}M+|\theta|\right)+|\vartheta|}{1-2\left(\frac{2(\log T)^{\alpha+\beta}\omega}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|(\log T)^{\beta}\varpi}{\Gamma(\beta+1)}\right)},$$

we show that $OB_r\subset B_r$, where $B_r=\{x\in X:\|x\|\leq r\}$.

For $x\in B_r$, we find the following estimates based on the hypothesis (H_1) and (H_2) :

$$\begin{aligned}|f(t,x(t),x(\eta t))| &\leq |f(t,x(t),x(\eta t))-f(t,0,0)|+|f(t,0,0)|\\ &\leq 2\omega\|x\|+L\leq 2\omega r+L,\end{aligned}\tag{12}$$

and

$$|g(t,x(\mu t))|\leq||g(t,x(\mu t))|-g(t,0)|+|g(t,0)|\leq\varpi\|x\|+M\leq\varpi r+M.\tag{13}$$

Thanks to(12) and (13), we obtain

$$\begin{aligned}|Ox(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|f(s,x(s),x(\eta s))|\frac{ds}{s}\\ &\quad+\frac{|\lambda|}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}|g(s,x(\mu s))|\frac{ds}{s}+|H_x(t)|\\ &\leq \frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}(2\omega r+L)+\frac{(\log T)^{\beta}}{\Gamma(\beta+1)}(\varpi r+M)+|H_x(t)|,\end{aligned}\tag{14}$$

where $H_x(t)$ is given by (8).

Then

$$\begin{aligned}|H_x(t)| &\leq |\theta|+\frac{(\log t)^{\beta}}{(\log T)^{\beta}}\left[\frac{1}{\Gamma(\alpha+\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}|f(s,x(s),x(\eta s))|\frac{ds}{s}\right.\\ &\quad\left.+\frac{|\lambda|}{\Gamma(\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\beta-1}|g(s,x(\mu s))|\frac{ds}{s}+|\theta|+|\vartheta|\right]\\ &\leq \frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}(2\omega r+L)+\frac{|\lambda|(\log T)^{\beta}}{\Gamma(\beta+1)}(\varpi r+M)+2|\theta|+|\vartheta|.\end{aligned}\tag{15}$$

Now, by (14) and (15), we can write

$$|Ox(t)|\leq\left(\frac{4(\log T)^{\alpha+\beta}\omega}{\Gamma(\alpha+\beta+1)}+\frac{2(\log T)^{\beta}\varpi}{\Gamma(\beta+1)}\right)r$$

$$+\frac{2\left(\log T\right)^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)}L+\frac{2\left|\lambda\right|\left(\log T\right)^{\beta}}{\Gamma\left(\beta+1\right)}M+2\left|\theta\right|+\left|\vartheta\right|\leq r$$

which implies that $OB_r \subset B_r$. Now, for $x, y \in B_r$ and for any $t \in J$, we get

$$\begin{aligned} & \left|Ox\left(t\right)-Oy\left(t\right)\right| & (16)\\ \leq\ & \frac{1}{\Gamma\left(\alpha+\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\left|f\left(s,x\left(s\right),x\left(\eta s\right)\right)-f\left(s,y\left(s\right),y\left(\eta s\right)\right)\right|\frac{ds}{s} \\ & \frac{\left|\lambda\right|}{\Gamma\left(\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}\left|g\left(s,x\left(\mu s\right)\right)-g\left(s,y\left(\mu s\right)\right)\right|\frac{ds}{s} \\ \leq\ & \left(\frac{2\left(\log T\right)^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)}\omega+\frac{\left|\lambda\right|\left(\log T\right)^{\beta}}{\Gamma\left(\beta+\right)}\varpi\right)\left\|x-y\right\|, \end{aligned}$$

and

$$\begin{aligned} & \left|H_x\left(t\right)-H_y\left(t\right)\right| & (17)\\ \leq\ & \frac{\left(\log t\right)^{\beta}}{\left(\log T\right)^{\beta}}\left[\frac{1}{\Gamma\left(\alpha+\beta\right)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}\left|f\left(s,x\left(s\right),x\left(\eta s\right)\right)\right.\right. \\ & -f\left(s,y\left(s\right),y\left(\eta s\right)\right)\Big|\frac{ds}{s} \\ & \left.+\frac{\left|\lambda\right|}{\Gamma\left(\beta\right)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\beta-1}\left|g\left(s,x\left(\mu s\right)\right)-g\left(s,y\left(\mu s\right)\right)\right|\frac{ds}{s}\right] \\ \leq\ & \left(\frac{2\left(\log T\right)^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)}\omega+\frac{\left|\lambda\right|\left(\log T\right)^{\beta}}{\Gamma\left(\beta+1\right)}\varpi\right)\left\|x-y\right\|. \end{aligned}$$

From the above inequalities, we get

$$\begin{aligned} \left\|Ox-Oy\right\| &\leq \left(\frac{2\left(\log T\right)^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)}\omega+\frac{\left|\lambda\right|\left(\log T\right)^{\beta}}{\Gamma\left(\beta+1\right)}\varpi\right)\left\|x-y\right\|+\left\|H_x-H_y\right\| \\ &\leq \left(\frac{4\left(\log T\right)^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)}\omega+\frac{2\left|\lambda\right|\left(\log T\right)^{\beta}}{\Gamma\left(\beta+1\right)}\varpi\right)\left\|x-y\right\|. \end{aligned}$$

which shows that O is a contraction in view of the assumptions $2\frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega + \frac{|\lambda|(\log T)^{\beta}}{\Gamma(\beta+1)}\varpi < \frac{1}{2}$. Hence, by Banach's fixed point theorem, the operator O has aunique fixed point which corresponds to the unique solution of problem (1).This completes the proof.

In the next result, we show the existence of solutions for the problem (1) by means of Krasnoselskii's fixed point theorem [19].

Theorem 4. *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the condition (H_1) and (H_2). In addition, we assume that:*

(H_3) : *For each $t \in J, (x, y) \in \mathbb{R}^2$ and $k_f, k_g \in C([0, T], \mathbb{R}^+)$, we have*

$$|f(t, x, y)| \leq k_f(t), \qquad |g(t, x)| \leq k_g(t).$$

If

$$\frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega < \frac{1}{2} - \frac{|\lambda| (\log T)^{\beta}}{2\Gamma(\beta+1)}\varpi. \tag{18}$$

Then the problem (1) has at least one solution on $[1, T]$.

Proof. Let us fix

$$\sigma \geq 2\left(\frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|k_f\| + \frac{|\lambda| (\log T)^{\beta}}{\Gamma(\beta+1)}\|kg\| + |\theta|\right) + |\vartheta|,$$

where $\|k_f\| = \sup_{t\in[0,T]}|k_f(t)|$ and $\|kg\| = \sup_{t\in[0,T]}|k_g(t)|$.

On $B_\sigma = \{x \in X : \|x\| \leq \sigma\}$, we define the operators O_1 and O_2, as

$$\begin{aligned} O_1 x(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} f(s, x(s), x(\eta s))\frac{ds}{s} \\ &\quad + \frac{\lambda}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s, x(\mu s))\frac{ds}{s}, \\ O_2 x(t) &= H_x(t). \end{aligned} \tag{19}$$

For $x, y \in B_\sigma$ and $t \in J$, we have

$$\begin{aligned} |O_1 x(t) + O_2 y(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} |f(s, x(s), x(\eta s))|\frac{ds}{s} \\ &\quad + \frac{|\lambda|}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} |g(s, x(\mu s))|\frac{ds}{s} + |H_y(t)|. \end{aligned}$$

Using (H_3), we obtain

$$\begin{aligned} &\|O_1(x) + O_2(y)\| \\ \leq\ & \frac{2(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|N_f\| + \frac{2|\lambda| (\log T)^{\beta}}{\Gamma(\beta+1)}\|Ng\| + 2|\theta| + |\vartheta| \leq \sigma. \end{aligned}$$

Thus $\|O_1(x)+O_2(y)\| \in B_\sigma$.

Next we prove that O_2 is a contraction. Let $x,y \in X$. Then for each $t \in J$, we have

$$\|O_2(x)-O_2(y)\| \leq \|H_x - H_y\|.$$

Thanks to (H_1) and (H_2), we get

$$\|O_2(x)-O_2(y)\| \leq \left(\frac{2(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega + \frac{|\lambda|(\log T)^{\beta}}{\Gamma(\beta+1)}\varpi\right)\|x-y\|.$$

Hence O_2 is a contraction. Continuities of f and g imply that the operator O_1 is continuous.

Also O_1 is uniformly bounded on B_σ as

$$\|O_1(x)\| \leq \frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|N_f\| + \frac{|\lambda|(\log T)^{\beta}}{\Gamma(\beta+1)}\|N_g\| < \infty.$$

Now we prove the compactness of the operator O_1.

We define $\sup_{(t,x,y)\in J\times B_\sigma\times B_\sigma}|f(t,x,y)| = N_1$ and $\sup_{(t,x)\in J\times B_\sigma}|g(t,x)| = N_2$. Let $t_1,t_2 \in J$ such that $t_2 < t_1$, then, we have

$$\begin{aligned} & |O_1x(t_1) - O_1x(t_2)| \\ \leq\ & \frac{N_1}{\Gamma(\alpha+\beta+1)}\left[(\log t_1)^{\alpha+\beta} - (\log t_2)^{\alpha+\beta}\right] + \frac{N_2|\lambda|}{\Gamma(\beta+1)}\left[(\log t_1)^{\beta} - (\log t_2)^{\beta}\right], \end{aligned}$$

which is independent of x and tends to zero as $t_1 - t_2 \to 0$. Thus O_1 is relatively compact on B_σ. Hence, by the Arzela-Ascoli theorem, O_1 is compact on B_σ. Consequently, by the Krasnoselskii's fixed point theorem, O has a solution on $[1,T]$. The proof is completed.

4. Stability results

In this section, we will study different types of Ulam stability for the considered fractional boundary value problem (1).

Definition 3. *The fractional boundary value problem (1) is Ulam-Hyers stable if there exists a real number $c_{f,g} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in X$ of the inequality*

$$\left|{}_{H}^{C}D^{\alpha}\left[{}_{H}^{C}D^{\beta}y(t) - \lambda g(t,y(\mu t))\right] - f(t,y(t),y(\eta t))\right| \leq \varepsilon, t \in J, \tag{20}$$

there exists a solution $x \in X$ of fractional boundary value problem (1) with

$$|y(t)-x(t)| \leq c_{f,g}\varepsilon, t \in J.$$

Definition 4. *The fractional boundary value problem (1) is generalized Ulam-Hyers stable if there exists* $\psi_{f,g} \in C(\mathbb{R}_+, \mathbb{R}_+), \psi_{f,g}(0) = 0$, *such that for each solution* $y \in X$ *of the inequality (20) there exists a solution* $x \in X$ *of the fractional boundary value problem (1) with*

$$|y(t) - x(t)| \leq \psi_{f,g}(\varepsilon), t \in J.$$

Definition 5. *The fractional boundary value problem (1) is Ulam-Hyers-Rassias stable with respect to* $\varphi \in X$ *if there exists a real number* $c_{f,g} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $y \in X$ *of the inequality*

$$\left| {}_{H}^{C}D^{\alpha} \left[{}_{H}^{C}D^{\beta} y(t) - \lambda g(t, y(\mu t)) \right] - f(t, y(t), y(\eta t)) \right| \leq \varepsilon \varphi(t), t \in J, \tag{21}$$

there exists a solution $x \in X$ *of problem (1) with*

$$|y(t) - x(t)| \leq c_{f,g} \varepsilon \varphi(t), t \in J.$$

Definition 6. *The fractional boundary value problem (1) is generalized Ulam-Hyers-Rassias stable with respect to* $\varphi \in X$ *if there exists a real number* $c_{f,g,\varphi} > 0$ *such that for each solution* $y \in X$ *of the inequality*

$$\left| {}_{H}^{C}D^{\alpha} \left[{}_{H}^{C}D^{\beta} y(t) - \lambda g(t, y(\mu t)) \right] - f(t, y(t), y(\eta t)) \right| \leq \varphi(t), t \in J, \tag{22}$$

there exists a solution $x \in X$ *of problem (1) with*

$$|y(t) - x(t)| \leq c_{f,g,\varphi} \varphi(t), t \in J.$$

Remark 1. *A function* $y \in X$ *is a solution of the inequality (20) if and only if there exists a function* $\psi : J \to \mathbb{R}$ *such that*

(1) : $|\psi(t)| \leq \varepsilon, t \in J.$

(2) : $\left| {}_{H}^{C}D^{\alpha} \left[{}_{H}^{C}D^{\beta} y(t) - \lambda g(t, y(\mu t)) \right] = f(t, y(t), y(\eta t)) \right| + \psi(t), t \in J, \lambda < 0, 0 < \mu, \eta < 1.$

Theorem 5. *Let* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $g : J \times \mathbb{R} \to \mathbb{R}$ *be continuous, suppose that* (H_1) *and* (H_2) *hold. Then if*

$$2 \frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} \omega < 1 - |\lambda| \frac{(\log T)^{\beta}}{\Gamma(\beta + 1)} \varpi, \tag{23}$$

the fractional boundary value problem (1) is Ulam-Hyers stable and consequently, generalized Ulam-Hyers stable.

Proof. Let $y \in X$ be a solution of the inequality (20), i.e.

$$\left| {}_{H}^{C}D^{\alpha}\left[{}_{H}^{C}D^{\beta}y\left(t\right)-\lambda g\left(t,y\left(\mu t\right)\right)\right]-f\left(t,y\left(t\right),y\left(\eta t\right)\right)\right| \leq \varepsilon,\ t\in J,$$

and let us denote by $x \in X$ the unique solution of the problem

$$\begin{cases} {}_{H}^{C}D^{\alpha}\left[{}_{H}^{C}D^{\beta}x\left(t\right)-\lambda g\left(t,x\left(\mu t\right)\right)\right]-f\left(t,x\left(t\right),x\left(\eta t\right)\right),t\in J, \\ \\ x\left(1\right)=y\left(1\right), \quad x\left(T\right)=y\left(T\right), \end{cases}$$

where $\lambda > 0, 0 < \alpha, \beta \leq 1$ and $0 < \mu, \eta < 1$.

By Lemma 2, we can write

$$\begin{aligned} x\left(t\right) &= \frac{1}{\Gamma\left(\alpha+\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}f\left(s,x\left(s\right),x\left(\eta s\right)\right)\frac{ds}{s} \\ &+\frac{\lambda}{\Gamma\left(\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}g\left(s,x\left(\mu s\right)\right)\frac{ds}{s}+H_{x}\left(t\right), \end{aligned}$$

by integration of the inequality (20), we obtain

$$\begin{aligned} &\left| y\left(t\right)-H_{y}\left(t\right)-\frac{1}{\Gamma\left(\alpha+\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}f\left(s,y\left(s\right),y\left(\eta s\right)\right)\frac{ds}{s}\right. \\ &\left.-\frac{\lambda}{\Gamma\left(\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}g\left(s,y\left(\mu s\right)\right)\frac{ds}{s}\right| \\ \leq\ & \frac{\varepsilon}{\Gamma\left(\alpha+\beta+1\right)}\left(\log t\right)^{\alpha+\beta}\leq\frac{\varepsilon}{\Gamma\left(\alpha+\beta+1\right)}\left(\log T\right)^{\alpha+\beta}, \end{aligned}$$

where

$$H_{x}\left(t\right)=\frac{c_{0}}{\Gamma\left(\beta+1\right)}\left(\log t\right)^{\beta}+c_{1} \text{ and } H_{y}\left(t\right)=\frac{c_{2}}{\Gamma\left(\beta+1\right)}\left(\log t\right)^{\beta}+c_{3}$$

On the other hand, if $x\left(1\right)=y\left(1\right)$ and $x\left(T\right)=y\left(T\right)$ then $c_0 = c_2$ and $c_1 = c_3$.

We have, for each $t \in [1, T]$

$$
\begin{aligned}
&|y(t) - x(t)| \\
&\leq \left| y(t) - H_y(t) - \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} f(s, y(s), y(\eta s)) \frac{ds}{s} \right. \\
&\left. - \frac{\lambda}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} g(s, y(\mu s)) \frac{ds}{s} \right| \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} |f(s, y(s), y(\eta s)) - f(s, x(s), x(\eta s))| \frac{ds}{s} \\
&+ \frac{|\lambda|}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} |g(s, y(\mu s)) - g(s, x(\mu s))| \frac{ds}{s}
\end{aligned}
$$

Thanks to (H_1) and (H_2), we obtain

$$
\begin{aligned}
|y(t) - x(t)| &\leq \frac{\varepsilon}{\Gamma(\alpha+\beta+1)} (\log T)^{\alpha+\beta} \\
&+ \left(\frac{2 (\log T)^{\alpha+\beta} \omega}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| (\log T)^{\beta} \varpi}{\Gamma(\beta+1)} \right) \|x - y\|,
\end{aligned}
$$

which implies that

$$
\|x - y\| \leq \frac{\varepsilon (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1) \left[1 - \left(\frac{2(\log T)^{\alpha+\beta} \omega}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| (\log T)^{\beta} \varpi}{\Gamma(\beta+1)} \right) \right]}.
$$

Then, for each $t \in [1, T]$

$$
|x(t) - y(t)| \leq \frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1) \left[1 - \left(\frac{2(\log T)^{\alpha+\beta} \omega}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| (\log T)^{\beta} \varpi}{\Gamma(\beta+1)} \right) \right]} \varepsilon = c_{f,g} \varepsilon.
$$

So, the fractional boundary value problem (1) is Ulam-Hyers stable. By putting $\varphi(\varepsilon) = \gamma \varepsilon, \varphi(0) = 0$ yields that the fractional boundary value problem (1) generalized Ulam-Hyers stable.

Theorem 6. *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, g : J \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $(H_1), (H_2)$ and (23) hold. In addition, we assume that:*

(H_4) : *There exists an function $\varphi \in C(J, \mathbb{R}_+)$ and there exists $\eta_\varphi > 0$ such that for any $t \in J$*

$$
\frac{1}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} \varphi(s) \frac{ds}{s} \leq \eta_\varphi \varphi(t). \tag{24}
$$

Then the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable.

Proof. Let $y \in X$ be a solution of the inequality (21), i.e.

$$\left| {}_{H}^{C}D^{\alpha}\left[{}_{H}^{C}D^{\beta}y(t) - \lambda g(t, y(\mu t))\right] - f(t, y(t), y(\eta t))\right| \leq \varepsilon\varphi(t), t \in J,$$

and let us denote by $x \in X$ the unique solution of the problem

$$\begin{cases} {}_{H}^{C}D^{\alpha}\left[{}_{H}^{C}D^{\beta}x(t) - \lambda g(t, x(\mu t))\right] - f(t, x(t), x(\eta t)), t \in J, \\ \\ x(1) = y(1), \quad x(T) = y(T), \end{cases}$$

where $\lambda > 0, 0 < \alpha, \beta \leq 1$ and $0 < \mu, \eta < 1$.

Thanks to Lemma 2, we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} f(s, x(s), x(\eta s))\frac{ds}{s} \\ &\quad + \frac{\lambda}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s, x(\mu s))\frac{ds}{s} + H_x(t). \end{aligned}$$

Now, by integration of the inequality (21), we obtain

$$\begin{aligned} &\left| y(t) - H_y(t) - \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} f(s, y(s), y(\eta s))\frac{ds}{s} \right. \\ &\left. -\frac{\lambda}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s, y(\mu s))\frac{ds}{s}\right| \\ \leq &\ \frac{\varepsilon}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} \varphi(s)\frac{ds}{s}. \end{aligned} \tag{25}$$

Thanks to $(H_1), (H_2)$ and (H_4), we can write

$$\begin{aligned}
&|y(t)-x(t)| \\
&\leq \left| y(t) - H_y(t) - \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} f(s, y(s), y(\eta s)) \frac{ds}{s} \right. \\
&\left. -\frac{\lambda}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} g(s, y(\mu s)) \frac{ds}{s} \right| \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} |f(s, y(s), y(\eta s)) - f(s, x(s), x(\eta s))| \frac{ds}{s} \\
&+ \frac{|\lambda|}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} |g(s, y(\mu s)) - g(s, x(\mu s))| \frac{ds}{s} \\
&\leq \varepsilon \eta_\varphi \varphi(t) + \left(\frac{2(\log T)^{\alpha+\beta}\omega}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| (\log T)^\beta \varpi}{\Gamma(\beta+1)} \right) \|x-y\|,
\end{aligned}$$

which implies that

$$\|y-x\| \left[1 - \left(\frac{2(\log T)^{\alpha+\beta}\omega}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| (\log T)^\beta \varpi}{\Gamma(\beta+1)} \right) \right] \leq \varepsilon \eta_\varphi \varphi(t).$$

Then, for each $t \in [1, T]$

$$|x(t) - y(t)| \leq \frac{\varepsilon \lambda_\varphi}{\left[1 - \left(\frac{2(\log T)^{\alpha+\beta}\omega}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| (\log T)^\beta \varpi}{\Gamma(\beta+1)} \right) \right]} \varphi(t) = \varepsilon c_{f,g,\varphi} \varphi(t).$$

Hence, the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable.

5. Example

Consider the neutral fractional pantograph equation with Caputo-Hadamard type fractional derivatives

$$\begin{cases} {}^C_H D^{\frac{1}{2}} \left[{}^C_H D^{\frac{2}{3}} x(t) - \frac{1}{23} \left(\frac{3}{16} \sin(t)\, x\left(\frac{t}{3}\right) + \frac{2}{3} \right) \right] \\ = \frac{1}{\sqrt{16+t^2}} \cos(t)\, x(t) + \frac{1}{4} x\left(\frac{t}{2}\right) + \frac{3}{5}, \ t \in [1, e], \\ x(1) = \sqrt{3}, \quad x(e) = \frac{5}{7}, \end{cases} \tag{26}$$

For this example, we have $\alpha = \frac{1}{2}, \beta = \frac{2}{3}, \lambda = \frac{1}{23}, \mu = \frac{1}{3}, \eta = \frac{1}{2}, T = e$, $g(t,x) = \frac{3}{16}\sin(t)\,x + \frac{2}{3}$ and $f(t,x,y) = \frac{1}{\sqrt{16+t^2}}\cos(t)\,x + \frac{1}{4}y + \frac{3}{5}$.

For each $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $t \in [1, e]$, we have

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq \frac{1}{4}\left(|x_1 - x_2| + |y_1 - y_2|\right).$$

Hence the condition (H_1) holds with $\omega = \frac{1}{4}$. Also, for any $x_1, y_1 \in \mathbb{R}$, we have

$$|g(x_1) - g(y_1)| \leq \frac{3}{16}|x_1 - y_1|.$$

So, (H_2) is satisfied with $\varpi = \frac{3}{16}$.

It follows that

$$2\frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega + \frac{|\lambda|(\log T)^{\beta}}{\Gamma(\beta+1)}\varpi \simeq 0.256\,13 < \frac{1}{2},$$

By Theorem 3, we conclude that the problem (26) has a unique solution on $[1, e]$.

Furthermore, we have

$$2\frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \simeq 0.461\,96 < 1 - \frac{|\lambda|(\log T)^{\beta}}{\Gamma(\beta+1)}\varpi \simeq 0.990\,97,$$

and from Theorem 5, the fractional problem (26) is Ulam-Hyers stable.

Let $\varphi(t) = t^2$. Then

$$\frac{1}{\Gamma(\alpha+\beta)}\int_1^t (\log\frac{t}{s})^{\alpha+\beta-1}\varphi(s)\frac{ds}{s} \leq \frac{3}{\Gamma\left(\frac{25}{6}\right)}t^2 = \eta_\varphi\varphi(t).$$

Thus, the hypothesis (H_4) of Theorem 6 is satisfied with $\varphi(t) = t^2$ and $\eta_\varphi = \frac{3}{\Gamma(\frac{25}{6})}$. Therefore, by Theorem 6 the problem (26) is Ulam-Hyers-Rassias stable.

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