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SOME YOUNG AND HÖLDER TYPE OPERATOR INEQUALITIES

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Abstract. In this paper we obtain some Young and Hölder type inequalities for the weighted geometric mean of positive operators on Hilbert spaces.

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1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notation

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted geometric mean.* When $\nu = \frac{1}{2}$ we write $A\sharp B$ for brevity.

In [4] the authors obtained the following Hölder's type inequality for the weighted geometric mean:

$$\left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \le \langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \tag{1}$$

for any $x \in H$.

Moreover, if $0 < m_1 I \le A \le M_1 I$, $0 < m_2 I \le B \le M_2 I$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, I is the identity operator and

$$\lambda(p; m, M) := \left[\frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p}(mM^p - Mm^p)^{1/q}}\right]^p$$

for $0 < m < M$, then the following reverse inequality also holds:

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \le \lambda^{1/p}\left(p; \frac{m_1}{M_2^{q-1}}, \frac{M_1}{m_2^{q-1}}\right)\left\langle B^q \sharp_{1/p} A^p x, x\right\rangle, \tag{2}$$

for any $x \in H$.

In particular, one can obtain from (2) the following noncommutative version of *Greub-Rheinboldt inequality*

$$\left\langle A^2 \sharp B^2 x, x\right\rangle \leq \left\langle A^2 x, x\right\rangle^{1/2} \left\langle B^2 x, x\right\rangle^{1/2} \leq \frac{m_1 m_2 + M_1 M_2}{2\sqrt{m_1 m_2 M_1 M_2}} \left\langle A^2 \sharp B^2 x, x\right\rangle \qquad (3)$$

for any $x \in H$.

Furthermore, if A and B are replaced by $C^{1/2}$ and $C^{-1/2}$ in (3), then we get the *Kantorovich inequality* [15]

$$(1 \leq) \left\langle Cx, x\right\rangle^{1/2} \left\langle C^{-1} x, x\right\rangle^{1/2} \leq \frac{m+M}{2\sqrt{mM}},\ x \in H \text{ with } \|x\| = 1,$$

provided $mI \leq C \leq MI$ for some $0 < m < M$.

For various related inequalities, see [1]-[2], [3]-[10], [12]-[13] and [14]-[17].

In this paper we obtain some new Young and Hölder type inequalities for the weighted geometric mean of positive operators on Hilbert spaces.

2. Some Young and Hölder Type Results

The following simple Young operator inequality follows from (1):

Proposition 1. *Let A, B be positive invertible operators and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$B^q \sharp_{1/p} A^p \leq \frac{1}{p} A^p + \frac{1}{q} B^q. \qquad (4)$$

In particular, we have

$$A^2 \sharp B^2 \leq \frac{1}{2}\left(A^2 + B^2\right). \qquad (5)$$

Proof. From (1) and the geometric mean-arithmetic mean inequality we have

$$\begin{aligned} \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle &\leq \left\langle A^p x, x\right\rangle^{1/p} \left\langle B^q x, x\right\rangle^{1/q} \leq \frac{1}{p}\left\langle A^p x, x\right\rangle + \frac{1}{q}\left\langle B^q x, x\right\rangle \\ &= \left\langle \left(\frac{1}{p} A^p + \frac{1}{q} B^q\right) x, x\right\rangle \end{aligned}$$

for any $x \in H$, which implies (4).

The following Hölder's type result for sums of operators holds:

Theorem 1. *Let A_k, B_k, $k \in \{1,...,n\}$ be positive invertible operators and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left\| \sum_{k=1}^{n} p_k B_k^q \sharp_{1/p} A_k^p \right\| \leq \left\| \sum_{k=1}^{n} p_k A_k^p \right\|^{1/p} \left\| \sum_{k=1}^{n} p_k B_k^q \right\|^{1/q}, \tag{6}$$

for any positive sequence p_k, $k \in \{1,...,n\}$.

In particular, we have

$$\left\| \sum_{k=1}^{n} p_k B_k^2 \sharp A_k^2 \right\|^2 \leq \left\| \sum_{k=1}^{n} p_k A_k^2 \right\| \left\| \sum_{k=1}^{n} p_k B_k^2 \right\|. \tag{7}$$

Proof. From (1) we have

$$\begin{aligned} \left\langle \sum_{k=1}^{n} p_k B_k^q \sharp_{1/p} A_k^p x, x \right\rangle &= \sum_{k=1}^{n} p_k \left\langle B_k^q \sharp_{1/p} A_k^p x, x \right\rangle \\ &\leq \sum_{k=1}^{n} p_k \left\langle A_k^p x, x \right\rangle^{1/p} \left\langle B_k^q x, x \right\rangle^{1/q} \end{aligned} \tag{8}$$

for any $x \in H$.

Using the weighted discrete Hölder inequality we have

$$\begin{aligned} &\sum_{k=1}^{n} p_k \left\langle A_k^p x, x \right\rangle^{1/p} \left\langle B_k^q x, x \right\rangle^{1/q} \\ &\leq \left(\sum_{k=1}^{n} p_k \left[\left\langle A_k^p x, x \right\rangle^{1/p} \right]^p \right)^{1/p} \left(\sum_{k=1}^{n} p_k \left[\left\langle B_k^q x, x \right\rangle^{1/q} \right]^q \right)^{1/q} \\ &= \left(\sum_{k=1}^{n} p_k \left\langle A_k^p x, x \right\rangle \right)^{1/p} \left(\sum_{k=1}^{n} p_k \left\langle B_k^q x, x \right\rangle \right)^{1/q} \\ &= \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^{1/p} \left\langle \sum_{k=1}^{n} p_k B_k^q x, x \right\rangle^{1/q} \end{aligned} \tag{9}$$

for any $x \in H$.

Then by (8) and (9) we get

$$\left\langle \sum_{k=1}^{n} p_k B_k^q \sharp_{1/p} A_k^p x, x \right\rangle \leq \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^{1/p} \left\langle \sum_{k=1}^{n} p_k B_k^q x, x \right\rangle^{1/q} \tag{10}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (10) we have

$$\begin{aligned}
\left\| \sum_{k=1}^{n} p_k B_k^q \sharp_{1/p} A_k^p \right\| &= \sup_{\|x\|=1} \left\langle \sum_{k=1}^{n} p_k B_k^q \sharp_{1/p} A_k^p x, x \right\rangle \\
&\leq \sup_{\|x\|=1} \left\{ \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^{1/p} \left\langle \sum_{k=1}^{n} p_k B_k^q x, x \right\rangle^{1/q} \right\} \\
&\leq \sup_{\|x\|=1} \left\{ \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^{1/p} \right\} \sup_{\|x\|=1} \left\{ \left\langle \sum_{k=1}^{n} p_k B_k^q x, x \right\rangle^{1/q} \right\} \\
&= \left\{ \sup_{\|x\|=1} \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \right\}^{1/p} \left\{ \sup_{\|x\|=1} \left\langle \sum_{k=1}^{n} p_k B_k^q x, x \right\rangle \right\}^{1/q} \\
&= \left\| \sum_{k=1}^{n} p_k A_k^p \right\|^{1/p} \left\| \sum_{k=1}^{n} p_k B_k^q \right\|^{1/q}
\end{aligned}$$

and the inequality (6) is proved.

3. Some Reverses

We need the following result that is of interest in itself as well:

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{I}, \tag{11}$$

then

$$\begin{aligned}
\frac{1}{2}\nu(1-\nu) d (b-a)^2 &\leq (1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b) \\
&\leq \frac{1}{2}\nu(1-\nu) D (b-a)^2
\end{aligned} \tag{12}$$

for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \leq \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)^2 D, \tag{13}$$

for any a, $b \in \mathring{I}$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (13).

Proof. We consider the auxiliary function $f_D : I \subset \mathbb{R} \to \mathbb{R}$ defined by $f_D(x) = \frac{1}{2}Dx^2 - f(x)$. The function f_D is differentiable on $\mathring{I}$ and $f_D''(x) = D - f''(x) \geq 0$, showing that f_D is a convex function on $\mathring{I}$.

By the convexity of f_D we have for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$ that

$$\begin{aligned}
0 &\leq (1-\nu) f_D(a) + \nu f_D(b) - f_D((1-\nu)a + \nu b) \\
&= (1-\nu)\left(\frac{1}{2}Da^2 - f(a)\right) + \nu\left(\frac{1}{2}Db^2 - f(b)\right) \\
&- \left(\frac{1}{2}D((1-\nu)a + \nu b)^2 - f_D((1-\nu)a + \nu b)\right) \\
&= \frac{1}{2}D\left[(1-\nu)a^2 + \nu b^2 - ((1-\nu)a + \nu b)^2\right] \\
&- (1-\nu) f(a) - \nu f(b) + f_D((1-\nu)a + \nu b) \\
&= \frac{1}{2}\nu(1-\nu) D(b-a)^2 - (1-\nu) f(a) - \nu f(b) + f_D((1-\nu)a + \nu b),
\end{aligned}$$

which implies the second inequality in (12).

The first inequality follows in a similar way by considering the auxiliary function $f_d : I \subset \mathbb{R} \to \mathbb{R}$ defined by $f_d(x) = f(x) - \frac{1}{2}dx^2$ that is twice differentiable and convex on $\mathring{I}$.

If we take $f(x) = x^2$, then (11) holds with equality for $d = D = 2$ and (13) reduces to an equality as well.

If $D > 0$, the second inequality in (12) is better than the corresponding inequality obtained by Furuichi and Minculete in [7] by applying Lagrange's theorem two times. They had instead of $\frac{1}{2}$ the constant 1. Our method also allowed to obtain, for $d > 0$, a lower bound that can not be established by Lagrange's theorem method employed in [7].

We have:

Lemma 3. *For any a, $b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\leq \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right]. \quad (14)
\end{aligned}$$

Proof. Now, if we write the inequality (12) for the convex function $f : (0,\infty) \to \mathbb{R}$,

$f(x) = -\ln x$, then we get for any $a, b > 0$ and $\nu \in [0,1]$ that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}} &\le \ln((1-\nu)a+\nu b) - (1-\nu)\ln a - \nu \ln b \qquad (15)\\ &\le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}.\end{aligned}$$

Since

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2 \text{ and } \frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2,$$

then by (15) we get the desired result (14).

The second inequalities in (14) is better than the corresponding results obtained by Furuichi and Minculete in [7] where instead of constant $\frac{1}{2}$ they had the constant 1.

Remark 1. *For $\nu = \frac{1}{2}$ we get the following inequalities of interest*

$$\exp\left[\frac{1}{8}\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \le \frac{\frac{a+b}{2}}{\sqrt{ab}} \le \exp\left[\frac{1}{8}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right], \qquad (16)$$

for any $a, b > 0$.

We have the following result that is of interest in itself as well:

Theorem 4. *Let A and B be two positive invertible operators, $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and m, $M > 0$ such that*

$$m^p B^q \le A^p \le M^p B^q. \qquad (17)$$

Then

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right] \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \qquad (18)$$

for any $x \in H$.

Proof. If a, $b \in [t, T] \subset (0, \infty)$ and since

$$0 < \frac{\max\{a,b\}}{\min\{a,b\}} - 1 \le \frac{T}{t} - 1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2 \le \left(\frac{T}{t}-1\right)^2.$$

Therefore, by (14) we get

$$(1-\nu)\,a+\nu b \le a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu\,(1-\nu)\left(\frac{T}{t}-1\right)^2\right], \tag{19}$$

for any $a,\, b\in[t,T]$ and $\nu\in(0,1)$.

Now, if C is an operator with $tI\le C\le TI$ then for $p>1$ we have $t^pI\le C^p\le T^pI$. Using the functional calculus we get from (19) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\left\langle C^py,y\right\rangle \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}\left\langle Cy,y\right\rangle, \tag{20}$$

for any $y\in H$, $\|y\|=1$ and $d\in[t^p,T^p]$.

Since $d=\langle C^py,y\rangle\in[t^p,T^p]$ for any $y\in H$, $\|y\|=1$, hence by (20) we have

$$\left(1-\frac{1}{p}\right)\left\langle C^py,y\right\rangle+\frac{1}{p}\left\langle C^py,y\right\rangle \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\left\langle C^py,y\right\rangle^{1-\frac{1}{p}}\left\langle Cy,y\right\rangle, \tag{21}$$

that is equivalent to

$$\left\langle C^py,y\right\rangle \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\left\langle C^py,y\right\rangle^{1-\frac{1}{p}}\left\langle Cy,y\right\rangle, \tag{22}$$

and by division with $\langle C^py,y\rangle^{1-\frac{1}{p}}>0$, $y\in H$, $\|y\|=1$, to

$$\left\langle C^py,y\right\rangle^{1/p} \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\left\langle Cy,y\right\rangle. \tag{23}$$

If $z \in H$ with $z \neq 0$, then by taking $y = \frac{z}{\|z\|}$ in (23) we get

$$\langle C^p z, z\rangle^{1/p} \langle z, z\rangle^{1/q} \leq \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p - 1\right)^2\right] \langle Cz, z\rangle, \tag{24}$$

for any $z \in H$.

Now, from (17) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M^p I$ and by taking the power $\frac{1}{p}$ we get $mI \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \leq MI$.

By writing the inequality (24) for $C = \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}}$, $t = m$, $T = M$ and $z = B^{\frac{q}{2}} x$, with $x \in H$, we have

$$\begin{aligned} &\left\langle B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle^{1/p} \left\langle B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle^{1/q} \\ &\leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right] \left\langle \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle, \end{aligned}$$

namely

$$\begin{aligned} &\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \\ &\leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right] \left\langle B^{\frac{q}{2}} \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}} x, x\right\rangle, \end{aligned}$$

for any $x \in H$, and the inequality (18) is proved.

Remark 2. *We observe, for A and B two positive invertible operators, that the condition (17) is equivalent to following condition*

$$mI \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \leq MI. \tag{25}$$

If we assume that

$$rB^q \leq A^p \leq RB^q, \tag{26}$$

then by (18) we have the inequality

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \leq \exp\left[\frac{1}{2pq}\left(\frac{R}{r} - 1\right)^2\right] \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \tag{27}$$

for any $x \in H$.

We have:

Corollary 5. *Let A and B be two positive invertible operators and m, $M > 0$ such that*

$$mI \leq \left(B^{-1}A^2B^{-1}\right)^{\frac{1}{2}} \leq MI, \tag{28}$$

then we have

$$\left\langle A^2x, x\right\rangle^{1/2} \left\langle B^2x, x\right\rangle^{1/2} \leq \exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right] \left\langle A^2\sharp B^2x, x\right\rangle \tag{29}$$

for any $x \in H$.

If $mI \leq C \leq MI$ for some m, M with $0 < m < M$, then by (29) we get

$$\langle Cx, x\rangle^{1/2} \left\langle C^{-1}x, x\right\rangle^{1/2} \leq \exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right] \|x\|^2, \tag{30}$$

for any $x \in H$.

Corollary 6. *Assume that A and B satisfy the conditions*

$$m_1I \leq A \leq M_1I,\ m_2I \leq B \leq M_2I \tag{31}$$

for some $0 < m_1 < M_1$ and $0 < m_2 < M_2$, then we have

$$\begin{aligned} &\langle A^px, x\rangle^{1/p} \langle B^qx, x\rangle^{1/q} \\ &\leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q - 1\right)^2\right] \left\langle B^q\sharp_{1/p}A^px, x\right\rangle, \end{aligned} \tag{32}$$

for any $x \in H$.

In particular, we have

$$\left\langle A^2x, x\right\rangle^{1/2} \left\langle B^2x, x\right\rangle^{1/2} \leq \exp\left[\frac{1}{8}\left(\left(\frac{M_1M_2}{m_1m_2}\right)^2 - 1\right)^2\right] \left\langle A^2\sharp B^2x, x\right\rangle, \tag{33}$$

for any $x \in H$.

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