

## CONVOLUTION OF LN-TRANSLATION SURFACES IN EUCLIDEAN 3-SPACE

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**ABSTRACT.** In this paper, we study LN-translation surfaces with zero mean curvature and zero Gaussian curvature. Further, we investigate the computation of parametrizations of convolution surfaces of paraboloids and translation surfaces. In addition, we give necessary and sufficient conditions for a convolution surface of LN-translation surfaces to become flat and minimal in Euclidean 3-space.

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### 1. INTRODUCTION

A surface that arises when a curve  $\alpha(u)$  is translated over another curve  $\beta(v)$ , is called a translation surface. A translation surface can be defined as the sum of the two generating curves  $\alpha(u)$  and  $\beta(v)$ . Therefore, translation surfaces are made up of quadrilateral, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures. A translation surface in a Euclidean 3-space  $\mathbb{E}^3$  formed by translating two curves lying in orthogonal planes is the graph of a function  $z = r(u, v) = f(u) + g(v)$ , where  $f(u)$  and  $g(v)$  are smooth functions on some interval of  $\mathbb{R}$  ([1]).

In 1835, H. F. Scherk studied translation surfaces in  $\mathbb{E}^3$  defined as graph of the function  $z(u, v) = f(u) + g(v)$  and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z(u, v) = \frac{1}{a} \log \left| \frac{\cos(au)}{\cos(av)} \right| = \frac{1}{a} \log |\cos(au)| - \frac{1}{a} \log |\cos(av)|,$$

where  $a$  is a non-zero constant. These surfaces are now referred as Scherk's minimal surfaces.

LN-surfaces, which were studied in ([5]) have sufficient flexibility to model smooth surfaces without parabolic points. Peternell and Odehnal generalized the concept of LN-surfaces to  $\mathbb{R}^4$  ([9]). Bulca calculated the Gaussian, normal and mean curvatures of LN- surfaces in  $\mathbb{E}^4$ , Further, she pointed out the flat and minimal points of the surfaces ([3]).

The computation of convolution curves-surfaces and Minkowski sums of objects occurs in various areas, like computer graphics, computational geometry and motion planning. Sampoli, Peternell and Jüttler showed that even the convolution surface of an LN-surface and any rational surface admits rational parametrization ([8, 10,11]). Aydoğan and Arslan studied with the convolution surface  $C$  of a paraboloid  $A \subset E^3$  and a parametric surface  $B \subset E^3$ . They took some spacial surfaces for  $B$  such as, surface of revolution, Monge patch and ruled surface and calculate the Gaussian curvature of the convolution surface  $C$ . Further, they gave necessary and sufficient conditions for a convolution surface  $C$  to become flat ([2]).

## 2. PRELIMINARIES

Let  $\mathbb{E}^3$  be a Euclidean 3-space with the scalar product given by

$$\langle , \rangle = dx^2 + dy^2 + dz^2$$

where  $(x, y, z)$  is a rectangular coordinate system of  $\mathbb{E}^3$ , In particular, the norm of a vector  $V \in \mathbb{E}^3$  is given by

$$\|V\| = \sqrt{\langle V, V \rangle}.$$

If  $V = (v_1, v_2, v_3)$  and  $W = (y_1, y_2, y_3)$  are arbitrary vectors in  $\mathbb{E}^3$ , the vector product of  $V$  and  $W$  is given by

$$V \wedge W = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$

Let  $S : \Psi := \Psi(u, v)$  be a surface in Euclidean 3-space. The normal vector field of  $S$  is given by  $\mathbf{N} = \Psi_u \wedge \Psi_v$  and the unit normal vector field of  $S$  can be defined by

$$U = \frac{\Psi_u \wedge \Psi_v}{\|\Psi_u \wedge \Psi_v\|}.$$

The first fundamental form  $I$  of the surface  $S$  is

$$I = Edu^2 + 2Fdudv + Gdv^2,$$

with the coefficients

$$E = \langle \Psi_u, \Psi_u \rangle, G = \langle \Psi_v, \Psi_v \rangle, F = \langle \Psi_u, \Psi_v \rangle.$$

The second fundamental form  $II$  of the surface  $S$  is given by

$$II = Ldu^2 + 2Mdudv + Ndv^2,$$

with the coefficients

$$L = \langle \Psi_{uu}, U \rangle, N = \langle \Psi_{vv}, U \rangle, M = \langle \Psi_{uv}, U \rangle.$$

Under this parametrization of the surface  $S$ , the Gaussian curvature  $K$  and the mean curvature  $H$  are given by

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2}, \\ H &= \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \end{aligned}$$

respectively ([7]).

**Definition 1.** Consider a polynomial (or, more general, a rational) surface  $\Psi(u, v)$ . This surface is said to be an LN-surface, if its normal vectors admit a linear representation of the form

$$\mathbf{N}(u, v) = \vec{a}u + \vec{b}v + \vec{c},$$

with certain constant coefficient vectors  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ . More precisely, it satisfies the equations the equations

$$\langle \Psi_u, \mathbf{N} \rangle = \langle \Psi_v, \mathbf{N} \rangle = 0. \quad (2.1)$$

The equations (2.1) can be seen as linear constraints on the space of polynomial or rational parametric surfaces.

**Remark 1.** If the three vectors  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$  are linearly dependent, then the surface  $\Psi(u, v)$  describes a general cylinder, since the unit normals  $U$  are contained in a great circle on the unit sphere. In the remainder of this paper we assume that the three vectors are linearly independent. Without loss of generality we may then assume that

$$\vec{a} = (1, 0, 0), \vec{b} = (0, 1, 0), \vec{c} = (0, 0, 1). \quad (2.2)$$

i.e.,  $\mathbf{N}(u, v) = (u, v, 1)$ . This situation can be achieved by a uniform scaling of  $\mathbb{R}^3$ , a suitable choice of Cartesian coordinates, and a linear parameter transformation  $u = u(u', v'), v = v(u', v')$ .

**Proposition 1.** *Under the assumptions of Remark , the tangent planes of an LN surface have the equations*

$$T(u, v) : r(u, v) = ux + vy + z = 0, \quad (2.3)$$

where  $f(u, v) = -\Psi(u, v) \cdot \mathbf{N}(u, v)$  is a polynomial or rational function, in the case of a polynomial or rational LN surface, respectively. On the other hand, given a system of tangent planes of the form (2.3) with a polynomial or rational function  $h(u, v)$ , the envelope surface

$$\Psi_L = (-z_u, -z_v, -z + uz_u + vz_v) \quad (2.4)$$

is a polynomial or rational LN surface.

**Remark 2.** *Due to (2.1), singular points of the envelope surface (2.4) are characterized by  $z_{uu}z_{vv} - z_{uv} = 0$ . In addition, the Gaussian curvature of the envelope equals*

$$K(u, v) = \frac{1}{(z_{uu}z_{vv} - z_{uv})(1 + u^2 + v^2)^2}. \quad (2.5)$$

Consequently, the algebraic curve  $z_{uu}z_{vv} - z_{uv} = 0$  in the  $(u, v)$ -parameter domain separates elliptic ( $K > 0$ ) and hyperbolic ( $K < 0$ ) points on the LN surface.

Given two surfaces  $A$  and  $B$  in  $\mathbb{R}^3$ , their Minkowski sum  $A \oplus B$  is defined to be the set

$$A \oplus B = \{a + b : a \in A, b \in B\},$$

where  $a$  and  $b$  denote position vectors of arbitrary points in  $A$  and  $B$ . The convolution surface is defined to be

$$A + B = \{a + b : a \in A, b \in B, n_A(a) \parallel n_B(b)\},$$

where  $n_A(a)$  and  $n_B(b)$  are parallel surface normal vectors at the points  $a \in A$  and  $b \in B$ . The convolution surface is often denoted by  $A \star B$ . Since we are working with parametrizations only it is more convenient to denote the convolution by  $A + B$  and we call it also sum of  $A$  and  $B$ . In general, the computation of the convolution surface  $A + B$  of two smooth surfaces  $A$  and  $B$  results in the following problem. Assume that the surfaces  $A$  and  $B$  are parametrized by  $a(u, v)$  and  $b(s, t)$ , respectively and that the normal vectors are denoted by  $n_A(u, v)$  and  $n_B(s, t)$ . The convolution surface  $A + B$  is formed by the sums of vectors  $a, b$  whose normal vectors  $n_A, n_B$  are parallel. Thus, we have to find parametrizations  $a(u(s, t), v(s, t)) = a(s, t)$  and  $b(s, t)$  of parts of  $A$  and  $B$  over a common parameter domain of the  $st$ -plane with the property that the normal vectors  $n_A(s, t)$  and  $n_B(s, t)$  at  $a$  and  $b$  are parallel. Let us point out that in case of an arbitrary surface  $B$  there is no one-one correspondence between points  $a \in A$  and  $b \in B$  with  $n_A(a) \parallel n_B(b)$ .

Now, we want to investigate parametrizations of the sum  $A + B$  of a paraboloid  $A$  and a parametrized surface  $B$ . We assume that a coordinate system has been chosen in a way that the paraboloid  $A$  is given by the equation

$$F_A = z - x^2 - cy^2 = 0, \quad c \neq 0.$$

This implies that  $A$  is representable by  $a(u, v) = (u, v, u^2 + cv^2)$  and it is either an elliptic or a hyperbolic paraboloid depending on whether  $c > 0$  or  $c < 0$ . The surface  $B$  is assumed to admit a local parametrization  $b : (s, t) \in G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , which is a smooth mapping. Two points  $a \in A$  and  $b \in B$  are corresponding if the normal vectors  $n_A(a)$  and  $n_B(b)$  at  $a$  and  $b$ , respectively, are linearly dependent,

$$n_A(a) = \lambda n_B(b), \quad \lambda \neq 0. \quad (2.6)$$

Then,  $a + b$  is a point of the convolution surface  $A + B$ . So, let  $n_B(s, t) = (n_1, n_2, n_3)(s, t)$  be a normal vector of  $B$ , we rewrite condition (2.6) in coordinates and obtain

$$(-2u, -2cv, 1) = \lambda (n_1, n_2, n_3)(s, t).$$

In case  $n_3(s, t) \neq 0$  we have  $\lambda = \frac{1}{n_3(s, t)}$  and

$$u(s, t) = -\frac{n_1}{2n_3}(s, t), \quad v(s, t) = -\frac{n_2}{2cn_3}(s, t). \quad (2.7)$$

The final representation of the sum  $A + B$  is

$$\Psi_C = (a + b)(s, t) = \left( \frac{-n_1}{2n_3} + y_1, \frac{-n_2}{2cn_3} + y_2, \frac{1}{4cn_3^2} (cn_1^2 + n_2^2) + y_3 \right) (s, t). \quad (2.8)$$

Also, the convolution surface  $A + B$  of an LN-surface has the reparameterization

$$\Psi_{CL} = (a + b)(s, t) = \left( \frac{n_1}{n_3} + y_1, \frac{n_2}{n_3} + y_2, \frac{1}{n_3^2} (n_1^2 + cn_2^2) + y_3 \right) (s, t). \quad (1)$$

See details in ([6, 8, 9, 10,11]).

### 3. LN-TRANSLATION SURFACES

In this chapter, we define the LN-translation surfaces in Euclidean 3-space. Consider a surface in as a the graph of a function  $z = r(u, v)$  of two variables, which is itself the sum of two functions  $f$  and  $g$  of one variable. Here, we restrict our topic to regular surfaces  $\Psi$ . Thus, we can express in open form as

$$\Psi: z = f(u) + g(v). \quad (2)$$

A translation surface is defined by a patch

$$\Psi(u, v) = (u, v, f(u) + g(v)). \quad (3)$$

So, using (2.4) and (3.1), we can define the LN- translation surfaces defined by as

$$S_L : \Psi_L(u, v) = (-f'(u), -g'(v), uf'(u) + vg'(v) - f(u) - g(v)). \quad (4)$$

The coefficients of the first and the second fundamental forms of the LN-translation surface given by

$$\begin{aligned} E_L &= (1 + u^2) f''^2, \quad G_L = (1 + v^2) g''^2, \quad F_L = uv f'' g'', \\ e_L &= \frac{1}{g'' \sqrt{1 + u^2 + v^2}}, \quad g_L = \frac{1}{f'' \sqrt{1 + u^2 + v^2}}, \quad f_L = 0. \end{aligned} \quad (5)$$

Thus, we have the unit normal vector

$$U_L = \frac{1}{f'' g'' \sqrt{1 + u^2 + v^2}} (u, v, 1). \quad (6)$$

**Proposition 2.** *Let  $S_L$  be a LN-translation surface in Euclidean 3-space. Then the Gaussian and the mean curvatures of  $S_L$  can be given by*

$$\begin{aligned} K_L &= \frac{1}{f'' g'' \sqrt{1 + u^2 + v^2}}, \\ H_L &= \frac{(1 + u^2) f''^2 + (1 + v^2) g''^2}{2 f''^2 g''^2 (1 + u^2 + v^2)^{\frac{3}{2}}}, \end{aligned} \quad (7)$$

respectively.

So we have the following result.

**Corollary 1.** *Let  $S_L$  be a LN -translation surface in Euclidean 3-space. Then there is no flat LN -translation surface.*

We assume that  $S_L$  is minimal. Hence, the mean curvature is zero if and only if

$$(1 + u^2) f''^2 + (1 + v^2) g''^2 = 0. \quad (8)$$

Then, the minimality condition (3.6) can be separated for the variables

$$(1 + u^2) f''^2 = - (1 + v^2) g''^2 = m, \quad (9)$$

where  $m \in \mathbb{R}$ . Solving this equation for  $f$  and  $g$ , we get

$$\begin{aligned} f(u) &= c_1 + c_2 u + m \left( u \arctan u - \frac{1}{2} \ln |1 + u^2| \right), \\ g(v) &= c_5 + c_6 v - m \left( v \arctan v - \frac{1}{2} \ln |1 + v^2| \right), \end{aligned} \quad (10)$$

where  $c_i, m \in \mathbb{R}$ . Thus we have following theorem.

**Theorem 2.** *A LN-translation surface  $S_L$  is minimal in Euclidean 3-space if and only if it is a part of the surface (3.3) with (3.9).*

#### 4. CONVOLUTION OF TRANSLATION SURFACES

Assume that  $S$  is a the translation surface given by (3.2). So, using (2.8), the convolution surface of a paraboloid and a translation surface has the parametrization

$$S_C : \Psi_C = \left( \frac{1}{2} (2u + f'), \frac{2cv + g'}{2c}, \frac{4cf + 4cg + cf'^2 + g'^2}{2c} \right). \quad (11)$$

**Proposition 3.** *Let  $S_C$  be a convolution of translation surface Euclidean 3-space. Then the Gaussian and the mean curvatures of  $S_C$  can be given by*

$$\begin{aligned} K_C &= \frac{4cf''g''}{(2 + f'')(2c + g'')(1 + f'^2 + g'^2)^2}, \\ H_C &= \frac{f''(2c + 2cg'^2) + g''(2c + 2cf'^2) + f''g''(1 + c + cf'^2 + g'^2)}{(2 + f'')(2c + g'')(1 + f'^2 + g'^2)^{\frac{3}{2}}}, \end{aligned} \quad (12)$$

respectively.

We suppose that the convolution of translation surface given by (4.1) has zero the Gaussian curvature. Therefore, either  $f'' = 0$  or  $g'' = 0$  which implies that the surface is flat. We assume that  $S_C$  is minimal. Hence, the mean curvature is zero if and only if

$$f''(2c + 2cg'^2) + g''(2c + 2cf'^2) + f''g''(c + cf'^2 + 1 + g'^2) = 0. \quad (13)$$

The equation (4.3) turns out to be

$$f''(2c + 2cg'^2) + (2c + 2cf'^2)g'' + f''(c + cf'^2)g'' + f''(1 + g'^2)g'' = 0. \quad (14)$$

The expression (4.4) is analyzed with the method as in ([4]). We write

$$\sum_{i=1}^4 f_i(u)g_i(v) = 0,$$

where

$$\begin{aligned} f_1 &= f_4 = f'', & f_2 &= 2c + 2cf'^2, & f_3 &= f''(c + cf'^2), \\ g_1 &= g_3 = g'', & g_2 &= 2c + 2cg'^2, & g_4 &= (1 + g'^2)g''. \end{aligned}$$

If  $f_1 = f_4 = f'' = 0$ , then we get  $f(u) = c_1u + c_2$ . Replacing  $f(u)$  in the equation (4.4), we obtain  $g(v) = c_3v + c_4$ . If  $f_2 = 0$  or  $f_3 = 0$ , then we get  $f(u) = \pm iu + c_1$ , which is a contradiction. Similarly, if  $g'' = 0$ , then we get  $f'' = 0$ . Thus we have:

**Theorem 3.** *Let  $S_C$  be a convolution of translation surface in Euclidean 3-space. If  $S_C$  is flat or minimal then it is parameterized by (4.1) with  $f(u) = c_1u + c_2$  and  $g(v) = c_3v + c_4$ , where  $c_i \in \mathbb{R}$ .*

## 5. CONVOLUTION OF LN-TRANSLATION SURFACES IN EUCLIDEAN 3-SPACE

Let  $S_{CL}$  be a convolution of LN-translation surface in  $\mathbb{E}^3$ . So, using (2.9) and (3.2), we can define the convolution of LN-translation surface is parameterized by as

$$\Psi_{CL}(u, v) = (u - f', v - g', u^2 + cv^2 - f - g + uf' + vg'). \quad (15)$$

The Gaussian curvature of  $S_{CL}$  is given by

$$K_{CL} = \frac{P(u)Q(v)R(u)S(v)}{W_1(u, v)}, \quad (16)$$

where

$$\begin{aligned} P &= (-1 + f''), \\ Q &= (-1 + g''), \\ R &= (-2 + f'' + f''^2 - 3uf'''), \\ S &= (-2c - g'' + 2cg'' + g''^2 - vg''' - 2cvg''') \end{aligned}$$

and the function  $W_1$  depend on the functions  $f$  and  $g$ . Also, the mean curvature of  $S_{CL}$  is given by

$$H_{CL} = \frac{Q(v)J(v)R(u) + P(u)V(u)Y(v)}{W_2(u, v)}, \quad (5.3)$$

where

$$\begin{aligned} V &= \left( (-1 + f'')^2 + u^2 (2 + f'')^2 \right), \\ Y &= \left( (-1 + g'') (2c + g'') - (1 + 2c) v g''' \right), \\ J &= \left( (-1 + g'')^2 + (2cv + v g'')^2 \right) \end{aligned}$$

and the funtion  $W_2$  depend on the functions  $f$  and  $g$ . So we have the following result.

**Corollary 4.** *Let  $S_{CL}$  be a convolution surface of LN -translation surface in Euclidean 3-space. If the convolution  $S_{CL}$  is a flat surface, then at least one of the following cases occur;*

$$\begin{aligned} f(u) &= c_1 + c_2 u + \frac{u^2}{2}, \\ f(u) &= c_3 + u (c_4 - u - 3e^{-3c_5}) - 3e^{-6c_5} (1 + e^{3c_5}) \ln |1 + ue^{3c_5}|, \\ g(v) &= c_6 + c_7 v + \frac{v^2}{2}, \\ g(v) &= c_8 + v \left( c_9 - cv + e^{-c_{10}(1+2c)} \right) \\ &\quad + e^{-2c_{10}(1+2c)} \left( \frac{1}{1+2c} + ve^{c_{10}(1+2c)} \right) \ln \left| 1 - (1+2c) ve^{c_{10}(1+2c)} \right|, \end{aligned}$$

where  $c_i \in \mathbb{R}$ .

*Proof.* If  $S_{CL}$  is a flat surface, then

$$P(u)Q(v)R(u)S(v) = 0$$

holds. So, we have the four possible cases;

$$\begin{aligned} P(u) &= 0, \\ R(u) &= 0, \\ Q(v) &= 0 \\ S(v) &= 0. \end{aligned}$$

Solving these differential equations we get the results. This completes the proof of the corollary.

We assume that  $S^*$  is minimal. Hence, the mean curvature is zero if and only if

$$Q(v)J(v)R(u) + P(u)V(u)Y(v) = 0. \quad (17)$$

Then, the minimality condition (3.12) can be separated for the variables

$$\frac{R(u)}{P(u)V(u)} = -\frac{Y(v)}{Q(v)J(v)} = m,$$

where  $m \in \mathbb{R}$ . Solving this equation for  $f$  and  $g$  for  $m = 0$ , we get

$$\begin{aligned} f(u) &= c_1 + u(c_2 - u - 3e^{-3c_3}) + 3e^{-6c_3}(1 + ue^{3c_3}) \ln |1 + ue^{3c_3}|, \\ g(v) &= c_4 + v(c_5 - cv + e^{-c_6(1+2c)}) \\ &\quad + e^{-2c_6(1+2c)} \left( \frac{1}{1+2c} - ve^{c_6(1+2c)} \right) \ln |1 - (1+2c)ve^{c_6(1+2c)}|, \end{aligned} \quad (18)$$

where  $c_i \in \mathbb{R}$ . In (5.4), if we take

$$\frac{Q(v)J(v)}{Y(v)} = -\frac{P(u)V(u)}{R(u)} = m,$$

then we have

$$\begin{aligned} f(u) &= c_1 + uc_2 + \frac{u^2}{2}, \\ g(v) &= c_3 + vc_4 + \frac{v^2}{2}, \end{aligned} \quad (19)$$

where  $c_i \in \mathbb{R}$  and  $m = 0$ . Thus we have following theorem.

**Theorem 5.** *A convolution surface of LN- translation surface  $S_{CL}$  is minimal in Euclidean 3-space if and only if it is a part of the surface (5.1) with (5.5) or (5.6).*

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