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ON A FAMILY OF TELESCOPIC NUMERICAL SEMIGROUPS

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Abstract. In this study, we will give some results about Frobenius number, gaps, and determine number of telescopic numerical semigroup S_k and Arf closure of S_k such that $S_k = \langle 8, 8k+4, 8k+9 \rangle$ where $k \geq 1, k \in \mathbb{Z}$.

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1. Introduction

Let $\mathbb{N} = \{a \in \mathbb{Z} : a \geq 0\}$ and $\mathbb{Z}$ be integers set. The subset S of $\mathbb{N}$ is a numerical semigroup if

$$
\begin{array}{ll}
(i) & 0 \in S \\
(ii) & y_1 + y_2 \in S \quad \text{for all} \quad y_1, y_2 \in S \\
(iii) & Card(\mathbb{N} \setminus S) < \infty.
\end{array}
$$

The condition (iii) is equivalent to $gcd(S) = 1$, $gcd(S) =$ greatest common divisor the element of S.

Let S be a numerical semigroup, then $F(S) = max\{a : a \in \mathbb{Z} \setminus S\}$ is called Frobenius number of S, $m(S) = min\{s \in S : s > 0\}$ is called multiplicity of S, $n(S) = Card(\{0, 1, 2, ..., F(S)\} \cap S)$ is called the number determine of S. If $F(S) - r \in S$ then S is called symmetric numerical semigroup, for all $r \in \mathbb{Z} \setminus S$. It is known that $S =< y_1, y_2 >$ is symmetric numerical semigroup and $F(S) = y_1 y_2 - y_1 - y_2$ (see [1]). If S is a numerical semigroup such that $S =< y_1, y_2, ..., y_n >$, then we observe that $S =< y_1, y_2, ..., y_n >= \{s_0 = 0, s_1, s_2, ..., s_{n-1}, s_n = F(S) + 1, \rightarrow \ldots\}$ where $s_i < s_{i+1}, n = n(S)$, and the arrow means that every integer greater than $F(S)+1$ belongs to S , for $i = 1, 2, ..., n = n(S)$. If $p \in \mathbb{N}$ and $p \notin S$,then p is called gap of S. We denote the set of gaps of S, by $H(S) = \{p : p \in \mathbb{N} \setminus S\}$,and the $G(S) = Card(H(S))$ is called the genus of S. Also, It is know that $F(S) = G(S)+n(S)-1$.If

S is a symmetric numerical semigroup then $n(S) = G(S) = \frac{F(S)+1}{2}$ (see [6]). $S =< y_1, y_2, y_3 >$ is called a triply-generated telescopic numerical semigroup if $y_3 \in < \frac{y_1}{d}, \frac{y_2}{d} >$ where $d = gcd(y_1, y_2)$ (see [5, 7, 2]). If S is a numerical semigroup such that $S =< y_1, y_2, y_3, ..., y_n >$,then $L(S) =< y_1, y_2 - y_1, y_3 - y_1, ..., y_n - y_1 >$ is called Lipman numerical semigroup of S,and it is known that $L_0(S) = S \subseteq L_1(S) = L(L_0(S)) \subseteq L_2(S) = L(L_1(S)) \subseteq ... \subseteq L_q(S) = L(L_{q-1}(S)) \subseteq \cdots \subseteq \mathbb{N}$. A numerical semigroup S is called Arf if $y_1 + y_2 - y_3 \in S$, for all $y_1, y_2, y_3 \in S$ such that $y_1 \geq y_2 \geq y_3$. The intersection of any family of Arf numerical semigroups is again an Arf numerical semigroup. Thus, since $\mathbb{N}$ is an Arf numerical semigroup, one can consider the smallest Arf numerical semigroup containing a given numerical semigroup. The smallest Arf numerical semigroup containing a numerical semigroup is called the Arf closure of S and it is denoted by $Arf(S)$. However, the Arf closure of S can also be expressed with Lipman numerical semigroup of S (for details see [3, 6]).

In this paper, we will give some results about Frobenius number, gaps, and determine number of telescopic numerical semigroup S_k and Arf closure of S_k such that $S_k =< 8, 8k + 4, 8k + 9 >$ where $k \geq 1, k \in \mathbb{Z}$. Here,$S_k$ is symmetric numerical semigroup,where $k \geq 1, k \in \mathbb{Z}$. But, any telescopic numerical semigroup is not symmetric.For example,

$$S =< 6, 9, 20 >= \{0, 6, 9, 12, 15, 18, 20, 21, 24, 26, 27, 29, 30, 32, 33, 35, 36, 38, \rightarrow \ldots\}$$

is telescopic numerical semigroup but it is not symmetric since $F(S) = 37$ and $F(S) - v = 37 - 3 = 34 \notin S$ for $v = 3 \in \mathbb{Z} \setminus S$.

2. Main Results

Proposition 1. *([8]) $S_k =< 8, 8k+4, y >$ is a telescopic numerical semigroup where $k \geq 1, k \in \mathbb{Z}$ and $y > 8k + 4$ is odd integer number.*

Proposition 2. *([4]) Let $S =< u_1, u_2, ..., u_n >$ be a numerical semigroup and $d = gcd\{u_1, u_2, ..., u_{n-1}\}$. If $T =< \frac{u_1}{d}, \frac{u_2}{d}, ..., \frac{u_{n-1}}{d} >$ is numerical semigroup then*

1. $F(S) = dF(T) + (d - 1)u_n$

2. $G(S) = dG(T) + \frac{(d-1)(u_n-1)}{2}$.

Proposition 3. *Let $S_k =< 8, 8k + 4, 8k + 9 >$ be a telescopic numerical semigroup where $k \geq 1, k \in \mathbb{Z}$. Then, we have*

(a) $F(S_k) = 32k + 23$
(b) $n(S_k) = 16k + 12$
(c) $G(S_k) = 16k + 12$.

Proof. (a) We find that $F(T) = 2(2k+1) - 2 - 2k - 1 = 2k - 1$ since $d = gcd(8, 8k+4) = 4$ and $T =< \frac{8}{4}, \frac{8k+4}{4} >=< 2, 2k+1 >$, where $k \geq 1, k \in \mathbb{Z}$. In this case, we obtain that $F(S_k) = 4(2k-1) + (4-1)(8k+9) = 32k+23$ from Proposition (2)-(1).

(b)-(c) It is trivial $n(S_k) = G(S_k) = \frac{F(S_k)+1}{2} = \frac{32k+24}{2} = 16k + 12$ from S_k is symmetric numerical semigroup.

Theorem 1. *Let* $S_k =< 8, 8k+4, 8k+9 >$ *be a telescopic numerical semigroup where* $k \geq 1, k \in \mathbb{Z}$. *Then* $Arf(S_k) = \{0, 8, 16, 24, ..., 8k, 8k+4, 8k+8, \rightarrow \ldots\}$.

Proof. It is trivial $m_0 = 8$ since $L_0(S_k) = S_k$. Thus, we write $L_1(S_k) =< 8, 8k-4, 8k+1 >$. In this case,

(1) If $8k - 4 < 8$ (if $k = 1$) then $S_1 =< 8, 12, 17 >$ and we obtain

$$L_1(S_1) =< 8, 4, 9 >=< 4, 9 >, m_1 = 4,$$

$$L_2(S_1) =< 4, 5 >, m_2 = 4$$

and

$$L_3(S_1) =< 4, 1 >=< 1 >= \mathbb{N}, m_3 = 1.$$

Thus, we have $Arf(S_1) = \{0, 8, 12, 16, \rightarrow ...\}$.

(2) If $8k - 4 > 8$ (if $k \geq 2$) then

$$L_1(S_k) =< 8, 8k-4, 8k+1 >, m_1 = 8.$$

In this case, we write $L_2(S_k) =< 8, 8k-12, 8k-7 >$.

(a) If $k = 2$ then

$$L_2(S_2) =< 8, 4, 9 >=< 4, 9 >, m_2 = 4,$$

$$L_3(S_2) =< 4, 5 >, m_3 = 4$$

and

$$L_4(S_2) =< 4, 1 >=< 1 >= \mathbb{N}, m_4 = 1.$$

Thus, we write $Arf(S_2) = \{0, 8, 16, 20, 24, \rightarrow ...\}$.

(b) If $k > 2$ then $L_2(S_k) =< 8, 8k-12, 8k-7 >, m_2 = 8$, and $L_3(S_k) =< 8, 8k-20, 8k-15 >$. In this case,
(i) If $k = 3$ then

$$L_3(S_3) =< 8, 4, 9 >=< 4, 9 >, m_3 = 4,$$

$$L_4(S_3) =< 4, 5 >, m_4 = 4$$

and

$$L_5(S_3) =< 4, 1 >=< 1 >= \mathbb{N}, m_5 = 1.$$

So, we find that $Arf(S_3) = \{0, 8, 16, 24, 28, 32, \rightarrow ...\}$.
(ii) If $k > 3$ then $L_3(S_k) =< 8, 8k-20, 8k-15 >$,$m_3 = 8$, and $L_4(S_k) =< 8, 8k-28, 8k-23 >$. In this case,

(1)If $k = 4$ then

$$L_4(S_4) =< 8, 4, 9 >=< 4, 9 >, m_4 = 4,$$

$$L_5(S_4) =< 4, 5 >, m_5 = 4$$

and

$$L_6(S_3) =< 4, 1 >=< 1 >= \mathbb{N}, m_6 = 1.$$

Thus, we have $Arf(S_4) = \{0, 8, 16, 24, 32, 36, 40, \rightarrow ...\}$.

(2) If $k > 4$ then $L_4(S_k) =< 8, 8k-28, 8k-23 >$, $m_4 = 8$, and we write $L_5(S_k) =< 8, 8k-36, 8k-31 >$. If we continue the operations then we obtain Arf closure of $S_k =< 8, 8k+4, 8k+9 >$ as follows

$Arf(S_k) = \{0, 8, 16, 24, ..., 8k, 8k+4, 8k+8, \rightarrow ...\}$.

Thus,the proof is completed.

Corollary 2. *Let $S_k =< 8, 8k+4, 8k+9 >$ be a telescopic numerical semigroup where $k \geq 1, k \in \mathbb{Z}$. Then, we have*
(a) $F(Arf(S_k) = 8k + 7$
(b) $n(Arf(S_k)) = k + 2$
(c) $G(Arf(S_k)) = 7k + 6$.

Proof. (a) It is clear.
(b) Let a and b be the cardinalities of the subsets $\{0, 8, 16, 24, ..., 8k\}$ and $\{8k + 4, 8k+8\}$ of $Arf(S_k) = \{0, 8, 16, 24, ..., 8k, 8k+4, 8k+8, \rightarrow ...\}$, respectively. In this case, we find that $a + b = k + 2$.
(c) $G(Arf(S_k)) = F(Arf(S_k)) + 1 - n(Arf(S_k)) = 8k + 7 + 1 - (k + 2) = 7k + 6$.

Corollary 3. *Let $S_k =< 8, 8k+4, 8k+9 >$ be a telescopic numerical semigroup where $k \geq 1, k \in \mathbb{Z}$. Then,*
(a) $F(S_k) = F(Arf(S_k) + 8(3k + 2)$

(*b*) $n(S_k) = n(Arf(S_k)) + 5(3k + 2)$
(*c*) $G(S_k) = G(Arf(S_k)) + 3(3k + 2)$.

Proof. It is trivial from Proposition 3 and Corollary 2.

The following corollaries are satisfied from Proposition 3 and Corollary 2.

Corollary 4. *Let* $S_k =< 8, 8k + 4, 8k + 9 >$ *be a telescopic numerical semigroup where* $k \geq 1, k \in \mathbb{Z}$. *Then, we have*
(*a*) $F(S_{k+1}) = F(S_k) + 32$
(*b*) $n(S_{k+1}) = n(S_k) + 16$
(*c*) $G(S_{k+1}) = G(S_k) + 16$.

Corollary 5. *Let* $S_k =< 8, 8k + 4, 8k + 9 >$ *be a telescopic numerical semigroup where* $k \geq 1, k \in \mathbb{Z}$. *Then,*
(*a*) $F(Arf(S_{k+1})) = F(Arf(S_k)) + 8$
(*b*) $n(Arf(S_{k+1})) = n(Arf(S_k)) + 1$
(*c*) $G(Arf(S_{k+1})) = G(Arf(S_k)) + 7$.

Example 1. *We put* $k = 1$ *in* $S_k =< 8, 8k + 4, 8k + 9 >$ *triply-generated telescopic numerical semigroup. Then, we have*

$$S_1 =< 8, 12, 17 > = \{0, 8, 12, 16, 17, 20, 24, 25, 28, 29, 30, 32, 33, 34, 36, 37, 40, 41, 42, 44, 45, 46, 48, 49, ..., 53, 54, 56, \rightarrow \ldots\}.$$

In this case, we obtain

$$F(S_1) = 55 \ , \ n(S_1) = 28$$

$H(S_1) = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 18, 19, 21, 22, 23, 26, 27, 30, 31, 35, 38, 39, 43, 47, 55\}$,

$G(S_1) = Card(H(S_1)) = 28$, $Arf(S_1) = \{0, 8, 12, 16, \rightarrow \ldots\}$, $F(Arf(S_1)) = 15$, $H(Arf(S_1)) = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15\}$, $G(Arf(S_1)) = Card(H(Arf(S_1))) = 13$ *and* $n(Arf(S_1)) = 3$. *If we take* $k = 2$ *then, we write*

$$S_2 =< 8, 20, 25 > = \{0, 8, 16, 20, 24, 25, 28, 32, 33, 40, 41, 44, 45, 48, 49, 50, 52, 53, 56, 57, 58, 60, 61, 64, 65, 68, 69, 70, 72, ..., 78, 80, ..., 86, 88, \rightarrow \ldots\}.$$

In this case, we find $F(S_2) = 87$, $n(S_2) = 44$,$G(S_2)) = 44$, $Arf(S_2) = \{0, 8, 18, 20, 24, \rightarrow \ldots\}$, $F(Arf(S_2)) = 23$, $G(Arf(S_2)) = 20$ *and* $n(Arf(S_2)) = 4$. *So, we obtain* $F(Arf(S_1)) + 40 = 15 + 40 = 55 = F(S_1)$, $G(Arf(S_1)) + 15 = 13 + 15 = 28 = G(S_1)$,

$n(Arf(S_1)) + 25 = 3 + 25 = 28 = n(S_1)$, $F(S_1) + 32 = 55 + 32 = 87 = F(S_2)$, $n(S_1) + 16 = 28 + 16 = 44 = n(S_2)$,$G(S_1) + 16 = 28 + 16 = 44 = G(S_2)$ *and* $F(Arf(S_1)) + 8 = 15 + 8 = 23 = F(Arf(S_2))$, $n(Arf(S_1)) + 1 = 3 + 1 = 4 = n(Arf(S_2))$, $G(Arf(S_1)) + 7 = 13 + 7 = 20 = G(Arf(S_2))$.

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