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ON SUBCLASS OF BAZILEVIC FUNCTIONS ASSOCIATED WITH CERTAIN CARATHEODORY-TYPE FUNCTIONS NORMALIZED BY OTHER THAN UNITY

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Abstract. In this research work, we study a new subclass of Bazilevic functions via Caratheodory maps with normalization by other than unity defined by new operator denoted by $B^n_{\sigma,\gamma}(\lambda)$. We obtain some basic properties of the new class, namely inclusion, closure under certain integral transformation, Coefficient bounds, bound on the Fekete Szego functional.

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1. Introduction

Let P_λ is the class of all functions of the form

$$h(z) = 1 + i\frac{\mu}{\eta} + p_1 z + p_2 z^2 + p_3 z^3 + \cdots . \tag{1}$$

such that for $z \in U$, a complex number $\lambda = \eta + i\mu$ with $\mu \geq 0$ and $\eta > 0$, then $h(z)$ is said to belong to P_λ if and only if $h(0) = \frac{\lambda}{\eta} = 1 + i\frac{\mu}{\eta}$ and $Reh(z) > 0$.
The class of functions of the form (1) is Caratheodory-type with normalization $1 + i\frac{\mu}{\eta}$ as against normalization $p(0) = 1$ for Caratheodory maps and the function $L_0(z) = \frac{1+z}{1-z} + \frac{i\mu}{\eta}$ plays a central role in the class P_λ especially with respect to extremal problems.
In [2], the Basilevic map given as follows

$$f(z) = \left[\frac{\alpha}{1+\beta^2}\int_0^z [p(t) - i\beta]t^{-(1+\frac{i\alpha\beta}{1+\beta^2})}g(t)^{(\frac{\alpha}{1+\beta^2})}\right]. \tag{2}$$

where $p \in P$ and $g(z) = z + b_2 z^2 + \cdots$ is starlike with the parameters $\alpha > 0$ and β are real and all powers mean principal determinations only. The Basilevic map was redefined to suit the caratheodory type normalised by other than unity in the following definition

DEFINITION 1[2] Let $\lambda = \alpha/(1+i\beta)$, the function $f(z)$ belong to the class $B(\lambda, g)$ if and only if

$$\frac{z(f(z)^{\lambda})}{\eta z^{iu} g(z)^{\eta}} \in P_{\lambda}. \tag{3}$$

By taking $g(z) = z$ and using the salagean differential operator, we have the following definition

DEFINITION 2[4] The function $f(z)$ belong to the class $B_n(\lambda)$ if and only if

$$\frac{D^n(f(z)^{\lambda})}{\eta \lambda^{n-1} z^{\lambda}} \in P_{\lambda}. \tag{4}$$

Using the salagean differential operator $D^n f(z)$ and and inverse of integral operator $\mathcal{L}_{\sigma,\gamma} f(z) = \frac{(\lambda+\gamma)^{-\sigma} t^{\gamma-1}}{z^{\gamma}\Gamma-\sigma} \int_0^z (log\frac{z}{t})^{-\sigma-1} f(t)^{\lambda} dt$ (see [12], [5]), on $f(z)^{\lambda}$, we have

$$D^n(\mathcal{L}_{\sigma,\gamma} f(z)^{\lambda}) = z^{\lambda}\lambda^n + \sum_{k=2}^{\infty} \left(\frac{\lambda+\gamma+k-1}{\lambda+\gamma}\right)^{\sigma} (\lambda+k-1)^n A_k(\lambda) z^{\lambda+k-1}. \tag{5}$$

where A_k for $k = 2, 3, \cdots$ depends on the coefficients a_k of $f(z)$ and the index λ.
We denote

$$\mathcal{L}_{\sigma,\gamma}(D^n f(z)^{\lambda}) = D^n(\mathcal{L}_{\sigma,\gamma} f(z)^{\lambda}) = L^n_{\sigma,\gamma} f(z)^{\lambda}. \tag{6}$$

$n \in N \cup \{0\}, \sigma > 0, \gamma > -1$.
Note that $L^n_{1,0} = D^{n+1} f(z)^{\lambda}$, $L^0_{1,0} = Df(z)^{\lambda} = zf'(z)^{\lambda}$. If $\eta = 1, \mu = 0$, then $L^0_{1,0} = zf'(z)$.
From the series expansions of the operator $\mathcal{L}_{\sigma,\gamma}$ on $f(z)^{\lambda}$, we have the recursive relation

$$z(\mathcal{L}_{\sigma,\gamma} f(z)^{\lambda})' = (\lambda+\gamma)\mathcal{L}_{\sigma+1,\gamma} f(z)^{\lambda} - (\lambda+\gamma)\mathcal{L}_{\sigma,\gamma} f(z)^{\lambda}. \tag{7}$$

Applying D^n on (7), we have

$$L^{n+1}_{\sigma,\gamma} f(z)^{\lambda} = (\lambda+\gamma) L^n_{\sigma+1,\gamma} f(z)^{\lambda} - (\lambda+\gamma) L^n_{\sigma,\gamma} f(z)^{\lambda}. \tag{8}$$

Using the salagean anti-derivative define as $I_n = I(I_{n-1} f(z)) = \int_0^z \frac{I_{n-1} f(t)}{t} dt$ and $\mathcal{J}_{\sigma,\gamma} f(z) = \frac{(\lambda+\gamma)^{\sigma} t^{\gamma-1}}{z^{\gamma}\Gamma\sigma} \int_0^z (log\frac{z}{t})^{\sigma-1} f(t) dt$. (see [12], [5]) on $f(z)^{\lambda}$.
Therefore

$$I_n(\mathcal{J}_{\sigma,\gamma} f(z)^{\lambda}) = \frac{z^{\lambda}}{\lambda^n} + \sum_{k=2}^{\infty} \left(\frac{\lambda+\gamma}{\lambda+\gamma+k-1}\right)^{\sigma} \frac{A_k(\lambda)}{(\lambda+k-1)^n} z^{\lambda+k-1}. \tag{9}$$

We denote
$$I_n(\mathcal{J}_{\sigma,\gamma}f(z)^{\lambda}) = \mathcal{J}_{\sigma,\gamma}(I_nf(z)^{\lambda}) = J^n_{\sigma,\gamma}f(z)^{\lambda}. \tag{10}$$
It can be seen that
$$L^n_{\sigma,\gamma}(J^n_{\sigma,\gamma}f(z)^{\lambda}) = J^n_{\sigma,\lambda}(L^n_{\sigma,\gamma}f(z)^{\lambda}) = f(z)^{\lambda}. \tag{11}$$
Using the operator $L^n_{\sigma,\gamma}$, we introduce a new class defined as.

Definition 3

An analytic function $f \in A$ is said to belong to the class $B^n_{\sigma,\gamma}(\lambda)$ if and only if
$$\frac{L^n_{\sigma,\gamma}f(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} \in P_{\lambda}. \tag{12}$$

and the integral representation is as follows
$$f(z) = \left\{\eta\lambda^{n-1}[J^n_{\sigma,\gamma}z^{\lambda}h(z)]\right\}^{1/\lambda}.$$

2. Preliminary Lemmas

Lemma 1. *[4] Let $u = u_1 + u_2i$ and $v = v_1 + v_2i$. Let a be a complex number with $Rea > 0$ and $\psi(u,v)$ a complex-valued function satisfying:*
(a) $\psi(u,v)$ is continuous in a domain of Ω of $\mathbb{C}^2$,
(b) $(a,0) \in \Omega$ and $Re(a;0) > 0$,
(c) $Re(u_2i, v_1) \leq 0$ when (u_2i, v_1) and $2v_1Rea \leq -|a-iu|^2$. If $h = a+c_1z+c_2z^2+\cdots$ such that $(h(z), zh^{'}(z)) \in \Omega$ and its real part is greater than zero, then $Reh(z) > 0$.

Lemma 2. *[4] Let $h \in P_{\lambda}$. Then,*
$$|p_k| \leq 2,\ k = 1,2,3\ldots$$
The result is sharp. Equality holds for the function $h(z) = \frac{1+z}{1-z} + \frac{i\mu}{\eta}$.

Lemma 3. *[3] Let $h \in P_{\lambda}$. Then, we have the sharp inequalities*
$$\left|p_2 - \sigma\frac{p_1^2}{2}\right| \leq 2\,max\{1, |1-\sigma|\}.$$

Lemma 4. *[6] Let $f \in A$, for any complex number ζ.*
(i) If for $z \in E$, $D^{n+1}f(z)^{\zeta}/D^nf(z)^{\zeta}$ is independent of n, then
$$\frac{D^{n+1}f(z)^{\zeta}}{D^nf(z)^{\zeta}} = \zeta\frac{D^{n+1}f(z)}{D^nf(z)}$$
(ii) The equality also holds if $D^{n+1}f(z)/D^nf(z)$ is independent of n, $z \in E$.

Lemma 5. *[2] Let $h \in P_\lambda$*

$$Re \quad \frac{zh^{'}(z)}{h(z)} \geq \frac{-2r}{1-r^2}$$

3. Main Results

Theorem 6. $B^{n+1}_{\sigma,\gamma}(\lambda) \subset B^n_{\sigma,\gamma}(\lambda)$

Proof. Let

$$\frac{L^n_{\sigma,\gamma} f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z) \tag{13}$$

$$L^n_{\sigma,\gamma} f(z)^\lambda = \eta\lambda^{n-1}(z^\lambda h(z)) \tag{14}$$

$$(L^n_{\sigma,\gamma} f(z)^\lambda)^{'} = \eta\lambda^{n-1}(z^\lambda h^{'}(z) + \lambda z^{\lambda-1}h(z)) \tag{15}$$

$$z(L^n_{\sigma,\gamma} f(z)^\lambda)^{'} = \eta\lambda^n z^\lambda \left(\frac{zh^{'}(z)}{\lambda} + h(z)\right) \tag{16}$$

which becomes

$$L^{n+1}_{\sigma,\gamma} f(z)^\lambda = \eta\lambda^n z^\lambda \left(\frac{zh^{'}(z)}{\lambda} + h(z)\right) \tag{17}$$

so that if $f \in B^{n+1}_{\sigma,\gamma}(\lambda)$ then

$$Re\frac{L^{n+1}_{\sigma,\gamma} f(z)^\lambda}{\eta\lambda^n z^\lambda} = Re\left(\frac{zh^{'}(z)}{\lambda} + h(z)\right) > 0 \tag{18}$$

Now define $\psi(u,v) = u + \dfrac{v}{\lambda}$, $Re\lambda > 0$. Noting that $a = 1 + i\dfrac{\mu}{\eta}$, then ψ satisfies all the conditions of Lemma 1, it follows that

$$Re\frac{L^{n+1}_{\sigma,\gamma} f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = Reh(z) > 0 \tag{19}$$

meaning that $f \in B^n_{\sigma,\gamma}(\lambda)$

Theorem 7. $B^n_{\sigma,\gamma}(\lambda) \subset B^n_{\sigma+1,\gamma}(\lambda)$

Proof. Let $\dfrac{L^n_{\sigma+1,\gamma}f(z)}{\eta z^\lambda \lambda^{n-1}} = h(z)$ then

$$L^n_{\sigma+1,\gamma}f(z) = \eta\lambda^{n-1}(z^\lambda h(z))$$

$$L^{n+1}_{\sigma+1,\gamma}f(z)^\lambda = \eta\lambda^n z^\lambda \left(\frac{zh'(z)}{\lambda} + h(z)\right)$$

since $B^{n+1}_{\sigma,\gamma}(\lambda) \subset B^n_{\sigma,\gamma}(\lambda)$ from theorem 1 then

$$Re\frac{L^{n+1}_{\sigma+1,\gamma}f(z)^\lambda}{\eta\lambda^n z^\lambda} = Re\left(\frac{zh'(z)}{\lambda} + h(z)\right) > 0$$

by lemma 1, it follows that

$$Re\frac{L^{n+1}_{\sigma+1,\gamma}f(z)^\lambda}{\eta\lambda^{n-1} z^\lambda} = Reh(z) > 0 \tag{20}$$

meaning that $f \in B^n_{\sigma+1,\gamma}(\lambda)$

Theorem 8. *Let $f \in B^n_{\sigma,\gamma}(\lambda)$, then f is a $\alpha-n$ spiral univalent in the disk $|z| < r_0(\eta)$ where given by*

$$r_0(\eta) = \frac{1}{\eta}\left(\sqrt{1+\eta^2} - 1\right)$$

Proof. By definition, let

$$\frac{L^n_{\sigma,\gamma}f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z)$$

by some simple calculation, we have

$$L^{n+1}_{\sigma,\gamma}f(z)^\lambda = \eta\lambda^{n-1}z^\lambda\left(zh'(z) + \lambda h(z)\right)$$

and

$$\frac{L^{n+1}_{\sigma,\gamma}f(z)^\lambda}{L^n_{\sigma,\gamma}f(z)^\lambda} = \frac{zh'(z)}{h(z)} + \lambda$$

Since $\mathcal{L}_{0,\gamma}f(z)^\lambda = f(z)^\lambda$, we have

$$\frac{D^{n+1}f(z)^\zeta}{D^n f(z)^\zeta} \Rightarrow \frac{L^{n+1}_{1,\gamma}f(z)^\lambda}{L^n_{1,\gamma}f(z)^\lambda} \Rightarrow \frac{L^{n+1}_{2,\gamma}f(z)^\lambda}{L^n_{2,\gamma}f(z)^\lambda} \Rightarrow \cdots$$

and so on for all $\sigma \in N$. By lemma 4, we have

$$\zeta \frac{D^{n+1}f(z)}{D^n f(z)} \Rightarrow \lambda \frac{L^{n+1}_{1,\gamma} f(z)}{L^n_{1,\gamma} f(z)} \Rightarrow \lambda \frac{L^{n+1}_{2,\gamma} f(z)}{L^n_{2,\gamma} f(z)} \Rightarrow \cdots$$

and so on for all $\sigma \in N$, therefore

$$\lambda \frac{L^{n+1}_{\sigma,\gamma} f(z)}{L^n_{\sigma,\gamma} f(z)} = \frac{zh'(z)}{h(z)} + \lambda$$

Taking $\alpha = tan^{-1}\frac{\mu}{\eta}$ and by Lemma 5, we have

$$Re \quad e^{i\alpha} \frac{L^{n+1}_{\sigma,\gamma} f(z)}{L^n_{\sigma,\gamma} f(z)} > \frac{\eta}{|\lambda|} - \frac{2r^2}{1-r^2}$$

$$\frac{\eta - \eta r^2 - 2r}{(1-r^2)|\lambda|} > 0$$

where $|z| < r_0(\eta)$

Theorem 9. *The class $B^n_{\sigma,\gamma}(\lambda)$ is closed under the integral*

$$F(z)^\lambda = \frac{\lambda + c}{z^c} \int_0^z t^{c-1} f(t)^\lambda dt, \lambda = \eta + i\mu \tag{21}$$

Proof. From

$$F(z)^\lambda = \frac{\lambda + c}{z^c} \int_0^z t^{c-1} f(t)^\lambda dt \tag{22}$$

we have that

$$z^c F(z)^\lambda = (\lambda + c) \int_0^z t^{c-1} f(t)^\lambda dt \tag{23}$$

by differentiation , we have

$$cz^{c-1} F(z)^\lambda + z^c (F(z)^\lambda)' = (\lambda + c) z^{c-1} f(z)^\lambda \tag{24}$$

multiplying through z and by simple computation

$$z(F(z)^\lambda)' + c(F(z)^\lambda = (\lambda + c) f(z)^\lambda \tag{25}$$

so that

$$\frac{L^{n+1}_{\sigma,\gamma} F(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} + c \frac{L^n_{\sigma,\gamma} F(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} = (\lambda + c) \frac{L^n_{\sigma,\gamma} f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \tag{26}$$

define $h(z) \in P_\lambda$ by

$$\frac{L^n_{\sigma,\gamma}F(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z) \tag{27}$$

we have that

$$\frac{L^{n+1}_{\sigma,\gamma}F(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = \lambda h(z) + zh'(z) \tag{28}$$

so that

$$\frac{L^\sigma_n F(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z) + \frac{zh'(z)}{\lambda + c} \tag{29}$$

which implies that

$$Re(\psi(h(z), zh(z)) = Re\left(h(z) + \frac{zh'(z)}{\lambda + c}\right) \tag{30}$$

by lemma 1, we have $Reh(z) > 0$ and the proof completes

Theorem 10. *Let $f \in B^n_{\sigma,\gamma}(\lambda)$, then*

$$|a_2| \leq \frac{2\eta|\lambda|^{n-2}}{|\lambda+1|^n}\left|\frac{\lambda+\gamma}{\lambda+\gamma+1}\right|^\sigma$$

$$|a_3| \leq \frac{2\eta|\lambda|^{n-2}|\lambda+\gamma|^\sigma}{|\lambda+2|^n|\lambda+\gamma+2|^\sigma} max\{1, |\mathbf{M_1}|\} \tag{31}$$

where $\mathbf{M_1} = \dfrac{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma} + (1-\lambda)\eta\lambda^{n-2}(\lambda+\gamma)^\sigma(\lambda+2)^n(\lambda+\gamma+2)^\sigma}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}$ *The bounds are best possible. Equalities are obtained also by*

$$f(z)^\lambda = \left\{\eta\lambda^{n-1}\left[J^n_{\sigma,\gamma}z^\lambda\left(\frac{1+z}{1-z} + i\frac{u}{\eta}\right)\right]\right\}^{\dfrac{1}{\lambda}}$$

$$= z + \frac{\eta\lambda^{n-2}}{(\lambda+1)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+1}\right)^\sigma z^2 +$$

$$\frac{\eta(\lambda)^{n-2}(\lambda+\gamma)^\sigma}{(\lambda+2)^n(\lambda+\gamma+2)^\sigma}\left\{\frac{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma} + (1-\lambda)\eta\lambda^{n-2}(\lambda+\gamma)^\sigma(\lambda+2)^n(\lambda+\gamma+2)^\sigma}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}\right\}z^3 + \cdots$$

Proof. Let $f \in B^n_{\sigma,\lambda}(\lambda))$, then there exists $h \in P_\lambda$ such that

$$\frac{L^n_{\sigma,\lambda}f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z) = 1 + \frac{i\mu}{\eta} + c_1 z + c_2 z^2 + c_3 c^3 + \cdots \tag{32}$$

$L^n_{\sigma,\gamma}f(z)^\lambda = \lambda^n z^\lambda + \eta\lambda^{n-1}c_1 z^{\lambda+1} + \eta\lambda^{n-1}c_2 z^{\lambda+2} + \eta\lambda^{n-1}c_3 z^{\lambda+3} + \eta\lambda^{n-1}c_4 z^{\lambda+4} + \cdots$

Using the anti-derivative of the operator $L^n_{\sigma,\gamma}$ denoted as $J^n_{\sigma,\gamma}$, we have that

$$\begin{aligned} f(z)^\lambda &= z^\lambda + \frac{\eta\lambda^{n-1}}{(\lambda+1)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+1}\right)^\sigma c_1 z^{\lambda+1} + \tfrac{\eta\lambda^{n-1}}{(\lambda+2)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+2}\right)^\sigma c_2 z^{\lambda+2} \\ &\quad + \tfrac{\eta\lambda^{n-1}}{(\lambda+3)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+3}\right)^\sigma c_3 z^{\lambda+3} + \tfrac{\eta\lambda^{n-1}}{(\lambda+4)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+4}\right)^\sigma c_4 z^{\lambda+4} \cdots \end{aligned}$$

Given that

$$\begin{aligned} f(z)^\lambda &= z^\lambda + \lambda a_2 z^{\lambda+1} + \left(\lambda a_3 + \frac{\lambda(\lambda-1)}{2}a_2^2\right) z^{\lambda+2} + \left(\lambda a_4 + \lambda(\lambda-1)a_2 a_3 + \tfrac{\lambda(\lambda-1)(\lambda-2)}{6}a_2^3\right) z^{\lambda+3} \\ &\quad + \left(\lambda a_5 + \lambda(\lambda-1)a_2 a_4 + \tfrac{\lambda(\lambda-1)}{2}a_3^2 + \tfrac{\lambda(\lambda-1)(\lambda-2)}{2}a_2^2 a_3 + \tfrac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{12}a_2^4\right) z^{\lambda+4} + \cdots \end{aligned}$$

$$a_3 = \frac{\eta\lambda^{n-2}(\lambda+\gamma)^\sigma c_2}{(\lambda+2)^n(\lambda+\gamma+2)^\sigma} - \frac{(\lambda-1)\eta^2\lambda^{2(n-2)}(\lambda+\gamma)^{2\sigma}}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}\frac{c_1^2}{2}$$

By comparing the coefficient, we have

$$a_2 = \frac{\eta\lambda^{n-2}}{(\lambda+1)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+1}\right)^\sigma c_1$$

By Lemma 2, we obtained the bound of a_2

$$a_3 = \frac{\eta\lambda^{n-2}(\lambda+\gamma)^\sigma}{(\lambda+2)^n(\lambda+\gamma+2)^\sigma}\left[c_2 - \frac{\eta\lambda^{n-2}(\lambda-1)(\lambda+\gamma)^\sigma(\lambda+2)^n(\lambda+\gamma+2)^\sigma}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}\frac{c_1^2}{2}\right]$$

By Lemma 3 and with $\rho = \dfrac{\eta\lambda^{n-2}(\lambda-1)(\lambda+\gamma)^\sigma(\lambda+2)^n(\lambda+\gamma+2)^\sigma}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}$, we obtained the bound on the third coefficient of these function. By letting

$$h(z) = \frac{1+z}{1-z} + i\frac{u}{\eta}$$

from the integral representation we have the equality attained by the extremal function given.

Theorem 11. *Let $f \in B^n_{\sigma,\gamma}(\lambda)$. Then*

$$|a_3 - \rho a_2^2| \leq \frac{2\eta\lambda^{n-2}(\lambda+\gamma)^\sigma}{(\lambda+2)^n(\lambda+\gamma+2)^\sigma} max\{1, |\mathbf{M_2}|\} \tag{33}$$

where $\mathbf{M_2} = \dfrac{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma} + \eta(1+2\rho-\lambda)\lambda^{n-2}(\lambda+\gamma)^\sigma(\lambda+2)^n(\lambda+\gamma+2)^\sigma}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}$

Proof. From the computation that

$$\begin{aligned} f(z)^{\lambda} &= z^{\lambda} + \frac{\eta\lambda^{n-1}}{(\lambda+1)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+1}\right)^{\sigma} c_1 z^{\lambda+1} + \frac{\eta\lambda^{n-1}}{(\lambda+2)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+2}\right)^{\sigma} c_2 z^{\lambda+2} \\ &\quad + \frac{\eta\lambda^{n-1}}{(\lambda+3)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+3}\right)^{\sigma} c_3 z^{\lambda+3} + \cdots \end{aligned}$$

and by comparing coefficient, then

$$a_2 = \frac{\eta\lambda^{n-2}}{(\lambda+1)^n}\left(\frac{\lambda+\gamma}{\lambda+\gamma+1}\right)^{\sigma} c_1 \tag{34}$$

and

$$a_3 = \frac{\eta\lambda^{n-2}(\lambda+\gamma)^{\sigma} c_2}{(\lambda+2)^n(\lambda+\gamma+2)^{\sigma}} + \frac{(1-\lambda)\eta^2\lambda^{2(n-2)}(\lambda+\gamma)^{2\sigma}}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}\frac{c_1^2}{2} \tag{35}$$

Hence

$$|a_3 - \rho a_2^2| = \frac{\eta\lambda^{n-2}(\lambda+\gamma)^{\sigma}}{(\lambda+2)^n(\lambda+\gamma+2)^{\sigma}} c_2 - \frac{(\lambda-1+2\rho)(\lambda+2)^n\eta\lambda^{n-2}(\lambda+\gamma)^{\sigma}(\lambda+\gamma+2)^{\sigma}}{(\lambda+1)^{2n}(\lambda+\gamma+1)^{2\sigma}}\frac{c_1^2}{2} \tag{36}$$

by lemma 3 we have the required inequality

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