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LYAPUNOV OPERATOR INEQUALITIES FOR EXPONENTIAL STABILITY OF LINEAR SKEW-PRODUCT SEMIFLOWS IN BANACH SPACES

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Abstract. In the present paper we prove a sufficient condition and a characterization for the stability of linear skew-product semiflows by using Lyapunov function in Banach spaces. These are generalizations of the results obtained in [**1**] and [**12**] for the case of C_0-semigroups. Moreover, there are presented the discrete variants of the results mentioned above.

1. Introduction

The theorem of A. M. Lyapunov establishes that if A is a $n \times n$ complex matrix then A has all its characteristics roots with real parts negative if and only if for any positive definite Hermitian matrix H, there exists a positive definite Hermitian matrix W satisfying the equation

$$(L_H) \qquad A^*W + WA = -H$$

(where $*$ denotes the conjugate transpose of a matrix) (see [**2**]).

The use of the above Lyapunov operator equation is extended on the infinite-dimensional framework by Daleckij and Krein [**4**] for the case of semigroups $T(t) = \mathrm{e}^{tA}$, where A is a bounded linear operator. The authors prove in [**4**] that $\{\mathrm{e}^{tA}\}_{t\geq 0}$, with $A \in \mathcal{B}(X)$ is exponentially stable if and only if there exists $W \in \mathcal{B}(X)$, $W >> 0$ (i.e., there exists $m > 0$ such that $\langle Wx, x\rangle \geq m\|x\|^2$ for any $x \in X$), solution of the Lyapunov equation $A^*W + WA = -I$.

This result is extended by R. Datko [**5**], for the general case of C_0-semigroups as it follows.

Theorem 1.1 ([**5**]). *A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is exponentially stable if and only if there exists $W \in \mathcal{B}(X)$, $W = W^*$, $W \geq 0$ such that*

$$(L) \qquad \langle Ax, Wx\rangle + \langle Wx, Ax\rangle = -\|x\|^2$$

for all $x \in D(A)$, where A denotes the infinitezimal generator of $\{T(t)\}_{t\geq 0}$.

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C. Chicone [**3**], Y. Latushkin [**3**], A. Pazy [**9**], J. Goldstein [**6**] and L. Pandolfi [**8**] studied the Lyapunov operator equations with unbounded A. All the above results are given in the setting of one-parameter semigroups acting on Hilbert spaces.

Moreover, in [**10**], an attempt to establish an equivalence between the solvability of the Lyapunov operator equation and the exponential stability of a C_0-semigroup in the general context of Banach spaces is presented.

Also in [**12**], C. Preda and P. Preda studied the case of the Lyapunov operator equation for the exponential stability of one-parameter semigroups acting on Banach spaces by using the idea of N.U. Ahmed (see [**1**]).

For the case of linear skew-product semiflows on real Hilbert spaces, a result which presents an equality of Lyapunov type can be found in [**15**]. In that paper, Pham Viet Hai and Le Ngoc Thanh present some characterizations for the uniform exponential stability of linear skew-product semiflows using a variant of Lyapunov equality.

Some necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows in Banach spaces are given in the paper [**7**]. The authors use Banach function spaces to obtain generalizations of some well-known results of Datko, Neerven, Rolewicz and Zabczyk.

On the other hand, in the paper [**14**], Pham Viet Hai extends the results of P. Preda, A. Pogan and C. Preda from [**11**] for the case of the uniform exponential stability of linear skew-product semiflows.

In the present paper, we try to go more general and find variants of Lyapunov operator equation for the exponential stability of linear skew-product semiflows acting on Banach spaces.

This paper extends for the case of linear skew-product semiflows the results obtained in [**12**] for the case of strongly continuous, one-parameter semigroups acting on Banach spaces by using analogous techniques.

In order to do that, we need to recall some notions about the adjoint of a linear operator on a Banach space.

Let X be a real or complex Banach space and X' its (dual) conjugate space consisting of all bounded and antilinear functionals on X. Also X^* will denote the classic dual space of all bounded and linear functionals on X.

If Y is also a Banach space, we will denote by $\mathcal{B}(X,Y)$ the Banach space of all linear and bounded operators from X to Y. If $X = Y$, we will write $\mathcal{B}(X)$.

The norms on X, X', Y and $\mathcal{B}(X,Y)$ will be denoted by the symbol $\|\cdot\|$.

We will use the symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ to denote the set of real, nonnegative real and natural numbers respectively and $\mathbb{N}^* = \mathbb{N} - \{0\}$.

We will present some definitions in what follows.

Let Θ be a metric space.

Definition 1.1. A map $\sigma\colon \Theta \times \mathbb{R}_+ \to \Theta$ is said to be a *continuous semiflow on* Θ if the following conditions hold

i) $\sigma(\theta, 0) = \theta$ for all $\theta \in \Theta$;

ii) $\sigma(\theta, t+s) = \sigma(\sigma(\theta, s), t)$ for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$;
iii) $(\theta, t) \mapsto \sigma(\theta, t)$ is continuous on $\Theta \times \mathbb{R}_+$.

If iii) holds for any $t, s \in \mathbb{R}$ then σ is said to be a *flow on* Θ.

Definition 1.2. Let σ be a continuous semiflow on Θ. A *strongly continuous cocycle over the continuous semiflow* σ is an operator-valued function

$$\Phi\colon \Theta \times \mathbb{R}_+ \to \mathcal{B}(X), \qquad (\theta, t) \mapsto \Phi(\theta, t)$$

that satisfies the following properties

i) $\Phi(\theta, 0) = I$ (I – the identity operator on X) for all $\theta \in \Theta$;
ii) $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous for each $\theta \in \Theta$ and $x \in X$;
iii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$ for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$ (the cocycle identity);

If, in addition,

iv) there exist constants $M, \omega > 0$ such that

$$\|\Phi(\theta, t)\| \le M\, \mathrm{e}^{\omega t} \qquad \text{for } t \ge 0 \text{ and } \theta \in \Theta,$$

then the strongly continuous cocycle is *exponentially bounded*.

Definition 1.3. The linear skew-product semiflow (LSPS) associated with the above cocycle is the dynamical system $\pi = (\Phi, \sigma)$ on $\varepsilon = X \times \Theta$ defined by

$$\pi\colon X \times \Theta \times \mathbb{R}_+ \to X \times \Theta, \quad (x, \theta, t) \mapsto \pi(x, \theta, t) = (\Phi(\theta, t)x, \sigma(\theta, t)).$$

We will give some examples of LSPS. First of all, we will define some notions used in the following examples.

Definition 1.4. A family $\{T(t)\}_{t\ge 0}$ of linear and bounded operators acting on X is said to be *a C_0-semigroup* or *a strongly continuous semigroup* on X if the following conditions hold:

i) $T(0) = I$;
ii) $T(t+s) = T(t)T(s)$ for all $t, s \ge 0$;
iii) there exists $\lim\limits_{t\to 0_+} T(t)x = x$ for all $x \in X$.

If the second property holds for any $t, s \in \mathbb{R}$, then $\{T(t)\}_{t\in\mathbb{R}}$ is called a C_0-group.

For a general presentation of the theory of C_0-semigroups, we refer the reader to [**9**].

Definition 1.5. A family of linear and bounded operators $\{U(t,s)\}_{t\ge s\ge 0}$ is said to be *a two-parameter evolution family* if the following conditions hold:

i) $U(t,t) = I$ for all $t \ge 0$;
ii) $U(t,t_0)U(t_0,s) = U(t,s)$ for all $t \ge t_0 \ge s \ge 0$;
iii) $U(\cdot, s)x$ is continuous on $[s, \infty)$ for all $s \ge 0$, $x \in X$;
$U(t, \cdot)x$ is continuous on $[0, t)$ for all $t \ge 0$, $x \in X$;
iv) there exist $M, \omega > 0$ such that

$$\|U(t,s)\| \le M\, \mathrm{e}^{\omega(t-s)} \quad \text{for all } t \ge s \ge 0.$$

For a general presentation of the theory of two-parameter evolution families, we refer the reader to [**3**] or [**4**].

Example 1.1. Let Θ be a metric space, σ a semiflow on Θ and $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X. The pair $\pi_T = (\Phi_T, \sigma)$ where $\Phi_T(\theta, t) = T(t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$ is a linear skew-product semiflow over σ on $\Theta \times X$.

Example 1.2. Let $\Theta = \mathbb{R}_+$, $\sigma(\theta, t) = \theta + t$ and let $\{U(t, s)\}_{t\geq s}$ be an evolution family on the Banach space X. We define

$$\Phi_U(\theta, t) = U(t + \theta, \theta) \qquad \text{for all } (\theta, t) \in \Theta_+ \times \mathbb{R}_+.$$

Then $\{\Phi_U(\theta, t)\}_{\theta\in\Theta, t\geq 0}$ is an exponentially bounded, strongly continuous cocycle (over the above semiflow σ) and the linear skew-product semiflow associated with it is the pair $\pi = (\Phi_U, \sigma)$.

Therefore, we can say that the notion of a cocycle generalizes the classic notion of a two-parameter evolution family.

Example 1.3. Let Θ be a metric space, σ a semiflow on Θ, X a Banach space and $A\colon \Theta \to \mathcal{B}(X)$ a continuous mapping. The problem

$$\begin{aligned} \dot{x}(t) &= A(\sigma(\theta, t))x(t) \\ x(t_0) &= x_0 \end{aligned}$$

has an unique solution for all $t_0 \in \mathbb{R}_+$ and $x_0 \in X$. For details we refer the reader to [**13**].

Definition 1.6. A linear skew-product semiflow (LSPS) $\pi = (\Phi, \sigma)$ on a Banach bundle $\varepsilon = X \times \Theta$ is said to be *exponentially stable* if there exist constants $N, \nu > 0$ such that

$$\|\Phi(\theta, t)x\| \leq N\,\mathrm{e}^{-\nu t}\,\|x\| \qquad \text{for all } t \geq 0,\ \theta \in \Theta,\ x \in X.$$

All the results concerning the Lyapunov inequality for the exponential stability of linear skew-product semiflows (LSPS), were acting on Hilbert spaces. We will try to go more general and find variants of Lyapunov operator equation for the exponential stability of linear skew-product semiflows (LSPS) acting on Banach spaces. This requires to recall some facts about the adjoint of a linear operator on a Banach space (see [**12**]).

Definition 1.7. Let X, Y be two Banach spaces and $A \in \mathcal{B}(X, Y)$. Then there exists an unique operator $A^* \in \mathcal{B}(Y', X')$ that satisfies $y(Ax) = A^*y(x)$ for all $x \in X$ and $y \in Y'$. A^* will be called *the adjoint of A*.

It can be easily checked that

- $\|A\| = \|A^*\|$;
- $(A + B)^* = A^* + B^*$;
- $(\lambda A)^* = \overline{\lambda} A^*$;
- If X, Y are reflexive, then $A^{**} = A$.

It is worth to note that the above notion of the adjoint of a linear and bounded operator between two Banach spaces allows us to create a definition of the adjoint that directly generalizes the definition of the adjoint of an operator on Hilbert spaces. In other words, if X and Y are Hilbert spaces and $A \in \mathcal{B}(X, Y)$, then there is no difference of the adjoint between the adjoint A^* defined by considering X, Y to be Hilbert spaces, and the adjoint A^* defined by considering X, Y to be Banach spaces. If we chose that $A^*: Y^* \to X^*$, then we would obtain a different definition compared to the Hilbert space definition.

For defining the concept of a self-adjoint operator on a Banach space, we recall that X is isomorphic and isometric with a subspace of X''.

Definition 1.8.

(i) An operator $A \in \mathcal{B}(X, X')$ is said to be *self-adjoint* if the restriction of A^* to X is A, and therefore,

$$Ay(x) = \overline{Ax(y)} \qquad \text{for all} \quad x, y \in X.$$

(ii) $A \in \mathcal{B}(X, X')$ is said to be *positive* if A is self-adjoint and $Ax(x) \geq 0$ for all $x \in X$.

Remark 1.1. It is easy to see that $A \in \mathcal{B}(X, X')$ is positive if and only if $Ax(x)$ is a positive real number for all $x \in X$.

In the following we will denote by

$$\mathcal{B}^+(X, X') = \{A \in \mathcal{B}(X, X') \ : \ A \text{ is positive}\}.$$

Following Lyapunov's idea, we obtain a Lyapunov-type operatorial equation for the case of linear skew-product semiflows acting on Banach spaces. Indeed, from the equation (L_H) and (L), taking into account the fact that any C_0-semigroup is a particular case of linear skew-product semiflows, we obtain for the case of Hilbert spaces that (see [**15**])

$$(L^*) \qquad \langle A(\sigma(\theta,t))x, W(\sigma(\theta,t))x\rangle + \langle W(\sigma(\theta,t))x, A(\sigma(\theta,t))x\rangle = -\|x\|^2.$$

If we assume that (L^*) holds for some conditions, let f be the function defined by

$$f(t) = \langle W(\sigma(\theta,t))\Phi(\theta,t)x, \Phi(\theta,t)x\rangle.$$

It can be easily seen that $f'(t) = -\|\Phi(\theta,t)x\|^2$. Integrating with respect to τ on the interval $[0, t]$, we have

$$\langle W(\sigma(\theta,t))\Phi(\theta,t)x, \Phi(\theta,t)x\rangle - \langle W(\theta)x, x\rangle = -\int_0^t \|\Phi(\theta,\tau)x\|^2 \mathrm{d}\tau,$$

which implies

$$\Phi^*(\theta,t)W(\sigma(\theta,t))\Phi(\theta,t)x + \int_0^t \Phi^*(\theta,\tau)\Phi(\theta,\tau)x\mathrm{d}\tau = W(\theta)x.$$

If we rewrite the equation above to the case of Banach spaces, using the considerations about the adjoint of an operator in Banach spaces, we have

$$(L') \qquad W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x)+\int_0^t \|\Phi(\theta,\tau)x\|^2 \mathrm{d}\tau = W(\theta)x(x).$$

Remark 1.2. The bounded function $W\colon \Theta \to \mathcal{B}^+(X,X')$ from the equation (L') is said *Lyapunov function* corresponding to linear skew-product semiflow $\pi = (\Phi,\sigma)$.

2. Results

In what follows it will be presented a sufficient condition for the exponential stability of linear skew-product semiflows acting on Banach spaces in terms of Lyapunov inequation.

Theorem 2.1. *Let $\pi = (\Phi,\sigma)$ be a linear skew-product semiflow (LSPS). If there exists $W\colon \Theta \to \mathcal{B}^+(X,X')$ bounded such that*

$$(1) \qquad W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x)+\int_0^t \|\Phi(\theta,\tau)x\|^2 \mathrm{d}\tau \le W(\theta)x(x)$$

for all $t \ge 0$, $\theta \in \Theta$ and $x \in X$, then $\pi = (\Phi,\sigma)$ is exponentially stable.

Proof. Let $x \in X$, $\theta \in \Theta$ and $t \ge 0$. From (1) we have that

$$\int_0^t \|\Phi(\theta,\tau)x\|^2 \mathrm{d}\tau \le W(\theta)x(x) - W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x)$$
$$\le W(\theta)x(x) = |W(\theta)x(x)| \le K\|x\|^2$$

for all $\theta \in \Theta$, $x \in X$ and $t \ge 0$, where $K = \sup_{\theta\in\Theta} \|W(\theta)\| > 0$.

Thus we get that

$$\int_0^t \|\Phi(\theta,\tau)x\|^2 \mathrm{d}\tau \le K\|x\|^2$$

for all $\theta \in \Theta$, $x \in X$ and $t \ge 0$, which implies the following relation for $t \to \infty$

$$\int_0^\infty \|\Phi(\theta,\tau)x\|^2 \mathrm{d}\tau \le K\|x\|^2 \qquad \text{for all } \theta \in \Theta \text{ and } x \in X.$$

From [**15**, Lemma 2.4], it results that the linear skew-product semiflow $\pi = (\Phi,\sigma)$ is exponentially stable. □

In what follows, it will be presented the necessary condition which needs a stronger hypothesis.

Theorem 2.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$, there exists $W \colon \Theta \to \mathcal{B}^+(X, X')$ bounded such that*

$$W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) + \int_0^t \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)\mathrm{d}\tau = W(\theta)x(x) \tag{2}$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.

Proof. The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable and therefore we have from Definition 1.6 that there exist the constants $N, \nu > 0$ such that

$$\|\Phi(\theta,t)x\| \leq N\,\mathrm{e}^{-\nu t}\,\|x\| \qquad \text{for all } t \geq 0,\ \theta \in \Theta,\ x \in X.$$

Now we consider $x, y \in X$, $\theta \in \Theta$ and

$$W(\theta)x(y) = \int_0^\infty \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)y)\mathrm{d}\tau.$$

Next we will show that $W \in \mathcal{B}^+(X, X')$.
Thus we have that

$$\begin{aligned} |W(\theta)x(y)| &= \left| \int_0^\infty \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)y)\mathrm{d}\tau \right| \leq \int_0^\infty |\Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)y)|\,\mathrm{d}\tau \\ &\leq \|\Gamma\| \int_0^\infty \|\Phi(\theta,\tau)x\|\|\Phi(\theta,\tau)y\|\mathrm{d}\tau \leq \|\Gamma\|N^2 \int_0^\infty \mathrm{e}^{-2\nu\tau}\,\mathrm{d}\tau\|x\|\|y\| \\ &= \frac{N^2}{2\nu}\|\Gamma\|\|x\|\|y\|, \end{aligned}$$

which shows that W is linear and bounded.

On the other hand,

$$\begin{aligned} \overline{W(\theta)y(x)} &= \int_0^\infty \overline{\Gamma(\Phi(\theta,\tau)y)(\Phi(\theta,\tau)x)}\mathrm{d}\tau \\ &= \int_0^\infty \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)y)\mathrm{d}\tau = W(\theta)x(y) \end{aligned}$$

for all $x, y \in X$ and $\theta \in \Theta$. Thus, W is self-adjoint.

Moreover,

$$W(\theta)x(x) = \int_0^\infty \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)\mathrm{d}\tau \geq \gamma \int_0^\infty \|\Phi(\theta,\tau)x\|^2\mathrm{d}\tau \geq 0,$$

which implies the fact that W is positive.

It results that $W \in \mathcal{B}^+(X, X')$. Now we have that

$$\begin{aligned} & W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) \\ & = \int_0^\infty \Gamma(\Phi(\sigma(\theta,t),\tau)\Phi(\theta,t)x)(\Phi(\sigma(\theta,t),\tau)\Phi(\theta,t)x)\mathrm{d}\tau \\ & = \int_0^\infty \Gamma(\Phi(\theta,t+\tau)x)(\Phi(\theta,t+\tau)x)\mathrm{d}\tau \\ & = \int_0^\infty \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)\mathrm{d}\tau - \int_0^t \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)\mathrm{d}\tau \\ & = W(\theta)x(x) - \int_0^t \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)\mathrm{d}\tau \end{aligned}$$

and therefore, we get the relation (2) and the proof is complete. □

As a result of the last two theorems, we now obtain the necessary and sufficient conditions for the exponential stability of a linear skew-product semiflow (LSPS) as follows.

Corollary 2.1. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable if and only if for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$ for all $x \in X$, there exists $W : \mathbb{R}_+ \to \mathcal{B}^+(X, X')$ bounded such that*

$$W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) + \int_0^t \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)\mathrm{d}\tau = W(\theta)x(x) \tag{3}$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.

Proof. Necessity results from Theorem 2.2.

Sufficiency results analogously with Theorem 2.1, by considering in addition $\Gamma \in \mathcal{B}^+(X, X')$ with the same property as in Theorem 2.2. □

In what follows we will also present the discrete versions of the above results.

A sufficient condition is given as follows

Theorem 2.3. *Let $\pi = (\Phi, \sigma)$ be linear skew-product semiflow. If there exists $W : \mathbb{N} \to \mathcal{B}^+(X, X')$ bounded such that*

$$W(\sigma(\theta,n))\Phi(\theta,n)x(\Phi(\theta,n)x) + \sum_{k=0}^{n-1} \|\Phi(\theta,k)x\|^2 \leq W(\theta)x(x) \tag{4}$$

for all $\theta \in \Theta$, $n \in \mathbb{N}^*$ and $x \in X$, then the linear skew-product semiflow is exponentially stable.

Proof. We take $n \in \mathbb{N}^*$ and $x \in X$. From relation (4), we have that

$$\sum_{k=0}^{n-1} \|\Phi(\theta,k)x\|^2 \leq W(\theta)x(x) - W(\sigma(\theta,n))\Phi(\theta,n)x(\Phi(\theta,n)x)$$
$$\leq W(\theta)x(x) = |W(\theta)x(x)| \leq L\|x\|^2$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$, where $L = \sup_{\theta\in\Theta} \|W(\theta)\| > 0$.

For $n \to \infty$ in the previous relation we obtain that

$$\sum_{k=0}^{\infty} \|\Phi(\theta,k)x\|^2 \leq L\|x\|^2 < \infty \qquad \text{for all } \theta \in \Theta \text{ and } x \in X.$$

Applying [**15**, Lemma 2.1 and Lemma 2.2], we get that the linear skew product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable. □

The sufficient condition is given in the following theorem

Theorem 2.4. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma\|x\|^2$ for all $x \in X$, there exists $W\colon \Theta \to \mathcal{B}^+(X, X')$ bounded such that*

$$W(\sigma(\theta,n))\Phi(\theta,n)x(\Phi(\theta,n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)x) = W(\theta)x(x) \tag{5}$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$.

Proof. As the linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable, we have from Definition 1.6 that there exist the constants N, $\nu > 0$ such that

$$\|\Phi(\theta,n)x\| \leq N\,\mathrm{e}^{-\nu n}\,\|x\|, \qquad \text{for all } n \in \mathbb{N}\ \theta \in \Theta \text{ and } x \in X.$$

We take now $x,\ y \in X$, $n \in \mathbb{N}^*$ and

$$W(\theta)x(y) = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)y).$$

Next it will be shown that $W \in \mathcal{B}^+(X, X')$.

Therefore, we have that

$$|W(\theta)x(y)| = \left|\sum_{k=0}^{\infty} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)y)\right| \leq \sum_{k=0}^{\infty} |\Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)y)|$$
$$\leq \|\Gamma\| \sum_{k=0}^{\infty} \|\Phi(\theta,k)x\|\|\Phi(\theta,k)y\| \leq \|\Gamma\|N^2 \sum_{k=0}^{\infty} \mathrm{e}^{-2\nu k}\,\|x\|\|y\|$$
$$\leq \frac{N^2}{1-\mathrm{e}^{-2\nu}}\|\Gamma\|\|x\|\|y\|,$$

which shows that W is linear and bounded.

Moreover,

$$\overline{W(\theta)y(x)} = \sum_{k=0}^{\infty} \overline{\Gamma(\Phi(\theta,k)y)(\Phi(\theta,k)x)}$$
$$= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)y) = W(\theta)x(y)$$

for all $x, y \in X$ and $\theta \in \Theta$. Thus, W is self-adjoint.

On the other hand,

$$W(\theta)x(x) = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)x) \geq \gamma \sum_{k=0}^{\infty} \|\Phi(\theta,k)x\|^2 \geq 0,$$

which implies the fact that W is positive.

It results that $W \in \mathcal{B}^+(X, X')$. Thus we have that

$$\begin{aligned}
&W(\sigma(\theta,n))\Phi(\theta,n)x(\Phi(\theta,n)x)\\
&= \sum_{k=0}^{\infty} \Gamma(\Phi(\sigma(\theta,n),k)\Phi(\theta,n)x)(\Phi(\sigma(\theta,n),k)\Phi(\theta,n)x)\\
&= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta,n+k)x)(\Phi(\theta,n+k)x)\\
&= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)x) - \sum_{k=0}^{n-1} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)x)\\
&= W(\theta)x(x) - \sum_{k=0}^{n-1} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)x)
\end{aligned}$$

and therefore, we get the relation (5). □

As a result of Theorems 2.3 and 2.4, it can be obtained the following corollary

Corollary 2.2. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable if and only if for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$ for all $x \in X$, there exists $W : \mathbb{R}_+ \to \mathcal{B}^+(X, X')$ bounded such that*

$$W(\sigma(\theta,n))\Phi(\theta,n)x(\Phi(\theta,n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)x) = W(\theta)x(x) \tag{6}$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$.

Proof. Necessity results from Theorem 2.4.

Sufficiency results analogously with Theorem 2.3. □

Remark 2.1. As a conclusion, we can mention here that it is interesting to note that the sufficient condition can be easily obtained, but for the necessary condition, we need a stronger hypothesis. Thus, in terms of the the existence of $\Gamma \in \mathcal{B}^+(X, X')$ with the properties presented above, the exponential stability of a linear skew-product semiflow implies the existence of a Lyapunov function that verifies the Lyapunov-type equation.

Also, the sufficient condition holds in terms of the existence of $\Gamma \in \mathcal{B}^+(X, X')$.

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Bibliography

1. Ahmed N. U., *Semigroups Theory with Applications to Systems and Control*, Pittman Research, Notes Math., 1991.

2. Bellman R., *Introduction to Matrix Analysis*, McGrew-Hill, New York, 1960.

3. Chicone C., and Latushkin Y., *Evolution semigroups in dynamical systems and differential equations*, Mathemaical Surveys and Monographs, vol. 70, Providence, RO: American Mathematical Society, 1999.

4. Daleckij J. L. and Krein M.G., *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, R.I. 1974.

5. Datko R., *Extending a Theorem of Lyapunov to Hilbert Spaces*, J. Math. Anal. Appl. **32** (1970), 610–616.

6. Goldstein J., *On the equation $AX + BX = Q$*, Proc. Amer. Math. Soc. **70** (1978), 31–34.

7. Megan M., Sasu A. L. and Sasu B., *On uniform exponential stability of linear skew-product semiflows in Banach spaces*, Bull. Belg. Math. Soc. Simon Stevin **9(1)** (2002), 143–154.

8. Pandolfi L., *Lyapunov theorems for semigroups of operators*, Syst. Control. Lett. **15** (1990), 147–151.

9. Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

10. Vu Ngoc Phat, Tran Tin Kiet, *On the Lyapunov equation in Banach spaces and applications to control problems*, International J. of Mathematics and Mathematical Sciences, **39(3)** (2002), 155–166.

11. Pogan A., Preda C. and Preda P., *A discrete Lyapunov theorem for the exponential stability of evolution families*, New York J. Math. **11** (2005), 457–463.

12. Preda C. and Preda P., *Lyapunov operator inequalities for exponential stability of Banach space semigroups of operators*, Appl. Math. Letters **25(3)** (2012), 401–403.

13. Sacker R. J. and Sell G.R., *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations **113** (1994), 17–67.

14. Pham Viet Hai, *An extension of P. Preda, A. Pogan, C. Preda, Timisoara's theorems for the uniformly exponential stability of linear skew-product semiflows*, Bull. Math. Soc. Sci. Math. Roumanie Tome **53(101)**, no. 1, (2010), 69–83.

15. Pham Viet Hai andLe Ngoc Thanh; *The uniform exponential stability of linear skew-product semiflows on real Hilber space*, Math. J. Okayama Univ. **53** (2011), 173–183.

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