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RESULTS ON DIMENSION THEORY AND SOME GENERALIZATIONS OF COMPACT SPACES

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Abstract. In this paper we introduce G_δ-sequential spaces as a generalization of sequential spaces, and obtain some product theorems for $[n, m]$-compact spaces and for spaces with large inductive dimension $\leq n$.

1. Introduction

Dimension theory dates back at least to the work of P. Urysohn [**11**] and K. Menger [**8**]. Since then many mathematicians have contributed to the development of this theory. There are three notions of dimension of a topological space X, small inductive dimension (denoted by $\mathrm{ind}(X)$), large inductive dimension (denoted by $\mathrm{Ind}(X)$) and covering dimension (denoted by $\dim(X)$). If $\mathrm{ind}(X) = 0$, then X is called a zero-dimensional space. If $\dim(X) = 0$, then X is called a strongly zero-dimensional space.

In Section 2, we introduce G_δ-sequential spaces as a generalization of sequential spaces, and obtain some product theorems for $[n, m]$-compact spaces and for spaces with large inductive dimension $\leq n$. Theorems 2.9, 2.10, 2.11, 2.13 and 2.17 formulate the main results of this paper. In this paper, all spaces are assumed to be T_1 topological spaces. For terminology not defined here, see Engelking [**3**] and Willard [**12**].

2. Product theorems

Franklin [**4**] introduced sequential spaces as generalization of first countable spaces. In this section, we define G_δ-sequential spaces as a generalization of sequential spaces. We also obtain some product theorems for $[n, m]$-compact spaces and spaces with large inductive dimension $\leq n$.

Definition 2.1 ([**4**])**.** A subset A of a space X is called sequentially open if each sequence in X converging to a point in A is eventually in A. A space X is called a sequential space if every sequentially open subset of X is open.

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Definition 2.2. A space X is called G_δ-sequential if every sequentially open subset is a G_δ-set.

Definition 2.3. Let X be an arbitrary space. The G_δ-topology of X is the topology generated by the G_δ-sets of X.

Definition 2.4 ([**7**])**.** A space X is called scattered if every non-empty closed subset A of X has an isolated point.

Definition 2.5 ([**1**])**.** A space X is called $[n,m]$-compact if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq m$ has a subcover of cardinality $< n$. If X is $[n,m]$-compact for all $m > n$, then it is called $[n,\infty]$-compact. $[\aleph_0, m]$-compact spaces will be called simply m-compact.

Definition 2.6 ([**2**])**.** A space X is called paracompact if every open cover $\mathcal{U}$ of X has a locally finite open refinement.

Definition 2.7. A mapping f from a space X onto a space Y is called σ-closed if f maps closed sets onto F_σ-sets.

It is clear that every sequential space is G_δ-sequential. However a G_δ-sequential space may fail to be sequential (see Arens-Fort example [**10**, page 54]).

Kramer [**6**] showed that if X is a sequential space and Y is a countably compact space, then the projection mapping $P\colon X \times Y \to X$ is closed. A similar theorem concerning σ-closed mappings can be obtained using G_δ-sequential spaces. For this purpose we need the following lemma which can be obtained by modifying the proof of Kramer [**6**, Lemma 5.3].

Lemma 2.8. *Let X be a G_δ-sequential space and Y be a countably compact space. Let F be a closed subset of $X \times Y$ and V be an open subset of Y. Let x be a point of X such that $F(x) = \{y \in Y \mid (x,y) \in F\} \subset V$. Then there is a G_δ-set U containing x such that $z \in U$ implies $F(z) \subset V$.*

Theorem 2.9. *Let X be a G_δ-sequential space and Y be a countably compact space. Then the projection mapping $P\colon X \times Y \to X$ is σ-closed.*

The proof follows from Lemma 2.8 by taking $x \in X - P(F)$ and $V = \phi$.

Theorem 2.10. *Let f be a continuous σ-closed mapping from a space X onto a space Y such that $f^{-1}(y)$ is m-compact for each $y \in Y$. Then X is $[n,m]$-compact if the G_δ-topology of Y is so.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$, $|\Lambda| \leq m$ be an open cover of X. Let Γ denote the family of all finite subsets of Λ. Then $|\Gamma| \leq m$. Since $f^{-1}(y)$ is m-compact, we have that for each $y \in Y$, there exists a finite subset γ of Λ such that $f^{-1}(y) \subset \bigcup\{U_\alpha \mid \alpha \in \gamma\}$. Let $V_\gamma = Y - f(X - \bigcup_{\alpha\in\gamma} U_\alpha)$. Then $y \in V_\gamma$, V_γ is a G_δ-set and $f^{-1}(V_\gamma) \subset \cup\{U_\alpha \mid \alpha \in \gamma\}$. Thus $\{V_\gamma \mid \gamma \in \Gamma\}$ cover of Y, of which each element is a G_δ-set, and $|\Gamma| \leq m$. Since the G_δ-topology of Y is $[n,m]$-compact, $\{V_\gamma \mid \gamma \in \Gamma\}$ has a subcover of cardinality $< n$. Therefore X is the union of less than n members of $\left\{f^{-1}(V_\gamma) \mid \gamma \in \Gamma\right\}$. But for each $\gamma \in \Gamma$, the set $f^{-1}(V_\gamma)$ is contained in the union of finitely many members of $\mathcal{U}$. Hence X is $[n,m]$-compact. $\square$

Theorem 2.11. *Let X be a scattered, paracompact Hausdorff space. Then the G_δ-topology of X is paracompact.*

Proof. Let $\mathcal{U}$ be a cover of X by G_δ-sets. Let

$$F = \{x \in X \mid x \in U \text{ and } U \text{ is open implies } U \text{ cannot be covered by a } \sigma\text{-locally finite open refinement of } \mathcal{U}\}.$$

Obviously F is closed. Suppose $F \neq \phi$. Since X is scattered, F has an isolated point x. Thus there exists an open set $V \subseteq X$ such that $V \cap F = \{x\}$. Choose $U^* \in \mathcal{U}$ such that $x \in U^*$. Without loss of generality we can assume that $U^* = \bigcap\{V_n \mid n = 1, 2, \ldots\}$ where V_n is open for each $n = 1, 2, \ldots$, and $V_{n+1} \subseteq \overline{V_{n+1}} \subseteq V_n \subseteq V$. For each $n = 1, 2, \ldots$, $(\overline{V_n} - V_{n+1}) \subseteq X - F$. Therefore each $y \in (\overline{V_n} - V_{n+1})$ has a neighborhood M_y which can be covered by a σ-locally finite open refinement of $\mathcal{U}$.

Now $\mathcal{M} = \left\{M_y \mid y \in (\overline{V_n} - V_{n+1})\right\}$ is an open cover of $\overline{V_n} - V_{n+1}$. Since $\overline{V_n} - V_{n+1}$ is closed and X is paracompact, $\mathcal{M}$ has a locally finite (in X) open (in X) refinement, say $\mathcal{H}_n = \{H_\alpha \mid \alpha \in \Lambda_n\}$. For each $\alpha \in \Lambda_n$, H_α is covered by a σ-locally finite open refinement of $\mathcal{U}$, say $\bigcup_{i=1}^{\infty} \mathcal{A}_i^\alpha$. Let $\mathcal{B}_i^\alpha = \{H_\alpha \cap A \mid A \in \mathcal{A}_i^\alpha\}$ and $\mathcal{K}_i^n = \{B \mid B \in \mathcal{B}_i^\alpha,\ \alpha \in \Lambda_n\}$. Then $\mathcal{K}_i^n$ is a locally finite open refinement of $\mathcal{U}$, because if $x \in X$, there exists an open set N_x such that $N_x \cap H_\alpha = \phi$ for all except finitely many indices, say $\alpha_1, \alpha_2, \ldots, \alpha_n$. Each one of the collections $\mathcal{B}_i^{\alpha_1}, \mathcal{B}_i^{\alpha_2}, \ldots, \mathcal{B}_i^{\alpha_n}$ is locally finite. Hence for each $j = 1, 2, \ldots, n$, there exists an open set W_i^j and each W_i^j intersects at most finitely many members of $\mathcal{B}_i^{\alpha_j}$. Hence $W_i^1 \cap \ldots \cap W_i^n \cap N_x$ is an open neighborhood of x which intersects finitely many members of $\mathcal{K}_i^n$.

Now $\bigcup_{i=1}^{\infty} \mathcal{K}_i^n$ is an open σ-locally finite open refinement of $\mathcal{U}$ which covers $\overline{V_n} - V_{n+1}$. Consequently, $(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \mathcal{K}_i^n\ ) \cup \{U^*\}$ is an open σ-locally finite open refinement of $\mathcal{U}$ which covers V. This contradicts the fact that $x \in V$. Thus $F = \phi$. Therefore, for each $x \in V$, there is an open neighborhood G_x of x such that G_x can be covered by a σ-locally finite open refinement of $\mathcal{U}$. Since X is paracompact, $\{G_x \mid x \in X\}$ has a locally finite open refinement $\{D_\beta \mid \beta \in \Gamma\}$ where for each $\beta \in \Gamma$, D_β is covered by a σ-locally finite open refinement of $\mathcal{U}$, say $\bigcup_{i=1}^{\infty} \mathcal{C}_i^\beta$.

Let $\mathcal{G}_i = \left\{C \mid C \in \mathcal{C}_i^\beta,\ \beta \in \Gamma\right\}$. Then it is easy to see that $\mathcal{G}_i$ is locally finite. Therefore $\bigcup_{i=1}^{\infty} \mathcal{G}_i$ is a σ-locally finite open refinement of $\mathcal{U}$ which covers X. Hence the G_δ-topology of X is paracompact. □

Theorem 2.12 ([**5**]). *Let X be an $[n, \infty]$-compact scattered space. Then the G_δ-topology of X is $[n, \infty]$-compact.*

The proof follows by a similar method used in Theorem 2.11.

Theorem 2.13. *Let Y be an m-compact space and X be a G_δ-sequential scattered space. Then $X \times Y$ is $[n, m]$-compact if X is $[n, \infty]$-compact.*

Proof. By Theorem 2.9, the projection mapping $P \colon X \times Y \to X$ is closed. By Theorem 2.10, $X \times Y$ is $[n, m]$-compact. □

Definition 2.14. An open (closed) rectangle in $X \times Y$ is a set of the form $U \times V$ where U is an open (closed) subset of X and V is an open (closed) subset of Y.

The following definition was introduced by Nagata [**9**] to study the dimension of the products.

Definition 2.15. Let X and Y be two spaces. Then the product space $X \times Y$ is called an F-product if whenever H and K are disjoint closed sets in $X \times Y$, then there is an open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ of $X \times Y$ and a closed cover $\mathcal{F} = \{F_\alpha \mid \alpha \in \Lambda\}$ of $X \times Y$ such that:

(i) $\mathcal{F}$ consists of closed rectangles and $\mathcal{U}$ consists of open rectangles.
(ii) $\mathcal{U}$ is σ-locally finite.
(iii) $F_\alpha \subset U_\alpha$ for all $\alpha \in \Lambda$.
(iv) $\mathcal{U}$ refines $\{(X \times Y) - H,\ (X \times Y) - K\}$.

Kramer [**6**] proved that if X is sequential, paracompact and Hausdorff while Y is countably compact and normal, then $X \times Y$ is an F-product.

In case X is a G_δ-sequential space, we have the following theorems

Theorem 2.16. *Let X be a G_δ-sequential, paracompact, scattered and Hausdorff space. Let Y be a countably compact normal space. Then $X \times Y$ is an F-product.*

The proof follows from Theorem 2.11 and a similar technique used in the proof of the above Theorem of Kramer.

Nagata [**9**] showed that if X and Y are non-empty with $\operatorname{Ind}(X) \leq n$ while $\operatorname{Ind}(Y) \leq m$ and $X \times Y$ is a totally normal F-product, then $\operatorname{Ind}(X \times Y) \leq n + m$. Using this result together with Theorem 2.16, we get the following theorem.

Theorem 2.17. *Suppose X and Y are given as in Theorem 2.16. If $\operatorname{Ind}(X) \leq n$, $\operatorname{Ind}(Y) \leq m$ and $X \times Y$ is a totally normal, then $\operatorname{Ind}(X \times Y) \leq n + m$.*

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