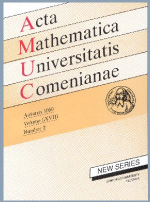

COMPOSITION OPERATOR ON THE SPACE OF FUNCTIONS TRIEBEL-LIZORKIN AND BOUNDED VARIATION TYPE

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Abstract. For a Borel-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$ and

$$\sup_{t>0} t^{-1} \int_{\mathbb{R}} \sup_{|h|\le t} |f'(x+h) - f'(x)|^p \, \mathrm{d}x < +\infty, \qquad (0 < p < +\infty),$$

we study the composition operator $T_f(g) := f \circ g$ on Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ in the case $0 < s < 1 + (1/p)$.

1. Introduction and the main result

The study of the composition operator $T_f\colon g \to f \circ g$ associated to a Borel-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ on Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$, consists in finding a characterization of the functions f such that

$$T_f(F^s_{p,q}(\mathbb{R}^n)) \subseteq F^s_{p,q}(\mathbb{R}^n). \tag{1.1}$$

The investigation to establish (1.1) was improved by several works, for example the papers of Adams and Frazier [1, 2], Brezis and Mironescu [6], Maz'ya and Shaposnikova [9], Runst and

Received August 8, 2011; revised February 17, 2012.
2010 *Mathematics Subject Classification.* Primary 46E35, 47H30.
Key words and phrases. Triebel-Lizorkin spaces; Besov spaces; functions of bounded variation; composition operators.

Sickel [12] and [10]. There were obtained some necessary conditions on f; from which we recall the following results. *For $s > 0$, $1 < p < +\infty$ and $1 \le q \le +\infty$*

- *if T_f takes $L_\infty(\mathbb{R}^n) \cap F^s_{p,q}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$, then f is locally Lipschitz continuous.*
- *if T_f takes the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$, then f belongs locally to $F^s_{p,q}(\mathbb{R})$.*

The first assertion is proved in [3, Theorem 3.1]. The proof of the second assertion can be found in [12, Theorem 2, 5.3.1].

Bourdaud and Kateb [4] introduced the functions class $U^1_p(\mathbb{R})$, the set of Lipschitz continuous functions f such that their derivatives, in the sense of distributions, satisfy

$$A_p(f') := \Big(\sup_{t>0} t^{-1} \int_{\mathbb{R}} \sup_{|h|\le t} |f'(x+h) - f'(x)|^p \, \mathrm{d}x\Big)^{1/p} < +\infty, \tag{1.2}$$

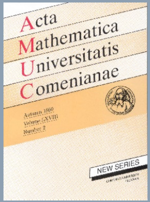
and are endowed with the seminorm

$$\|f\|_{U^1_p(\mathbb{R})} := \inf(\|g\|_\infty + A_p(g)),$$

where the infimum is taken over all functions g such that f is a primitive of g. In [4] the authors, proved the acting of the operator T_f on Besov space $B^s_{p,q}(\mathbb{R}^n)$ for $1 \le p < +\infty$, $1 < s < 1 + (1/p)$ and $f \in U^1_p(\mathbb{R})$ with $f(0) = 0$. In [5] the same result holds for $0 < s < 1 + (1/p)$.

In this work we will study the composition operator T_f on $F^s_{p,q}(\mathbb{R}^n)$ for a function f which belongs to $U^1_p(\mathbb{R})$, then we will obtain a result of type (1.1). To do this, we introduce the set $\mathcal{V}_p(\mathbb{R}^n)$ of the functions $g\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\|g\|_{\mathcal{V}_p(\mathbb{R}^n)} := \sum_{j=1}^{n} \Big(\int_{\mathbb{R}^{n-1}} \|g_{x'_j}\|^p_{BV^1_p(\mathbb{R})} \mathrm{d}x'_j\Big)^{1/p} < +\infty$$



where $BV_p^1(\mathbb{R})$ is the Wiener space of the primitives of functions of bounded p-variation (see Subsection 2.2 below for the definition) and

$$g_{x'_j}(y) := g(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n), \qquad y \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{1.3}$$

We will prove the following statement.

Theorem 1.1. *Let $0 < p, q < +\infty$ and $0 < s < 1 + (1/p)$. Then there exists a constant $c > 0$ such that the inequality*

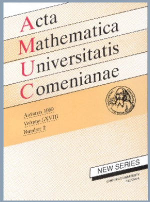
$$\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\, \|f\|_{U^1_p(\mathbb{R})} \left(\|g\|_p + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)} \right) \tag{1.4}$$

holds for all functions $g \in L_p(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ and all $f \in U^1_p(\mathbb{R})$ satisfying $f(0) = 0$. Moreover, for all such f, the operator T_f takes $L_p(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$.

Remark. (i) Since $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$, then T_f maps from $F^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$ under the assumptions of Theorem 1.1.
(ii) Since the Bessel potential spaces $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, Theorem 1.1 covers the results of composition operators in case $H^s_p(\mathbb{R}^n)$ instead of $F^s_{p,q}(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2 we collect some properties of the needed function spaces $F^s_{p,q}(\mathbb{R}^n)$ and $BV^1_p(\mathbb{R})$. Section 3 is devoted to the proof of the main result where in a first step we study the case of 1-dimensional which is the main tool when we prove Theorem 1.1. Also, our proof uses various Sobolev and Peetre embeddings, Fubini and Fatou properties, etc. In Section 4 we give some corollaries and prove the sharp estimate of (1.4).

Notation. We work with functions defined on the Euclidean space $\mathbb{R}^n$. All spaces and functions are assumed to be real-valued. We denote by $C_b(\mathbb{R}^n)$ the Banach space of bounded continuous functions on $\mathbb{R}^n$ endowed with the supremum. Let $\mathcal{D}(\mathbb{R}^n)$ (resp. $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$) denotes the C^∞-functions with compact support (resp. the Schwartz space of all C^∞ rapidly decreasing



functions and its topological dual). With $\|\cdot\|_p$ we denote the L_p-norm. We define the differences by $\Delta_h f := f(\cdot + h) - f$ for all $h \in \mathbb{R}^n$. If E is a Banach function space on $\mathbb{R}^n$, we denote by $E^{\ell oc}$ the collection of all functions f such that $\varphi f \in E$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. As usual, constants $c, c_1, \dots$ are strictly positive and depend only on the fixed parameters n, s, p, q; their values may vary from line to line.

2. Function spaces

2.1. Triebel-Lizorkin spaces

Let $0 < a \leq \infty$. For all measurable functions f on $\mathbb{R}^n$, we set

$$M_{p,q}^{s,u,a}(f) := \Big(\int_{\mathbb{R}^n} \Big(\int_0^a t^{-sq} \Big(\frac{1}{t^n} \int_{|h| \leq t} |\Delta_h f(x)|^u \, \mathrm{d}h \Big)^{q/u} \frac{\mathrm{d}t}{t} \Big)^{p/q} \mathrm{d}x \Big)^{1/p}.$$

Definition 2.1. Let $0 < p < +\infty$ and $0 < q \leq +\infty$. Let s be a real satisfying

$$1 < s < 2 \qquad \text{and} \qquad s > n \max\Big(\frac{1}{p} - 1, \frac{1}{q} - 1\Big).$$

Then, a function $f \in L_p(\mathbb{R}^n)$ belongs to $F_{p,q}^s(\mathbb{R}^n)$ if

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n)} := \|f\|_p + \sum_{j=1}^n M_{p,q}^{s-1,1,\infty}(\partial_j f) < +\infty.$$

The set $F_{p,q}^s(\mathbb{R}^n)$ is a quasi Banach space for the quasi-norm defined above. For the equivalence of the above definition with other characterizations we refer to [**15**, Theorem 3.5.3] from which we recall the following statement.

Proposition 2.2. *Let $0 < p < +\infty$ and $0 < q, u \leq +\infty$. Let s be a real satisfying*

$$1 < s < 2 \qquad \text{and} \qquad s > n \max\Big(\frac{1}{p} - \frac{1}{u}, \frac{1}{q} - \frac{1}{u}\Big).$$

Then, a function $f \in L_p(\mathbb{R}^n)$ belongs to $F^s_{p,q}(\mathbb{R}^n)$ if and only if

$$\|f\|_p + M^{s,u,\infty}_{p,q}(f) < +\infty \tag{2.1}$$

and the expression (2.1) is an equivalent quasi-norm in $F^s_{p,q}(\mathbb{R}^n)$. Moreover, this assertion remains true if one replaces $M^{s,u,\infty}_{p,q}$ by $M^{s,u,a}_{p,q}$ for any fixed $a > 0$.

The argument of the equivalence of above quasi-norms that we can replace the integration for $t \in]0, +\infty[$ by $t \leq a$ for a fixed positive number a is the part of the integral for which $t > a$ can be easily estimated by the L_p-norm.

Embeddings. Triebel-Lizorkin spaces are spaces of equivalence classes w.r.t. almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and call also the equivalence class a continuous function. Later on we need the following continuous embeddings:

(i) The spaces $F^s_{p,q}(\mathbb{R}^n)$ are monotone with respect to s and q, more exactly $F^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow F^t_{p,q}(\mathbb{R}^n) \hookrightarrow F^t_{p,\infty}(\mathbb{R}^n)$ if $t < s$ and $0 < q \leq \infty$.
(ii) With Besov spaces, we have $B^s_{p,1}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n)$.
(iii) If either $s > n/p$ or $s = n/p$ and $0 < p \leq 1$, then $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n)$.

For various further embeddings we refer to [14, 2.3.2, 2.7.1] or [12, 2.2.2, 2.2.3].

The Fatou property. Well-known the Triebel-Lizorkin space has the Fatou property, cf. [8]. We will briefly recall it. Any $f \in F^s_{p,q}(\mathbb{R}^n)$ can be approximated (in the weak sense in $\mathcal{S}'(\mathbb{R}^n)$) by a

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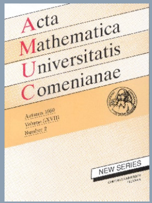


sequence $(f_j)_{j\geq 0}$ such that any f_j is an entire function of exponential type

$$f_j \in F^s_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad \limsup_{j\to+\infty} \| f_j\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\| f\|_{F^s_{p,q}(\mathbb{R}^n)}$$

with a positive constant c independent of f. Vice versa, if for a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, there exists a sequence $(f_j)_{j\geq 0}$ such that

$$f_j \in F^s_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad A := \limsup_{j\to+\infty} \| f_j\|_{F^s_{p,q}(\mathbb{R}^n)} < +\infty\,,$$

and $\lim_{j\to+\infty} f_j = f$ in the sense of distributions, then f belongs to $F^s_{p,q}(\mathbb{R}^n)$ and there exists a constant $c > 0$ independent of f such that $\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq cA$.

2.2. Functions of bounded variation

For a function $g\colon \mathbb{R} \to \mathbb{R}$, we set

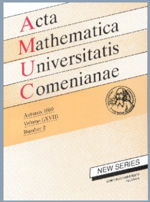
$$\nu_p(g) := \sup \Big(\sum_{k=1}^{N} |g(b_k) - g(a_k)|^p \Big)^{1/p}, \tag{2.2}$$

taken over all finite sets $\{]a_k, b_k[\,;\, k = 1, \ldots, N\}$ of pairwise disjoint open intervals. A function g is said to be of *bounded p-variation* if $\nu_p(g) < +\infty$. Clearly, by considering a finite sequence with only two terms, we obtain $|g(x) - g(y)| \leq \nu_p(g)$, for all $x, y \in \mathbb{R}$, hence g is a bounded function. The set of (generalized) primitives of functions of bounded p-variation is denoted by $BV^1_p(\mathbb{R})$ and endowed with the seminorm

$$\|f\|_{BV^1_p(\mathbb{R})} := \inf \nu_p(g),$$

where the infimum is taken over all functions g whose f is the primitive. For more details about this space we refer to [11] or [5]. However, we need to recall some embeddings

$$BV^1_p(\mathbb{R}) \hookrightarrow U^1_p(\mathbb{R}) \tag{2.3}$$



(equality in case $p = 1$), see [5, Theorem 5] for the proof which is given for $1 < p < +\infty$ and can be easily extended to $0 < p \leq 1$, see also [7, Theorem 9.3]. The Peetre embedding theorem

$$\dot{B}^{1+(1/p)}_{p,1}(\mathbb{R}) \hookrightarrow BV^1_p(\mathbb{R}) \hookrightarrow \dot{B}^{1+(1/p)}_{p,\infty}(\mathbb{R})\,, \quad (1 \leq p < +\infty), \tag{2.4}$$

where the dotted space is the *homogeneous* Besov space.

Example. Let $\alpha \in \mathbb{R}$. We put $u_\alpha(x) := |x + \alpha| - |\alpha|$ for all $x \in \mathbb{R}$, and

$$f_\alpha(x, y) := u_\alpha(x)\chi_{[0,1]}(y) + u_\alpha(y)\chi_{[0,1]}(x), \quad \forall x, y \in \mathbb{R},$$

where $\chi_{[0,1]}$ denotes the indicatrix function of $[0, 1]$. Clearly that $\nu_p(u'_\alpha) = 2$ and $\|\chi_{[0,1]}\|_{BV^1_p(\mathbb{R})} = 0$. Then it holds $f_\alpha \in \mathcal{V}_p(\mathbb{R}^2)$ with $\|f_\alpha\|_{\mathcal{V}_p(\mathbb{R}^2)} = 4$. The $\mathcal{V}_p(\mathbb{R}^n)$ space is defined in Section 1.

3. Proof of the result

Theorem 1.1 can be obtained from the following statement.

Proposition 3.1. *Let $0 < p, q < +\infty$, $0 < u < \min(p, q)$ and $0 < s < 1/p$. Then there exists a constant $c > 0$ such that the inequality*

$$M^{s,u,\infty}_{p,q}((f \circ g)') \leq c\|f\|_{U^1_p(\mathbb{R})}\, \|g\|_{BV^1_p(\mathbb{R})} \tag{3.1}$$

holds for all $f \in U^1_p(\mathbb{R}) \cap C^1(\mathbb{R})$ and all real analytic functions g in $BV^1_p(\mathbb{R})$.

Proof. For a better readability we split our proof in two steps.
Step 1. Let us prove

$$M^{s,u,a}_{p,q}((f \circ g)') \leq c\, a^{(1/p)-s}\, \|f\|_{U^1_p(\mathbb{R})}\, \|g\|_{BV^1_p(\mathbb{R})} \tag{3.2}$$

for all $a > 0$ and all $f \in U^1_p(\mathbb{R}) \cap C^1(\mathbb{R})$ and all real analytic functions g in $BV^1_p(\mathbb{R})$.

Assume first $a = 1$. By the assumptions on f and g it holds $(f \circ g)' = (f' \circ g)\, g'$. We have $\|(f \circ g)'\|_\infty \leq \|f'\|_\infty \, \|g'\|_\infty$ and

$$|\Delta_h((f' \circ g)g')(x)| \leq \|f'\|_\infty \, |\Delta_h g'(x)| \; + \; |g'(x)| \, |\Delta_h (f' \circ g)(x)|.$$

Hence

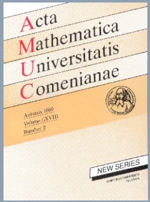
$$M^{s,u,1}_{p,q}((f \circ g)') \leq \|f'\|_\infty \, M^{s,u,1}_{p,q}(g') + V(f;g),$$

where

$$\begin{aligned} &V(f;g) \\ &:= \Big(\int_{\mathbb{R}} \Big(\int_0^1 t^{-sq} \Big(\frac{1}{t} \int_{-t}^{t} |\Delta_h (f' \circ g)(x)|^u |g'(x)|^u \mathrm{d}h \Big)^{q/u} \frac{\mathrm{d}t}{t} \Big)^{p/q} \mathrm{d}x \Big)^{1/p}. \end{aligned} \tag{3.3}$$

Estimate of $M^{s,u,1}_{p,q}(g')$. By writing $\int_0^1 \cdots = \sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} \cdots$ and by an elementary computation, we have

$$\begin{aligned} \int_0^1 t^{-sq} \Big(\frac{1}{t} \int_{-t}^{t} |\Delta_h g'(x)|^u \, \mathrm{d}h \Big)^{q/u} \frac{\mathrm{d}t}{t} &\leq c_1 \sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} t^{-sq} \sup_{|h| \leq t} |\Delta_h g'(x)|^q \, \frac{\mathrm{d}t}{t} \\ &\leq c_2 \sum_{j=0}^\infty 2^{jsq} \sup_{|h| \leq 2^{-j}} |\Delta_h g'(x)|^q. \end{aligned}$$



Let $\alpha := \min(1, p/q)$. By using the monotonicity of the ℓ_r-norms (i.e. $\ell_1 \hookrightarrow \ell_{1/\alpha}$) and by the Minkowski inequality w.r.t $L_{p/(\alpha q)}$, since $q < +\infty$, we obtain

$$\begin{aligned} M^{s,u,1}_{p,q}(g') &\le c_1 \Big(\int_{\mathbb{R}} \Big(\sum_{j=0}^{\infty} 2^{js\alpha q} \sup_{|h| \le 2^{-j}} |\Delta_h g'(x)|^{\alpha q} \Big)^{p/(\alpha q)} \,\mathrm{d}x \Big)^{1/p} \\ &\le c_2 \Big(\sum_{j=0}^{\infty} 2^{js\alpha q} \Big(\int_{\mathbb{R}} \sup_{|h| \le 2^{-j}} |\Delta_h g'(x)|^{p} \,\mathrm{d}x \Big)^{(\alpha q)/p} \Big)^{1/(\alpha q)} \\ &\le c_3 \Big(\sum_{j=0}^{\infty} 2^{j(s-(1/p))\alpha q} \Big)^{1/(\alpha q)} \|g\|_{U^1_p(\mathbb{R})}. \end{aligned}$$

From the embedding (2.3) and the assumption on s, the desired estimate holds.

Estimate of $V(f;g)$. In (3.3) the integral with respect to h can be limited to the interval $[0,t]$ denoting the corresponding expression by $V_+(f;g)$. Let us notice that the estimate with respect to $[-t,0]$ will be completely similar.

Again, by applying the Minkowski inequality twice, it holds

$$\begin{aligned} &V_+(f;g) \\ &\le \Big(\int_{\mathbb{R}} \Big(\int_0^1 \Big(\int_h^1 t^{-(s+(1/u))q} |\Delta_h (f' \circ g)(x)|^q \, |g'(x)|^q \frac{\mathrm{d}t}{t} \Big)^{u/q} \mathrm{d}h \Big)^{p/u} \mathrm{d}x \Big)^{1/p} \\ &\le \Big(\int_0^1 \Big(\int_{\mathbb{R}} |\Delta_h (f' \circ g)(x)|^p \, |g'(x)|^p \,\mathrm{d}x \Big)^{u/p} \Big(\int_h^{\infty} t^{-(s+(1/u))q} \frac{\mathrm{d}t}{t} \Big)^{u/q} \mathrm{d}h \Big)^{1/u} \\ &\le c \Big(\int_0^1 h^{-(su+1)} \Big(\int_{\mathbb{R}} |\Delta_h (f' \circ g)(x)|^p \, |g'(x)|^p \,\mathrm{d}x \Big)^{u/p} \mathrm{d}h \Big)^{1/u}. \end{aligned}$$

Case 1: Assume that g' does not vanish on $\mathbb{R}$. By the Mean Value Theorem and by the change of variable $y = g(x)$, we find

$$\begin{aligned}
&V_+(f;g)\\
&\le c_1 \|g'\|_\infty^{1-(1/p)} \Big(\int_0^1 h^{-(su+1)} \Big(\int_{\mathbb{R}} \sup_{|v| \le h\|g'\|_\infty} |f'(v+y) - f'(y)|^p \mathrm{d}y \Big)^{u/p} \mathrm{d}h \Big)^{1/u}\\
&\le c_2 \|f\|_{U_p^1(\mathbb{R})} \|g'\|_\infty \Big(\int_0^1 h^{u((1/p)-s)-1} \mathrm{d}h \Big)^{1/u}\\
&\le c_3 \|f\|_{U_p^1(\mathbb{R})} \|g\|_{BV_p^1(\mathbb{R})} .
\end{aligned}$$

Case 2: Assume that the set of zeros of g' is nonempty. Then it is a discrete set whose complement in $\mathbb{R}$ is the union of a family $(I_l)_l$ of open disjoint intervals. For any $h > 0$, we denote by $I'_{l,h}$ the set of $x \in I_l$ whose distance to the boundary of I_l is greater than h. We set

$$I''_{l,h} := I_l \setminus I'_{l,h} \quad \text{and} \quad g_l := g_{|I_l} .$$

Clearly the function g_l is a diffeomorphism of I_l onto $g(I_l)$. Let us notice that $I'_{l,h}$ is an open interval, possibly empty. In case it is not empty, we have

$$| g(g_l^{-1}(y) + h) - y | \le h \sup_{I_l} |g'|, \qquad \forall y \in g_l(I'_{l,h}). \tag{3.4}$$

The set $I''_{l,h}$ is an interval of length at most $2h$ or the union of two such intervals, and g' vanishes at one of the endpoints of these or those intervals.

We write $V_+(f;g) \le V_1(f;g) + V_2(f;g)$, where

$$V_1(f;g) := \Big(\int_0^1 h^{-(su+1)} \Big(\sum_l \int_{I'_{l,h}} |\Delta_h(f' \circ g)(x)|^p \, |g'(x)|^p \, \mathrm{d}x \Big)^{u/p} \mathrm{d}h \Big)^{1/u}$$

Go back
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and $V_2(f;g)$ is defined in the same way by replacing $I'_{l,h}$ by $I''_{l,h}$.

Estimate of $V_1(f;g)$. By the change of variable $y = g_l(x)$ and by (3.4), we deduce

$$\begin{aligned} V_1(f;g) \leq \Bigg(\int_0^1 h^{-(su+1)} \Bigg(\sum_l \sup_{I_l} |g'|^{p-1} \\ \times \int_{g(I'_{l,h})} \sup_{|v| \leq h \sup_{I_l} |g'|} |f'(v+y) - f'(y)|^p \,\mathrm{d}y \Bigg)^{u/p} \mathrm{d}h \Bigg)^{1/u} \\ \leq c_1 \, \|f\|_{U^1_p(\mathbb{R})} \Bigg(\sum_l \sup_{I_l} |g'|^p \Bigg)^{1/p} \Bigg(\int_0^1 h^{u((1/p)-s)-1} \,\mathrm{d}h \Bigg)^{1/u} \\ \leq c_2 \, \|f\|_{U^1_p(\mathbb{R})} \Bigg(\sum_l \sup_{I_l} |g'|^p \Bigg)^{1/p}. \end{aligned}$$

Hence it suffices to show

$$\Bigg(\sum_l \sup_{t \in I_l} |g'(t)|^p \Bigg)^{1/p} \leq c \, \|g\|_{BV^1_p}. \tag{3.5}$$

Indeed, by the assumption on g, for any I_l there exists $\xi_l \in I_l$ such that

$$|g'(\xi_l)| = \sup_{t \in I_l} |g'(t)|.$$

Furthermore, set β_l the right endpoint of I_l. The open intervals $\{\,]\xi_l, \beta_l[\,\}_l$ are pairwise disjoint. Then the assertion (3.5) follows from

$$\sum_l \sup_{t \in I_l} |g'(t)|^p = \sum_l |g'(\xi_l) - g'(\beta_l)|^p \leq \nu_p(g')^p.$$

(See (2.2) for the definition of ν_p).

Estimate of $V_2(f;g)$. Using both the elementary inequality $|\Delta_h(f'\circ g)(x)| \le 2\|f'\|_\infty$ and the properties of $I''_{l,h}$, it holds

$$V_2(f;g) \le c_1\,\|f'\|_\infty \Big(\sum_l \sup_{I_l} |g'|^p\Big)^{1/p} \Big(\int_0^1 h^{u((1/p)-s)-1}\,\mathrm{d}h\Big)^{1/u}$$
$$\le c_2\,\|f\|_{U^1_p(\mathbb{R})}\|g\|_{BV^1_p(\mathbb{R})}.$$

Hence we obtain (3.2) with $a=1$. We put $g_\lambda(x) := g(\lambda x)$ for all $x\in\mathbb{R}$ and all $\lambda>0$. Then (3.2) can be obtained for all $a>0$ since $\|g_a\|_{BV^1_p(\mathbb{R})} = a\|g\|_{BV^1_p(\mathbb{R})}$ and

$$M^{s,u,a}_{p,q}((f\circ g)') = a^{(1/p)-s-1}\,M^{s,u,1}_{p,q}((f\circ g_a)').$$

Step 2: Proof of (3.1). Let $a>0$. Let f and g be as in Proposition 3.1. By Proposition 2.2 it holds

$$M^{s,u,\infty}_{p,q}((f\circ g)') \le \|(f\circ g)'\|_{F^s_{p,q}(\mathbb{R})} = \|(f\circ g)'\|_p + M^{s,u,a}_{p,q}((f\circ g)').$$

Applying (3.2), we obtain

$$M^{s,u,\infty}_{p,q}((f\circ g)') \le \|f'\|_\infty\|g'\|_p + c_1\,a^{(1/p)-s}\,\|f\|_{U^1_p(\mathbb{R})}\,\|g\|_{BV^1_p(\mathbb{R})} \tag{3.6}$$

with a positive constant c_1 depending only on s,p and q (see the end of Step 1). Now, by replacing g by g_λ in (3.6), (g_λ is defined in Step 1), and by using the equality

$$M^{s,u,\infty}_{p,q}((f\circ g_\lambda)') = \lambda^{s+1-(1/p)}\,M^{s,u,\infty}_{p,q}((f\circ g)'),$$

we deduce

$$M^{s,u,\infty}_{p,q}((f\circ g)') \leq \lambda^{-s}\|f'\|_\infty\|g'\|_p + c_1 a^{(1/p)-s}\lambda^{(1/p)-s}\|f\|_{U^1_p(\mathbb{R})}\|g\|_{BV^1_p(\mathbb{R})} \tag{3.7}$$

for all $a, \lambda > 0$. Taking $a = 1/\lambda$. Now letting $\lambda \to +\infty$ in (3.7), we obtain the desired result. □

Remark. Proposition 3.1 is also valid in the n-dimensional case. The inequality (3.1) becomes

$$M^{s-1,u,\infty}_{p,q}(\partial_j(f\circ g)) \leq c\|f\|_{U^1_p(\mathbb{R})}\,\|g\|_{\mathcal{V}_p(\mathbb{R}^n)}, \qquad (j = 1,\dots,n)$$

for all $f \in U^1_p(\mathbb{R}) \cap C^1(\mathbb{R})$ and all real analytic functions g in $\mathcal{V}_p(\mathbb{R}^n)$.

Proof of Theorem 1.1. Step 1. Observe that the conditions $f(0) = 0$ and $f' \in L_\infty(\mathbb{R})$ imply

$$\|f\circ g\|_p \leq \|f'\|_\infty\,\|g\|_p$$

which is sufficient for the estimate $T_f(g)$ with respect to $L_p(\mathbb{R}^n)$-norm.

Step 2: The case $1 < s < 1 + (1/p)$ and $n = 1$. We first consider a function $f \in U^1_p(\mathbb{R})$, of class C^1 and a function g real analytic in $L_p(\mathbb{R}) \cap BV^1_p(\mathbb{R})$. By Proposition 3.1, it holds

$$\|f\circ g\|_{F^s_{p,q}(\mathbb{R})} \leq c\,\|f\|_{U^1_p(\mathbb{R})}\Big(\|g\|_p + \|g\|_{BV^1_p(\mathbb{R})}\Big). \tag{3.8}$$

Now we prove (3.8) in the general case. Let $g \in L_p(\mathbb{R}) \cap BV^1_p(\mathbb{R})$ and $f \in U^1_p(\mathbb{R})$. We introduce a function $\rho \in \mathcal{D}(\mathbb{R})$ satisfying $\rho(0) = 1$, and we set $\varphi_j(x) := 2^{jn}\mathcal{F}^{-1}\rho(2^j x)$ for all $x \in \mathbb{R}$ and all $j \in \mathbb{N}$; here $\mathcal{F}^{-1}\rho$ denotes the inverse Fourier transform of ρ. We set also

$$f_j := \varphi_j * f - \varphi_j * f(0) \quad \text{and} \quad g_j := \varphi_j * g.$$

Then the function g_j is real analytic and $g_j \to g$ in $L_p(\mathbb{R})$. We have also

$$\|g_j\|_{BV^1_p(\mathbb{R})} \leq c\,\|g\|_{BV^1_p(\mathbb{R})}\ , \qquad \forall j \in \mathbb{N}. \tag{3.9}$$

To prove (3.9), let $\{]a_k, b_k[,\ k = 1, \dots, N\}$ be a set of pairwise disjoint intervals. By the Minkowski inequality, it holds

$$\Big(\sum_{k=1}^{N}\Big|\int_{\mathbb{R}}\varphi_j(y)\left(g'(b_k-y)-g'(a_k-y)\right)\mathrm{d}y\Big|^p\Big)^{1/p} \leq \int_{\mathbb{R}}|\varphi_j(y)|\Big(\sum_{k=1}^{N}\Big|g'(b_k-y)-g'(a_k-y)\Big|^p\Big)^{1/p}\mathrm{d}y.$$

Now, for all $y \in \mathbb{R}$, the intervals $]a_k - y, b_k - y[\ (k = 1, \dots, N)$ are pairwise disjoint. Then

$$\Big(\sum_{k=1}^{N}|g_j'(b_k)-g_j'(a_k)|^p\Big)^{1/p} \leq \|\mathcal{F}^{-1}\rho\|_1\nu_p(g')\,, \qquad \forall j \in \mathbb{N}.$$

Hence we obtain (3.9).

The functions f_j are C^∞ such that $f_j(0) = 0$ and satisfy

$$\|f_j\|_{U_p^1(\mathbb{R})} \leq c\,\|f\|_{U_p^1(\mathbb{R})}\,, \qquad \forall j \in \mathbb{N}. \tag{3.10}$$

To prove (3.10), for all $t > 0$ and all $h \in [-t, t]$ we trivially have

$$|\varphi_j * f'(x+h) - \varphi_j * f'(x)| \leq \int_{\mathbb{R}}|\varphi_j(y)|\sup_{|z|\leq t}|f'(x-y+z)-f'(x-y)|\mathrm{d}y.$$

Acta Mathematica Universitatis Comenianae

By the Minkowski inequality, we have

$$\begin{aligned}\int_{\mathbb{R}} \sup_{|h|\le t} |\varphi_j * f'(x+h) - \varphi_j * f'(x)|^p \mathrm{d}x \\ \le \Big(\int_{\mathbb{R}} |\varphi_j(y)| \Big(\int_{\mathbb{R}} \sup_{|z|\le t} |f'(x-y+z) - f'(x-y)|^p \mathrm{d}x \Big)^{1/p} \mathrm{d}y \Big)^p \\ \le t \|\mathcal{F}^{-1}\rho\|_1^p \, A_p(f')^p, \qquad \text{(see (1.2) for the definition of } A_p).\end{aligned}$$

Consequently,

$$A_p(f_j') + \|f_j'\|_\infty \le \|\mathcal{F}^{-1}\rho\|_1 (A_p(f') + \|f'\|_\infty)$$

and we obtain the desired result.

On the other hand, we have

$$\lim_{j\to+\infty} \|f_j - f\|_\infty = 0. \tag{3.11}$$

To prove (3.11), since $\lim_{j\to+\infty} \varphi_j * f(0) = f(0) = 0$, the Lipschitz continuous of f yields

$$\begin{aligned}|f_j(x) - f(x)| &\le \|f'\|_\infty \int_{\mathbb{R}} |x-y||\varphi_j(x-y)|\mathrm{d}y \, + |\varphi_j * f(0)| \\ &\le c\, 2^{-j} \|f'\|_\infty \, + |\varphi_j * f(0)|.\end{aligned}$$

Then the desired result holds. By the same argument, we obtain

$$\|g_j - g\|_\infty \le c\, 2^{-j} \|g'\|_\infty. \tag{3.12}$$

Now we apply (3.8) to f_j and g_j. Then by (3.9) and (3.10), we obtain

$$\|f_j \circ g_j\|_{F^s_{p,q}(\mathbb{R})} \le c \, \|f\|_{U^1_p(\mathbb{R})} \Big(\|g\|_p + \|g\|_{BV^1_p(\mathbb{R})} \Big). \tag{3.13}$$

The elementary inequality

$$\|f \circ g - f_j \circ g_j\|_\infty \le \|f'\|_\infty \|g - g_j\|_\infty + \|f - f_j\|_\infty$$

complemented by (3.11)–(3.12) yields the convergence of the sequence $\{f_j \circ g_j\}_{j\in\mathbb{N}}$ to $f \circ g$ in $L_\infty(\mathbb{R})$. Since

$$|\langle f_j \circ g_j - f \circ g, \psi\rangle| \le \|f_j \circ g_j - f \circ g\|_\infty \|\psi\|_1, \qquad \forall \psi \in \mathcal{D}(\mathbb{R}),$$

thus we conclude that $\lim_{j\to+\infty} f_j \circ g_j = f \circ g$ in the sense of distributions. Hence, by the Fatou property of $F^s_{p,q}(\mathbb{R})$, see Subsection 2.1, we deduce (3.8).

Step 3: The case $1 < s < 1 + (1/p)$ *and* $n \ge 2$. We use the notation (1.3). Since Triebel-Lizorkin space has the Fubini property (see [**12**, p. 70]), by (3.1) it holds

$$\begin{aligned}
\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} &\le c_1 \sum_{j=1}^n \Big(\int_{\mathbb{R}^{n-1}} \|f \circ g_{x'_j}\|^p_{F^s_{p,q}(\mathbb{R})} \mathrm{d}x'_j \Big)^{1/p} \\
&\le c_2 \, \|f\|_{U^1_p(\mathbb{R})} \sum_{j=1}^n \Big(\int_{\mathbb{R}^{n-1}} \Big(\|g_{x'_j}\|^p_p + \|g_{x'_j}\|^p_{BV^1_p(\mathbb{R})} \Big) \mathrm{d}x'_j \Big)^{1/p} \\
&\le c_3 \, \|f\|_{U^1_p(\mathbb{R})} \Big(\|g\|_p + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)} \Big).
\end{aligned}$$

Step 4: The case $0 < s \le 1$. Due to the monotonicity of the Triebel-Lizorkin scale with respect to the smoothness parameter s, the result holds. Indeed, let $1 < t < 1 + (1/p)$. From Step 3, we have (1.4) with $\|f \circ g\|_{F^t_{p,q}(\mathbb{R}^n)}$ instead of $\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)}$. Now we apply the continuous embedding $F^t_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n)$. This completes the proof. $\square$

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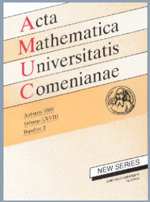
Go back
Full Screen
Close
Quit



Remark. In case $n = 1$ and $1 \le p, q < +\infty$ the inequality (1.4) becomes

$$\|f \circ g\|_{F^s_{p,q}(\mathbb{R})} \le c\|f\|_{U^1_p(\mathbb{R})}\Big(\|g\|_{F^s_{p,q}(\mathbb{R})} + \|g\|_{BV^1_p(\mathbb{R})}\Big)$$

for all $g \in L_p(\mathbb{R}) \cap BV^1_p(\mathbb{R})$, since $F^s_{p,q}(\mathbb{R}) \cap BV^1_p(\mathbb{R}) = L_p(\mathbb{R}) \cap BV^1_p(\mathbb{R})$ if $s < 1 + (1/p)$. To prove this equality, we have $\dot{B}^{1+(1/p)}_{p,\infty}(\mathbb{R}) \cap L_p(\mathbb{R}) = B^{1+(1/p)}_{p,\infty}(\mathbb{R})$ (see [**12**, 2.6.2, p. 95]). Then by (2.4) and by both $B^{1+(1/p)}_{p,\infty}(\mathbb{R}) \hookrightarrow B^s_{p,1}(\mathbb{R})$ and $B^s_{p,1}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n)$, it holds $L_p(\mathbb{R}) \cap BV^1_p(\mathbb{R}) \hookrightarrow F^s_{p,q}(\mathbb{R})$.

4. Concluding remarks

4.1. Some corollaries

In this section we fix a smooth cut-off function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(x) = 1$ for $|x| \le 1$. We put $\varphi_t(x) := \varphi\big(t^{-1}x\big)$, $\forall x \in \mathbb{R}$ and for all $t > 0$. Also for brevity we introduce the space $\mathcal{F}^s_{p,q}(\mathbb{R}^n) := F^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ endowed with the quasi-norm

$$\|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} := \|f\|_{F^s_{p,q}(\mathbb{R}^n)} + \|f\|_\infty.$$

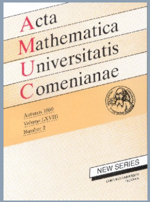
Theorem 1.1 has a consequence for the case of functions f which are only locally in $U^1_p(\mathbb{R})$.

Corollary 4.1. *Let s, p, q be real numbers as in Theorem 1.1. Then there exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} \le c\|f\varphi_{\|g\|_\infty}\|_{U^1_p(\mathbb{R})}\Big(\|g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)}\Big) \tag{4.1}$$

holds for all functions $g \in \mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ and all $f \in U^{1,\ell oc}_p(\mathbb{R})$ satisfying $f(0) = 0$. Moreover, for all such functions f, the composition operator T_f takes $\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ to $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$.

Proof. Since $f \circ g = (f\varphi_{\|g\|_\infty}) \circ g$ and $(f\varphi_t)(0) = 0$, the result follows from Theorem 1.1. □



There is consequence of Theorem 1.1 that we can obtain the equivalence of acting condition and boundedness.

Corollary 4.2. *Let s, p, q be real numbers as in Theorem 1.1. Let f be a function in $U_p^{1,\ell oc}(\mathbb{R})$ satisfying $f(0) = 0$. Then the following assertions are equivalent:*

(i) *T_f satisfies the acting condition $T_f(\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)) \subseteq \mathcal{F}^s_{p,q}(\mathbb{R}^n)$.*

(ii) *T_f maps bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ into bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$.*

Proof. Let $t > 0$. By (4.1), it holds

$$\|f \circ g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} \leq c\, t \|f \varphi_t\|_{U^1_p(\mathbb{R})} \tag{4.2}$$

for all $g \in \mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ such that $\|g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)} \leq t$. Now, from (4.2), we conclude that T_f maps bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ into bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$. □

Remark. If $n/p < s < 1 + (1/p)$, then we can replace $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$ by $F^s_{p,q}(\mathbb{R}^n)$ in Corollaries 4.1–4.2, since $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n)$.

We show that Theorem 1.1 can be extended to the case of the boundedness between Besov spaces and Triebel-Lizorkin spaces.

Corollary 4.3. *Let $1 \leq p, q < +\infty$ and $0 < s < 1 + (1/p)$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\, \|f\|_{U^1_p(\mathbb{R})}\, \|g\|_{B^{1+(1/p)}_{p,1}(\mathbb{R}^n)}$$

holds for all functions $g \in B^{1+(1/p)}_{p,1}(\mathbb{R}^n)$ and all $f \in U^1_p(\mathbb{R})$ satisfying $f(0) = 0$. Moreover, for all such functions f, the operator T_f takes $B^{1+(1/p)}_{p,1}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$.

Go back
Full Screen
Close
Quit

Proof. This is an easy consequence of Theorem 1.1 and the following continuous embedding

$$B^{1+(1/p)}_{p,1}(\mathbb{R}^n) \hookrightarrow \mathcal{V}_p(\mathbb{R}^n). \tag{4.3}$$

To prove (4.3), we use the notation (1.3) and the equivalent norm in Besov space given by

$$\|f\|_p + \sum_{j=1}^{n} \left(\int_0^1 t^{-sq} \, \|\Delta^2_{te_j} f\|_p^q \, \frac{\mathrm{d}t}{t} \right)^{1/q}, \qquad (0 < s < 2),$$

where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, see [**15**, p. 96].

Let $f \in B^{1+(1/p)}_{p,1}(\mathbb{R}^n)$. Since $\dot{B}^{1+(1/p)}_{p,1}(\mathbb{R}) \cap L_p(\mathbb{R}) = B^{1+(1/p)}_{p,1}(\mathbb{R})$ (in the sense of equivalent norms, see, e.g. [**15**]), then by (2.4), we get

$$\|f\|_{\mathcal{V}_p(\mathbb{R}^n)} \le c \sum_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} \|f_{x'_j}\|^p_{B^{1+(1/p)}_{p,1}(\mathbb{R})} \mathrm{d}x'_j \right)^{1/p}.$$

Using the Minkowski inequality with respect to $L_p(\mathbb{R}^{n-1})$, it follows

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^1 t^{-(1+(1/p))} \, \|\Delta^2_{te_k} f_{x'_j}\|_p \, \frac{\mathrm{d}t}{t} \right)^p \mathrm{d}x'_j \le \left(\int_0^1 t^{-(1+(1/p))} \, \|\Delta^2_{te_k} f\|_p \, \frac{\mathrm{d}t}{t} \right)^p$$

for $j, k \in \{1, \ldots, n\}$. Then we obtain the desired result. □

Remark. As in Corollary 4.1 we can see the case when the function f associated to the composition operator T_f belongs locally to $U^1_p(\mathbb{R})$. Indeed, if $1 \le p, q < +\infty$ and $0 < s < 1 + (1/p)$, it holds that

$$\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \le c \, \|f \varphi_{\|g\|_\infty}\|_{U^1_p(\mathbb{R})} \, \|g\|_{B^{1+(1/p)}_{p,1}(\mathbb{R}^n)}$$

for all $f \in U^{1,\ell oc}_p(\mathbb{R})$ such that $f(0) = 0$ and all $g \in B^{1+(1/p)}_{p,1}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$.

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4.2. Sharpness of estimate

For simplicity we define

$$\|g\| := \|g\|_{F^s_{p,q}(\mathbb{R}^n)} + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)}.$$

According to Corollary 4.1, there is a substantial class of *nonlinear* functions f for which there exist constants $c_f = c(f) > 0$ such that

$$\|\, f \circ g \,\|_{F^s_{p,q}(\mathbb{R}^n)} \le c_f \, \|g\|, \qquad \forall g \in F^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n).$$

In this form the inequality is *optimal* if we avoid *linear* functions in the following sense.

Proposition 4.4. *Let* $\Omega\colon [0,+\infty) \to [0,+\infty)$ *be a continuous function satisfying*

$$\lim_{t\to+\infty} t^{1/p}\,\Omega(t) = 0. \tag{4.4}$$

If f is a function such that the inequality

$$\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \le \Omega(\|g\|) \tag{4.5}$$

holds for all $g \in F^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$, then f is an affine function (linear, if we assume that $f(0) = 0$).

Proof. Let us define a smooth function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi(x) = 1$ on the cube $Q := [-1,1]^n$ and $\varphi(x) = 0$ if $x \notin 2Q$. We put $\Delta_h^2 := \Delta_h \circ \Delta_h$ and

$$g_a(x) := a x_1 \varphi(x)\,, \qquad (x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1},\ a > 0).$$

We have $\|g_a\| \sim a$ and

$$\Delta_h^2 (f \circ g_a)(x) = \Delta^2_{a h_1} f(a x_1), \qquad (\forall x \in \tfrac{1}{2(\sqrt{n})} Q,\ \forall h \in \tfrac{1}{4(\sqrt{n})} Q,\ \forall a > 0).$$

On the other hand, for all $h \in \frac{1}{4(\sqrt{n})}Q$ (i.e. $|h| \le 1/4$), we have

$$\|\Delta_h^2(f\circ g_a)\|_p \ge \Big(\int_{x\in(1/(2\sqrt{n}))Q} |\Delta_h^2(f\circ g_a)(x)|^p \mathrm{d}x\Big)^{1/p}$$
$$\ge c\,a^{-1/p}\Big(\int_{-a/(2\sqrt{n})}^{a/(2\sqrt{n})} |\Delta_{ah_1}^2 f(y)|^p \mathrm{d}y\Big)^{1/p}.$$

By the above inequality, the embedding $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n)$ and the assumption (4.5), we obtain

$$\Big(\int_{-a/(2\sqrt{n})}^{a/(2\sqrt{n})} |\Delta_{ah_1}^2 f(y)|^p \,\mathrm{d}y\Big)^{1/p} \le c_1\,|h|^s\,a^{1/p}\,\Omega(\|g_a\|)$$
$$\le c_2\,a^{1/p}\,\Omega(\|g_a\|), \qquad (\forall h : |h| \le 1/4).$$

By setting $u := ah_1$, we deduce that

$$\Big(\int_{-a/(2\sqrt{n})}^{a/(2\sqrt{n})} |\Delta_u^2 f(y)|^p \,\mathrm{d}y\Big)^{1/p} \le c_1\,a^{1/p}\,\Omega(c_2 a), \quad \forall a > 0,\ \forall u : |u| \le a.$$

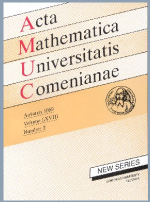
By applying the assumption (4.4) on Ω and taking a to $+\infty$, we obtain

$$\int_{-\infty}^{+\infty} |f(y+2u) - 2f(y+u) + f(y)|^p \,\mathrm{d}y = 0, \qquad \forall u \in \mathbb{R}.$$

Hence $f(y+2u) - 2f(y+u) + f(y) = 0$ a.e., $\forall y, u \in \mathbb{R}$. Then

$$f'(y+2u) - f'(y+u) = 0,\ \textit{i.e.},$$

it implies $f'(u) = f'(0)$ $(\forall u \in \mathbb{R})$. We deduce that f' is a constant. □



1. Adams D. and Frazier M., *BMO and smooth truncation in Sobolev spaces*, Studia Math. **89** (1988), 241–260.
2. ———, *Composition operators on potential spaces*, Proc. Amer. Math. Soc. **114** (1992), 155–165.
3. Allaoui S. E., *Remarques sur le calcul symbolique dans certains espaces de Besov à valeurs vectorielles*, Ann. Math. Blaise Pascal **16(2)** (2009), 399–429.
4. Bourdaud G. and Kateb M. E. D., *Fonctions qui opèrent sur les espaces de Besov*, Math. Ann. **303** (1995) 653–675.
5. Bourdaud G., Lanza de Cristoforis M. and Sickel W., *Superposition operators and functions of bounded p-variation*, Rev. Mat. Iberoamer. **22** (2006), 455–485.
6. Brezis H. and Mironescu P., *Gagliardo-Nirenberg, compositions and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001), 387–404.
7. DeVore R. and Lorentz G. G., *Constructive Approximation*, Springer, Berlin, 1993.
8. Franke J., *On the spaces* $F^s_{p,q}(\mathbb{R})$ *of Triebel-Lizorkin type: Pointwise multipliers and spaces on domains*, Math. Nachr. **125** (1986), 29–68.
9. Maz'ya V. and Shaposnikova T., *An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces*, J. Evol. Equ. **2** (2002), 113–125.
10. Moussai M., *The composition in multidimensional Triebel-Lizorkin spaces*, Math. Nachr. **284(2-3)** (2011), 317–331.
11. Peetre J., *New thoughts on Besov spaces*, Duke Univ. Math. Series I, Durham, N. C., 1976.
12. Runst T. and Sickel W., *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, de Gruyter, Berlin, 1996.
13. Triebel H., *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
14. ———, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
15. ———, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

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Go back
Full Screen
Close
Quit