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APPROXIMATION FOR PERIODIC FUNCTIONS VIA STATISTICAL A-SUMMABILITY

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Abstract. In this paper, using the concept of statistical A-summability which is stronger than the A-statistical convergence, we prove a Korovkin type approximation theorem for sequences of positive linear operator defined on $C^*(\mathbb{R})$ which is the space of all 2π-periodic and continuous functions on $\mathbb{R}$, the set of all real numbers. We also compute the rates of statistical A-summability of sequence of positive linear operators.

1. Introduction

The idea of statistical convergence was introduced by Fast [**5**], which is closely related to the concept of natural density or asymptotic density of subsets of the set of natural numbers $\mathbb{N}$. Let K be a subset of $\mathbb{N}$. The natural density of K is the nonnegative real number given by $\delta(K) := \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in K\}|$ provided that the limit exists, where $|B|$ denotes the cardinality of the set B (see [**14**] for details). Then, a sequence $x = \{x_k\}$ is called statistically convergent to a number L if for every $\varepsilon > 0$,

$$\delta(\{k : |x_k - L| \ge \varepsilon\}) = 0.$$

This is denoted by $st - \lim_{k\to\infty} x_k = L$ (see [**5**], [**7**]). It is easy to see that every convergent sequence is statistically convergent, but not conversely.

If $x = \{x_k\}$ is a number sequence and $A = \{a_{jk}\}$ is an infinite matrix, then Ax is the sequence whose j-th term is given by

$$A_j(x) := \sum_{k=1}^{\infty} a_{jk} x_k$$

provided that the series converges for each $j \in \mathbb{N}$. Thus we say that x is A-summable to L if

$$\lim_{j\to\infty} A_j(x) = L.$$

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We say that A is regular if $\lim_{j\to\infty} A_j(x) = L$ whenever $\lim_{k\to\infty} x_k = L$. The well-known necessary and sufficient conditions [**1**] (Silverman-Toeplitz) for A to be regular are:

R1) $\|A\| = \sup_{j\to\infty} \sum_{k=1}^{\infty} |a_{jk}| < \infty$,

R2) $\lim_{j\to\infty} a_{jk} = 0$ for each $k \in \mathbb{N}$,

R3) $\lim_{j\to\infty} \sum_{k=1}^{\infty} a_{jk} = 1$.

Freedman and Sember [**6**] introduced the following extension of statistical convergence. Let $A = \{a_{jk}\}$ be a nonnegative regular matrix. The A-density of K is defined by

$$\delta_A(K) := \lim_{j\to\infty} \sum_{k=1}^{\infty} a_{jk}\chi_K(k)$$

provided that the limit exists, where χ_K is the characteristic function of K. Then the sequence $x = \{x_k\}$ is said to be A-statistically convergent to the number L if for every $\varepsilon > 0$,

$$\delta_A(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

or equivalently

$$\lim_{j\to\infty} \sum_{k:|x_k - L|\geq\varepsilon} a_{jk} = 0.$$

We denote this limit by $st_A - \lim_{k\to\infty} x_k = L$ (see [**6**], [**8**], [**9**]). The case in which $A = C_1$, the Cesàro matrix of order one, reduces to the statistical convergence, and also if $A = I$, the identity matrix, then it coincides with the ordinary convergence.

Recently, the idea of statistical $(C, 1)$-summability was introduced in [**11**] and of statistical $(H,1)$-summability in [**12**] by Moricz, and of statistical $(\overline{N}, p)$-summability by Moricz and Orhan [**13**]. Then these statistical summability methods were generalized by defining the statistical A-summability in [**4**].

Now we recall statistical A-summability for a nonnegative regular matrix A.

Definition 1.1. Let $A = \{a_{jk}\}$ be a nonnegative regular matrix and $x = \{x_k\}$ be a sequence. We say that x is statistically A-summable to L if for every $\varepsilon > 0$,

$$\delta(\{j \in \mathbb{N} : |A_j(x) - L| \geq \varepsilon\}) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |\{j \leq n : |A_j(x) - L| \geq \varepsilon\}| = 0.$$

Thus $x = \{x_k\}$ is statistically A-summable to L if and only if Ax is statistically convergent to L. In this case we write $(A)_{st}-\lim_{k\to\infty} x_k = L$ or, $st - \lim_{j\to\infty} A_j(x) = L$.

Using the Definition 1.1, we see that if a sequence is bounded and A-statistically convergent to L, then it is A-summable to L, and hence statistically A-summable to L. However, its converse is not always true. Such examples were given in [**4**].

In this paper, using the concept of statistical A-summability where $A = \{a_{jk}\}$ is a nonnegative regular matrix, we give a generalization of the classical Korovkin approximation theorem by means of sequences of positive linear operators defined on the space of all real valued continuous and 2π periodic functions on $\mathbb{R}$. We also compute the rates of statistical A-summability of sequence of positive linear operators.

2. A Korovkin Type Theorem

We denote $C^*(\mathbb{R})$, the space of all real valued continuous and 2π periodic functions on $\mathbb{R}$. We recall that if a function f in $\mathbb{R}$ has period 2π, then for all $x \in \mathbb{R}$,

$$f(x) = f(x + 2\pi k)$$

holds for $k = 0, \pm 1, \pm 2, \ldots$. This space is equipped with he supremum norm

$$\|f\|_{C^*(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)|, \qquad (f \in C^*(\mathbb{R})).$$

Let L be a linear operator from $C^*(\mathbb{R})$ into $C^*(\mathbb{R})$. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in \mathbb{R}$ by $L(f(u); x)$ or, briefly, $L(f; x)$.

Throughout the paper, we also use the following test functions

$$f_0(x) = 1, \quad f_1(x) = \cos x \quad f_2(x) = \sin x.$$

We also have to recall the classical Korovkin theorem [**10**].

Theorem A. *Let $\{L_k\}$ be a sequence of positive linear operators acting from $C^*(\mathbb{R})$ into itself. Then, for all $f \in C^*(\mathbb{R})$,*

$$\lim_{k\to\infty} \|L_k(f) - f\|_{C^*(\mathbb{R})} = 0$$

if and only if

$$\lim_{k\to\infty} \|L_k(f_i) - f_i\|_{C^*(\mathbb{R})} = 0, \qquad (i = 0, 1, 2).$$

Recently, the statistical analog of Theorem A was studied by Duman [**3**]. It will be read as follows.

Theorem B. *Let $A = \{a_{jk}\}$ be a nonnegative regular matrix and let $\{L_k\}$ be a sequence of positive linear operators acting from $C^*(\mathbb{R})$ into itself. Then for all $f \in C^*(\mathbb{R})$,*

$$st_A - \lim_{k\to\infty} \|L_k(f) - f\|_{C^*(\mathbb{R})} = 0$$

if and only if

$$st_A - \lim_{k\to\infty} \|L_k(f_i) - f_i\|_{C^*(\mathbb{R})} = 0, \qquad (i = 0, 1, 2).$$

Now we study the approximation properties of sequence of positive linear operators on the space $C^*(\mathbb{R})$ via statistical A-summability where $A = \{a_{jk}\}$ is a nonnegative regular matrix.

Theorem 2.1. *Let $A = \{a_{jk}\}$ be a nonnegative regular matrix and let $\{L_k\}$ be a sequence of positive linear operators acting from $C^*(\mathbb{R})$ into itself. Then, for all $f \in C^*(\mathbb{R})$,*

$$st - \lim_{j\to\infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{C^*(\mathbb{R})} = 0 \tag{2.1}$$

if and only if

$$st - \lim_{j\to\infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_i) - (f_i) \right\|_{C^*(\mathbb{R})} = 0 \qquad (i = 0, 1, 2). \tag{2.2}$$

Proof. Since each f_i $(i = 0, 1, 2)$ belongs to $C^*(\mathbb{R})$, the implication (2.1) $\Rightarrow$ (2.2) is clear. Now, to prove the implication (2.2) $\Longrightarrow$ (2.1), assume that (2.2) holds. Let $f \in C^*(\mathbb{R})$ and let I be a closed subinterval of length 2π of $\mathbb{R}$. Fix $x \in I$. By the continuity of f at x, for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon$$

for all t satisfying $|t - x| < \delta$. On the other hand, by the boundedness of f, we have

$$|f(t) - f(x)| \le 2\|f\|_{C^*(\mathbb{R})}$$

for all $t \in \mathbb{R}$. Now consider the subintervals $(x - \delta, 2\pi + x - \delta]$ of length 2π. From [**3**] we can see that

$$|f(t) - f(x)| < \varepsilon + \frac{2\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}} \psi(t) \tag{2.3}$$

holds for all $t \in \mathbb{R}$, where $\psi(t) := \sin^2\left(\frac{t-x}{2}\right)$.

By using (2.3) and the positivity and monotonicity of L_k we have

$$\begin{aligned}
&\left| \sum_{k=1}^{\infty} a_{jk} L_k(f; x) - f(x) \right| \\
&\le \sum_{k=1}^{\infty} a_{jk} L_k(|f(t) - f(x)|\,; x) + |f(x)| \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\
&\le \sum_{k=1}^{\infty} a_{jk} L_k \left(\varepsilon + \frac{2\,\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}} \psi(t); x \right) + |f(x)| \left| \sum_{k=1}^{\infty} a_{jk} L_k\left(f_0; x\right) - f_0(x) \right| \\
&\le \varepsilon + \varepsilon \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| + \|f\|_{C^*(\mathbb{R})} \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\
&\quad + \frac{2\,\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}} \sum_{k=1}^{\infty} a_{jk} L_k\left(\psi(t); x\right).
\end{aligned}$$

After some simple calculations, we also get

$$\psi(t) = \frac{1}{2}\left(1 - \cos t \cos x - \sin t \sin x\right).$$

So we can get

$$\sum_{k=1}^{\infty} a_{jk} L_k(\psi(t);x) \leq \frac{1}{2}\left\{\left|\sum_{k=1}^{\infty} a_{jk} L_k(f_0;x) - f_0(x)\right|\right. \tag{2.4}$$
$$\left. + |\cos x| \left|\sum_{k=1}^{\infty} a_{jk} L_k(f_1;x) - f_1(x)\right| + |\sin x| \left|\sum_{k=1}^{\infty} a_{jk} L_k(f_2;x) - f_2(x)\right|\right\}.$$

Then, using (2.4), we obtain

$$\left|\sum_{k=1}^{\infty} a_{jk} L_k(f;x) - f(x)\right|$$
$$\leq \varepsilon + \left(\varepsilon + \|f\|_{C^*(\mathbb{R})} + \frac{\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}}\right)\left\{\left|\sum_{k=1}^{\infty} a_{jk} L_k(f_0;x) - f_0(x)\right|\right.$$
$$\left. + \left|\sum_{k=1}^{\infty} a_{jk} L_k(f_1;x) - f_1(x)\right| + \left|\sum_{k=1}^{\infty} a_{jk} L_k(f_2;x) - f_2(x)\right|\right\}.$$

Then, we obtain

$$\left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{C^*(\mathbb{R})} \leq \varepsilon + U\left\{\left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})}\right. \tag{2.5}$$
$$\left. + \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_1) - f_1\right\|_{C^*(\mathbb{R})} + \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_2) - f_2\right\|_{C^*(\mathbb{R})}\right\}$$

where $U := \varepsilon + \|f\|_{C^*(\mathbb{R})} + \dfrac{\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. By (2.5), it is easy to see that

$$\frac{1}{n}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{C^*(\mathbb{R})} \geq r\right\}\right|$$
$$\leq \frac{1}{n}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})} \geq \frac{r-\varepsilon}{3U}\right\}\right|$$
$$+ \frac{1}{n}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k\left(f_1\right) - f_1\right\|_{C^*(\mathbb{R})} \geq \frac{r-\varepsilon}{3U}\right\}\right|$$
$$+ \frac{1}{n}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k\left(f_2\right) - f_2\right\|_{C^*(\mathbb{R})} \geq \frac{r-\varepsilon}{3U}\right\}\right|.$$

Then using the hypothesis (2.2), we get

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{j\le n:\left\|\sum_{k=1}^{\infty}a_{jk}L_k(f)-f\right\|_{C^*(\mathbb{R})}\ge r\right\}\right|=0$$

for every $r>0$ and the proof is compete. □

3. Rate of Convergence

In this section, using statistical A-summability we study the rate of convergence of positive linear operators defined $C^*(\mathbb{R})$ into itself with the help of the modulus of continuity.

Demirci and Karakuş [**2**] introduced the rates of statistical A-summability of sequence as follows.

Definition 3.1 ([**2**])**.** Let $A=\{a_{jk}\}$ be a nonnegative regular matrix. A sequence $x=\{x_k\}$ is statistical A-summable to a number L with the rate of $\beta\in(0,1)$ if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{|\{j\le n:|A_j(x)-L|\ge\varepsilon\}|}{n^{1-\beta}}=0.$$

In this case, it is denoted by

$$x_k-L=o\left(n^{-\beta}\right)\quad((A)_{st}).$$

Using this definition, we obtain the following auxiliary result.

Lemma 3.2 ([**2**])**.** *Let $A=\{a_{jk}\}$ be a nonnegative regular matrix. Let $x=\{x_k\}$ and $y=\{y_k\}$ be bounded sequences. Assume that $x_k-L_1=o\left(n^{-\beta_1}\right)$ $((A)_{st})$ and $y_k-L_2=o\left(n^{-\beta_2}\right)$ $((A)_{st})$. Let $\beta:=\min\{\beta_1,\beta_2\}$. Then we have:*

(i) $(x_k-L_1)\mp(y_k-L_2)=o\left(n^{-\beta}\right)\quad((A)_{st})$

(ii) $\lambda(x_k-L_1)=o\left(n^{-\beta_1}\right)$ $((A)_{st})$ *for any real number* λ.

Now we remind the concept of modulus of continuity. For $f\in C^*(\mathbb{R})$, the modulus of continuity of f, denoted by $\omega(f;\delta_1)$, is defined by

$$\omega(f;\delta_1):=\sup_{|t-x|\le\delta_1}|f(t)-f(x)|\qquad(\delta_1>0).$$

It is also well know that, for any $\lambda>0$ and for all $f\in C^*(\mathbb{R})$

$$\omega(f;\lambda\delta_1)\le(1+[\lambda])\,\omega(f;\delta_1)\tag{3.1}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.

Then we have the following result.

Theorem 3.1. *Let $A=\{a_{jk}\}$ be a nonnegative regular matrix and let $\{L_k\}$ be a sequence of positive linear operators acting from $C^*(\mathbb{R})$ into itself. Assume that the following conditions holds:*

(i) $\|L_k(f_0)-f_0\|_{C^*(\mathbb{R})}=o\left(n^{-\beta_1}\right)\quad((A)_{st})$ *on* $\mathbb{R}$,

(ii) $\omega(f;\gamma_j) = o\left(n^{-\beta_2}\right) \quad ((A)_{st})$ *on* $\mathbb{R}$ *where* $\gamma_j := \sqrt{\|\sum_{k=1}^{\infty} a_{jk} L_k(\varphi)\|_{C^*(\mathbb{R})}}$ *with* $\varphi(t) = \sin^2\left(\frac{t-x}{2}\right)$.

Then we have for all $f \in C^*(\mathbb{R})$,

$$\|L_k(f) - f\|_{C^*(\mathbb{R})} = o\left(n^{-\beta}\right) \quad ((A)_{st}) \quad on \ \mathbb{R}$$

where $\beta := \min\{\beta_1, \beta_2\}$.

Proof. Let $f \in C^*(\mathbb{R})$ and $x \in \mathbb{R}$ be fixed. Using (3.1) and the positivity and monotonicity of L_k, we get for any $\delta_1 > 0$ and $j \in \mathbb{R}$,

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty} a_{jk} L_k(f;x) - f(x)\right| \\
&\le \sum_{k=1}^{\infty} a_{jk} L_k(|f(t) - f(x)|;x) + |f(x)| \left|\sum_{k=1}^{\infty} a_{jk} L_k(f_0;x) - f_0(x)\right| \\
&\le \sum_{k=1}^{\infty} a_{jk} L_k\left(\left(1 + \frac{(t-x)^2}{\delta_1^2}\right);x\right)\omega(f;\delta_1) + \|f\|_{C^*(\mathbb{R})}\left|\sum_{k=1}^{\infty} a_{jk} L_k(f_0;x) - f_0(x)\right| \\
&\le \sum_{k=1}^{\infty} a_{jk} L_k\left(\left(1 + \frac{\pi^2}{\delta_1^2}\sin^2\left(\frac{t-x}{2}\right)\right);x\right)\omega(f;\delta_1) \\
&\quad + \|f\|_{C^*(\mathbb{R})}\left|\sum_{k=1}^{\infty} a_{jk} L_k(f_0;x) - f_0(x)\right| \\
&\le \left|\sum_{k=1}^{\infty} a_{jk} L_k(f_0;x) - f_0(x)\right|\omega(f;\delta_1) + \omega(f;\delta_1) \\
&\quad + \frac{\pi^2}{\delta_1^2}\omega(f;\delta_1)\sum_{k=1}^{\infty} a_{jk} L_k\left(\sin^2\left(\frac{t-x}{2}\right);x\right) \\
&\quad + \|f\|_{C^*(\mathbb{R})}\left|\sum_{k=1}^{\infty} a_{jk} L_k(f_0;x) - f_0(x)\right|.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{C^*(\mathbb{R})} &\le \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})} \omega(f;\gamma_j) + (1+\pi^2)\omega(f;\gamma_j) \\
&\quad + \|f\|_{C^*(\mathbb{R})}\left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})}
\end{aligned}$$

where $\delta_1 := \gamma_j := \sqrt{\left\|\sum_{k=1}^{\infty} a_{jk} L_k(\varphi)\right\|_{C^*(\mathbb{R})}}$. Then, we obtain

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{C^*(\mathbb{R})} &\leq K\left\{\left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})} \omega(f;\gamma_j)\right.\\ &\left. + \omega(f;\gamma_j) + \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})}\right\}\end{aligned} \tag{3.2}$$

where $K = \max\left\{\|f\|_{C^*(\mathbb{R})}, 1+\pi^2\right\}$. Hence, for given $\varepsilon > 0$, from (3.2) and Lemma 3.2, it follows

$$\begin{aligned}&\frac{1}{n^{1-\beta}}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{C^*(\mathbb{R})} \geq \varepsilon\right\}\right|\\ &\leq \frac{1}{n^{1-\beta_1}}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})} \geq \sqrt{\frac{\varepsilon}{3K}}\right\}\right|\\ &\quad + \frac{1}{n^{1-\beta_2}}\left|\left\{j \leq n : \omega(f;\gamma_j) \geq \sqrt{\frac{\varepsilon}{3K}}\right\}\right|\\ &\quad + \frac{1}{n^{1-\beta_2}}\left|\left\{j \leq n : \omega(f;\gamma_j) \geq \frac{\varepsilon}{3K}\right\}\right|\\ &\quad + \frac{1}{n^{1-\beta_1}}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0\right\|_{C^*(\mathbb{R})} \geq \frac{\varepsilon}{3K}\right\}\right|\end{aligned} \tag{3.3}$$

where $\beta := \min\{\beta_1, \beta_2\}$. Letting $n \to \infty$ in (3.3), from (i) and (ii), we conclude that

$$\lim_{n\to\infty} \frac{1}{n^{1-\beta}}\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{C^*(\mathbb{R})} \geq \varepsilon\right\}\right| = 0,$$

which means

$$\|L_k(f) - f\|_{C^*(\mathbb{R})} = o\left(n^{-\beta}\right) \quad ((A)_{st}) \quad \text{on} \quad \mathbb{R}.$$

The proof is completed. □

Now we give the following classical rates of convergence of a sequence of positive linear operators defined on $C^*(\mathbb{R})$.

Corollary 1. *Let $\{L_k\}$ be a sequence of positive linear operators acting from $C^*(\mathbb{R})$ into itself. Assume that the following conditions holds:*

(i) $\lim_{k\to\infty} \|L_k(f_0) - f_0\|_{C^*(\mathbb{R})} = 0$,

(ii) $\lim_{k\to\infty} \omega(f;\delta_k) = 0$ *on* $\mathbb{R}$ *where* $\delta_k := \sqrt{\|L_k(\varphi)\|_{C^*(\mathbb{R})}}$ *with* $\varphi(t) = \sin^2\left(\frac{t-x}{2}\right)$.

Then for all $f \in C^*(\mathbb{R})$, we have

$$\lim_{k\to\infty} \|L_k(f) - f\|_{C^*(\mathbb{R})} = 0.$$

4. An Application to Theorem 2.1 and Theorem 3.1

In this section, we display an example of a sequence of positive linear operators. First of all, we show that Theorem 2.1 holds, but Theorem A and Theorem B do not hold. Then, using the same sequence of positive linear operators, we show that Theorem 3.1 holds but, Corollary 1 does not hold.

Let A be Cesàro matrix, i.e.,

$$a_{jk} = \begin{cases} \frac{1}{j}, & 1 \le k \le j, \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$\xi_k = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ -1, & \text{if } k \text{ is even.} \end{cases} \tag{4.1}$$

Then, we observe that, $A = \{a_{jk}\}$ is a nonnegative regular matrix and for the sequence $\xi := \{\xi_k\}$

$$st - \lim_{j\to\infty} A_j(\xi) = 0.$$

However, the sequence $\{\xi_k\}$ is not convergent in the usual sense and A-statistical convergent to 0. Then, consider the following Fejér operators

$$F_k(f;x) := \frac{1}{k\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin^2\left(\frac{k}{2}(t-x)\right)}{2\sin^2\left[\frac{t-x}{2}\right]} \mathrm{d}t \tag{4.2}$$

where $k \in \mathbb{N}$, $f \in C^*[-\pi,\pi]$. Then, we get (see [**10**])

$$F_k(f_0;x) = 1, \quad F_k(f_1;x) = \frac{k-1}{k}\cos x, \quad F_k(f_2;x) = \frac{k-1}{k}\sin x.$$

Now, using (4.1) and (4.2), we introduce the following positive linear operators defined on the space $C^*[-\pi,\pi]$

$$L_k(f;x) = (1+\xi_k)F_k(f;x). \tag{4.3}$$

(*i*) Now, we consider the positive linear operators defined by (4.3) on $C^*[-\pi,\pi]$. Since $st - \lim_{j\to\infty} A_j(\xi) = 0$, we conclude that

$$st - \lim_{j\to\infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_i) - (f_i) \right\|_{C^*[-\pi,\pi]} = 0, \qquad (i = 0,1,2).$$

Then, by Theorem 2.1, for all $f \in C^*[-\pi,\pi]$, we obtain

$$st - \lim_{j\to\infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - (f) \right\|_{C^*[-\pi,\pi]} = 0.$$

However, since $\{\xi_k\}$ does not converge in the usual sense and A-statistical converges to 0, we conclude that Theorem A and Theorem B do not work for the operators L_k in (4.3) while our Theorem 2.1 still works.

(ii) Now, we consider the positive linear operators defined by (4.3) on $C^*[-\pi,\pi]$. We observe that

$$\left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*[-\pi,\pi]} = \left| \frac{1}{j} \sum_{k=1}^{j} (1+\xi_k) - 1 \right| = \left| \frac{1}{j} \sum_{k=1}^{j} \xi_k \right|.$$

Since

$$\lim_{j\to\infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*[-\pi,\pi]} = 0,$$

then we get

$$\lim_{n\to\infty} \frac{1}{n^{1-\beta_1}} \left| \left\{ j \le n : \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*[-\pi,\pi]} \ge \varepsilon \right\} \right| = 0,$$

which means that

$$\|L_k(f_0) - f_0\|_{C^*[-\pi,\pi]} = o\left(n^{-\beta_1}\right) \quad ((A)_{st}). \tag{4.4}$$

Now, we compute the quantity $L_k(\varphi; x)$ where $\varphi(t) = \sin^2\left(\frac{t-x}{2}\right)$. After some calculations, we get

$$L_k(\varphi; x) = \frac{1+\xi_k}{2k}.$$

Then, we obtain $\gamma_j := \sqrt{\|\sum_{k=1}^{\infty} a_{jk} L_k(\varphi)\|_{C^*[-\pi,\pi]}} = \sqrt{\left|\frac{1}{j}\sum_{k=1}^{j} \frac{1+\xi_k}{2k}\right|}$. Since $\lim_{j\to\infty} \sqrt{\left|\frac{1}{j}\sum_{k=1}^{j} \frac{1+\xi_k}{2k}\right|} = 0$, we get $st - \lim_{j\to\infty} \sqrt{\left|\frac{1}{j}\sum_{k=1}^{j} \frac{1+\xi_k}{2k}\right|} = 0$. By the uniform continuity of f on $[-\pi,\pi]$, we write

$$\omega(f; \gamma_j) = o\left(n^{-\beta_2}\right) \quad ((A)_{st}). \tag{4.5}$$

From (4.4) and (4.5), the sequence of positive linear operators $\{L_k\}$ satisfies all hypotheses of Theorem 3.1. So, for all $f \in C^*[-\pi,\pi]$, we have

$$\|L_k(f) - f\|_{C^*[-\pi,\pi]} = o\left(n^{-\beta}\right) \quad ((A)_{st}).$$

However, since $\{\xi_k\}$ is not convergent, the conditions (i) and (ii) of Corollary 1 do not hold. So, the sequence $\{L_k\}$ given by (4.3) does not converge uniformly to the function $f \in C^*[-\pi,\pi]$.

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