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A TWO PARAMETER FAMILY OF PIECEWISE LINEAR TRANSFORMATIONS WITH NEGATIVE SLOPE

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Abstract. We study a two parameter family of piecewise linear transformations on the interval $[0,1]$ which have negative slope. We show that the nonwandering set consists of finitely many periodic orbits and an invariant set L which is topologically transitive and the disjoint union of finitely many closed intervals. We determine the number of these intervals.

1. Introduction

For a real number $\beta > 1$, the beta transformation is defined by $x \mapsto \beta x \bmod 1$ on the unit interval $[0,1]$. It can be used to generate β-expansions of real numbers. It is always topologically transitive and was first investigated in [**9**] and [**8**]. More recently, in [**5**] and [**1**], a beta transformation with negative slope was used to generate expansions with negative bases. It is defined on $[0,1]$ by $x \mapsto -\beta x \bmod 1$. It has more complicated dynamics as shown in [**7**]. Here we introduce a two parameter generalization of this negative beta transformation.

Set $G = \{(\alpha,\beta) : \alpha > 1,\ \beta > 1,\ \alpha\beta - \alpha - \beta < 0\}$. We choose this set as the parameter space. We define $T : [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} 1 - \alpha x & \text{if } x \in M_0 = \left[0, \dfrac{1}{\alpha}\right], \\ 1 + \dfrac{\beta}{\alpha} - \beta x & \text{if } x \in M_1 = \left(\dfrac{1}{\alpha}, 1\right]. \end{cases} \tag{1}$$

If $\beta \in (0,1)$, then there is an attracting fixed point in M_1, which attracts all orbits except the fixed point in M_0. If $\beta = 1$, then T^2 is the identity on M_1. Therefore, we assume $\beta > 1$. For $\alpha = \beta$, we get the negative beta transformation.

We consider the nonwandering set $\Omega(T)$ of the transformation T defined by (1). In particular, we are interested in the dependence of $\Omega(T)$ on the parameters α and β. This dependence was already investigated for other two parameter families of piecewise linear transformations, for tent maps in [**4**] and for the transformations $x \mapsto \beta x + \alpha \bmod 1$ in [**2**].

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Before we state the results, we need some definitions. We set $\delta_n = 0$ if n is even and $\delta_n = 1$ if n is odd. We define sequences $(c_n)_{n\geq 0}$ and $(d_n)_{n\geq 0}$ by

$$c_0 = d_0 = 1 \quad \text{and} \quad c_{n+1} = 2c_n - 2\delta_n,\ d_{n+1} = 2d_n + \delta_n - 1 \quad \text{for} \quad n \geq 0. \tag{2}$$

Using these sequences we define the sets

$$G_0 = G \quad \text{and} \quad G_n = \{(\alpha, \beta) \in G : \alpha^{c_n}\beta^{d_n} - \alpha^2 - \beta < 0\} \quad \text{for} \quad n \geq 1.$$

We have $G_{n+1} \subset G_n$ and we define

$$H_n = G_n \setminus G_{n+1} \quad \text{for} \quad n \geq 0$$

which is a nonempty set. Then $(H_n)_{n\geq 0}$ is a partition of the parameter space G. Furthermore, set $s_n = c_n + d_n - 1$ for $n \geq 0$. It follows from (2) that

$$s_0 = 1 \quad \text{and} \quad s_{n+1} = 2s_n - \delta_n \quad \text{for} \quad n \geq 0. \tag{3}$$

We prove the following results.

The transformation $T : [0, 1] \to [0, 1]$ defined by (1) is topologically transitive, if $(\alpha, \beta) \in H_0$. For $n \geq 1$ and $(\alpha, \beta) \in H_n$, we have

$$\Omega(T) = L \cup \bigcup_{k=1}^{n} P_k$$

where P_k is a periodic orbit of period $s_k - s_{k-1}$ and L is a topologically transitive T-invariant subset of $[0, 1]$ which is the disjoint union of s_n closed intervals.

Figure 1 shows the parameter space G and the curves $\alpha^{c_n}\beta^{d_n} - \alpha^2 - \beta = 0$ for $1 \leq n \leq 4$. The parameter space G is partitioned into the sets H_0, H_1, ... by

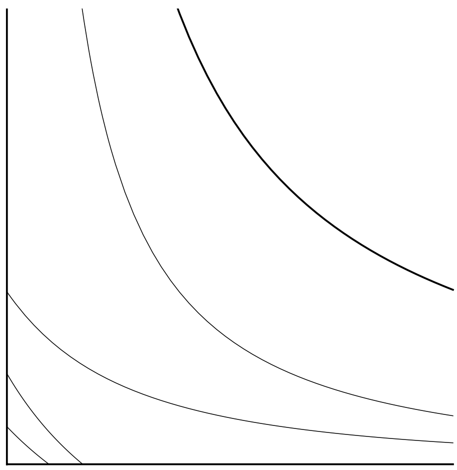

Figure 1. Parameter space.

these curves. One sees that H_0, H_1 and H_2 are unbounded, whereas G_3 and hence also the sets H_n with $n \geq 3$ are bounded.

It is well known that a piecewise linear transformation whose slopes have absolute values > 1, in particular, the transformation T defined by (1), has an absolutely continuous invariant measure whose support is a finite union of closed intervals (see e.g. [**6**]). Since this support is T-invariant and a subset of $\Omega(T)$, it must coincide with L. In the topologically transitive case it coincides with $[0,1]$. Hence the above result gives also the number of intervals of which the support of the absolutely continuous invariant measure consists.

The negative beta transformation is the special case of (1) where $\alpha = \beta$. Set $\beta_0 = 2$ and for $n \geq 1$, let β_n be the largest solution of $\beta^{s_n} - \beta - 1 = 0$. Then $(\beta, \beta) \in H_n$ is equivalent to $\beta_{n+1} \leq \beta < \beta_n$. Therefore, for the negative beta transformation the support L of the absolutely continuous invariant measure consists of s_n disjoint closed intervals if $\beta \in [\beta_{n+1}, \beta_n)$. This is proved in [**7**]. It is also proved there that $T|_L$ is topologically exact which implies topological transitivity.

The paper is organized as follows. In Section 2 we prove that T is topologically transitive if $(\alpha, \beta) \in H_0$ and investigate $\Omega(T)$ for $(\alpha, \beta) \in H_1$. For $(\alpha, \beta) \in G_1$, we find $T^3(0) < T^2(0) < \frac{1}{\alpha}$ and $T^2(0) < T^4(0)$. The fixed point $P_1 = \{\frac{1}{1+\alpha}\}$ lies between $T^3(0)$ and $T^2(0)$ and all points in $(T^3(0), T^2(0)) \setminus P_1$ are wandering. For $(\alpha, \beta) \in H_1$ we show then that the T-invariant set $L = [0, T^3(0)] \cup [T^2(0), 1]$, which is the disjoint union of $s_1 = 2$ closed intervals, is topologically transitive. Hence $\Omega(T) = P_1 \cup L$.

If $(\alpha, \beta) \in G_2$, we also have $\frac{1}{\alpha} < T^5(0) < T^4(0) < T^6(0)$. In particular, there exists a fixed point P_2 which lies between $T^5(0)$ and $T^4(0)$, and the set $L = [0, T^3(0)] \cup [T^2(0), T^5(0)] \cup [T^4(0), 1]$ is T-invariant and the disjoint union of $s_2 = 3$ closed intervals. If (α, β) moves from H_2 to G_3, then each of the three intervals of which L consists splits into two intervals such that L is then the disjoint union of $s_3 = 6$ closed intervals, and there is a periodic orbit P_3 of period $s_3 - s_2 = 3$ in the three gaps which emerge. If (α, β) moves from H_3 to G_4, then five of the six intervals of which L consists split into two intervals, such that L is then the disjoint union of $s_4 = 11$ closed intervals and there is a periodic orbit P_4 of period $s_4 - s_3 = 5$ in the five gaps which emerge. It continues in this way.

These further steps for $(\alpha, \beta) \in G_n$ with $n \geq 2$ are treated in Sections 3 and 4 using induction. In Section 3 the orbit of the point 0 is investigated, in particular, its dependence on the parameters α and β. In Section 4 we find T-invariant subsets which are finite unions of intervals. This leads then to the proof of the results for $\Omega(T)$ stated above.

This proof is inspired by the Markov graph which was developed in [**3**], although we do not introduce it here. The intervals defined in Section 4 are those which occur as vertices in this graph and the course of the proof follows its recursive structure.

2. First steps

Let $R\colon [0,1] \to [0,1]$ be a transformation with two monotone pieces, which means that there is $\gamma \in (0,1)$ such that $R|_{[0,\gamma]}$ and $R|_{(\gamma,1]}$ are continuous and monotone. We assume that R is expanding, which means that there exists $\kappa > 1$ such that $|R(I)| \geq \kappa |I|$ holds for all intervals I with $\gamma \notin I$. Here $|I|$ denotes the length of the interval I. Then we have

(4) $I \subset [0,1]$ a nonempty open interval $\Rightarrow$ $\gamma \in R^n(I)$ for some $n \geq 0$

since otherwise $R^n(I)$ would be an interval satisfying $|R^n(I)| \geq \kappa^n |I|$ for all $n \geq 1$, which is impossible as $\kappa > 1$. If $R : [0,1] \to [0,1]$ is not topologically transitive, it follows from general results for piecewise monotone transformations (see [**3**]), that there is a set $A \subsetneq [0,1]$ which is R-invariant and a finite union of nondegenerate closed intervals. Using this we can show the following theorem.

Theorem 1. *For* $(\alpha,\beta) \in H_0$ *the transformation* T *defined by* (1) *is topologically transitive.*

Proof. The transformation T has two monotone pieces and is expanding with $\kappa = \min(\alpha,\beta)$. We assume that T is not topologically transitive and hence there is a set $A \subsetneq [0,1]$ which is T-invariant and a finite union of nondegenerate closed intervals. Let ϑ be the fixed point of T in $[0,\frac{1}{\alpha}]$. We have $\vartheta \notin A$ since otherwise the T-invariance of A would imply $A = [0,1]$. Using (4) we get also $\frac{1}{\alpha} \in \operatorname{int} A$. Let J be the maximal subinterval of A, which contains $\frac{1}{\alpha}$. We have $J = [\frac{1}{\alpha} - q, \frac{1}{\alpha} + p]$ with $p > 0$ and $q > 0$. Set $U = (\frac{1}{\alpha}, \frac{1}{\alpha} + p]$ and $V = [\frac{1}{\alpha} - q, \frac{1}{\alpha})$.

By (4) there is a minimal $n \geq 1$ satisfying $\frac{1}{\alpha} \in T^n(U)$. Then $T^n(U) \subset J$ since A is T-invariant and J is the maximal subinterval of A which contains $\frac{1}{\alpha}$. Since $U \subset (\frac{1}{\alpha}, 1]$, we have $|T(U)| = \beta|U|$ and hence also $|J| \geq |T^n(U)| \geq \beta|U|$. This means $\beta p \leq p + q$.

Since $\vartheta \notin A$, we have $V \subset (\vartheta, \frac{1}{\alpha})$ and $T(V) \subset (0,\vartheta)$. Hence $|T^2(V)| = \alpha^2|V|$ and the left endpoint of $T^2(V)$ is larger than $\frac{1}{\alpha} - q$ since $\alpha > 1$. If $\frac{1}{\alpha} \in T^2(V)$, then $T^2(V) \subset J$ and we get $\alpha^2|V| < |J|$ which means $\alpha^2 q < p + q$. If $\frac{1}{\alpha} \notin T^2(V)$, then there is a minimal $n \geq 3$ satisfying $\frac{1}{\alpha} \in T^n(V)$ which implies $T^n(V) \subset J$. We get $\alpha^2|V| = |T^2(V)| < |T^n(V)| \leq |J|$ and again we have $\alpha^2 q < p + q$.

We have shown $(\beta - 1)p \leq q$ and $(\alpha^2 - 1)q < p$. This implies $(\beta-1)(\alpha^2-1) < 1$, which is equivalent to $(\alpha,\beta) \in G_1$. It contradicts $(\alpha,\beta) \in H_0$ and hence topological transitivity of T for $(\alpha,\beta) \in H_0$ is shown. □

We need a similar result for tent maps. We use the same parameter space G as for the map T in (1). For $(\alpha,\beta) \in G$ we define the tent map $S : [0,1] \to [0,1]$ by

$$S(x) = \begin{cases} 1 - \beta + \dfrac{\beta}{\alpha} + \beta x & \text{if } x \in [0,\gamma] \\ \alpha - \alpha x & \text{if } x \in (\gamma, 1] \end{cases} \quad \text{where} \quad \gamma = 1 - \frac{1}{\alpha}. \tag{5}$$

The point γ is called the critical point. This tent map has a unique fixed point in $(\gamma, 1]$ which we denote by ϱ.

Proposition 1. *Let* $S : [0,1] \to [0,1]$ *be a tent map as defined in* (5) *with* $(\alpha,\beta) \in G$. *If* $S(0) \leq \varrho$, *then* S *is topologically transitive.*

Proof. The tent map S is expanding with $\kappa = \min(\alpha,\beta)$. We proceed as in the last proof. We assume that S is not topologically transitive and hence there is a set $A \subsetneq [0,1]$ which is S-invariant and a finite union of nondegenerate closed intervals.

We have $\varrho \notin A$ since otherwise the S-invariance of A would imply $A = [0, 1]$. Using (4) we get also $\gamma \in \operatorname{int} A$. Let J be the maximal subinterval of A which contains γ. We have $J = [\gamma - q, \gamma + p]$ with $p > 0$ and $q > 0$. Set $U = [\gamma - q, \gamma)$ and $V = (\gamma, \gamma + p]$.

We have $U \subset [0, \gamma)$ and $S(U) \subset (\varrho, 1]$ since $\varrho \notin A$. We get $|S^2(U)| = \alpha\beta|U|$. By (4) there is a minimal $n \geq 2$ with $\gamma \in S^n(U)$. Then $S^n(U) \subset J$. It follows that $\alpha\beta|U| = |S^2(U)| \leq |S^n(U)| \leq |J|$. This means $\alpha\beta q \leq p + q$.

Since $\varrho \notin A$, we have $V \subset (\gamma, \varrho)$ and $S(V) \subset (\varrho, 1)$. This gives $|S^2(V)| = \alpha^2|V|$ and the right endpoint of $S^2(V)$ is less than $\gamma + p$ since $\alpha > 1$. If $\gamma \in S^2(V)$, then $S^2(V) \subset J$ and we get $\alpha^2|V| < |J|$ which means $\alpha^2 p < p + q$. If $\gamma \notin S^2(V)$, then there is a minimal $n \geq 3$ with $\gamma \in S^n(V)$ which implies $S^n(V) \subset J$. We get $\alpha^2|V| = |S^2(V)| < |S^n(V)| \leq |J|$ and again we have $\alpha^2 p < p + q$.

We have shown $(\alpha\beta - 1)q \leq p$ and $(\alpha^2 - 1)p < q$. Hence $(\alpha\beta - 1)(\alpha^2 - 1) < 1$. We assume $S(0) \leq \varrho$ which is equivalent to $1 - \beta + \frac{\beta}{\alpha} \leq \frac{\alpha}{1+\alpha}$. This contradicts $(\alpha\beta - 1)(\alpha^2 - 1) < 1$ and therefore, topological transitivity of S is shown. □

The behavior of the transformation T defined by (1) is determined by the orbit of the point 0. In order to find $\Omega(T)$ for $(\alpha, \beta) \in H_1$ we need to know how an initial segment of this orbit is ordered. Notice that $G_1 = \{(\alpha, \beta) \in G : \alpha^2\beta - \alpha^2 - \beta < 0\}$ and $G_2 = \{(\alpha, \beta) \in G : \alpha^2\beta^2 - \alpha^2 - \beta < 0\}$.

Lemma 1. *If $(\alpha, \beta) \in G_1$, then $T^3(0) < T^2(0) < \frac{1}{\alpha} < T(0)$ and $T^2(0) < T^4(0)$. If $(\alpha, \beta) \in G_2$, then we have $T^3(0) < T^2(0) < \frac{1}{\alpha} < T^5(0) < T^4(0) < T^6(0) < T(0)$. If $(\alpha, \beta) \in H_1$ and $T^4(0) > \frac{1}{\alpha}$, then we have $T^5(0) \geq T^4(0)$.*

Proof. For all $(\alpha, \beta) \in G$ we have

$$T(0) = 1 > \frac{1}{\alpha}, \quad T^2(0) = 1 + \frac{\beta}{\alpha} - \beta < \frac{1}{\alpha} \quad \text{and} \quad T^3(0) = 1 - \alpha - \beta + \alpha\beta.$$

Suppose that $(\alpha, \beta) \in G_1$. Then $\alpha^2\beta - \alpha^2 - \beta < 0$. This implies $T^3(0) < T^2(0)$ and $T^3(0) < \frac{1}{\alpha}$ follows. Hence we get

$$T^4(0) = 1 - \alpha T^3(0) = 1 - \alpha + \alpha^2 + \alpha\beta - \alpha^2\beta$$

and we have $T^2(0) < T^4(0)$ which is equivalent to $(\alpha - 1)(\alpha^2\beta - \alpha^2 - \beta) < 0$.

Additionally to $(\alpha, \beta) \in G_1$ we assume now either $(\alpha, \beta) \in G_2$ or $T^4(0) > \frac{1}{\alpha}$. We have $T^4(0) > \frac{1}{\alpha}$ also in the case, where $(\alpha, \beta) \in G_2$, since this inequality is equivalent to $(\alpha - 1)(\alpha^2\beta^2 - \alpha^2\beta - \beta) < 0$. Because $T^4(0) > \frac{1}{\alpha}$ we get

$$T^5(0) = 1 + \frac{\beta}{\alpha} - \beta T^4(0) = 1 + \frac{\beta}{\alpha} - \beta + \alpha\beta - \alpha^2\beta - \alpha\beta^2 + \alpha^2\beta^2$$

and $T^5(0) > \frac{1}{\alpha}$ holds since it is equivalent to $(\alpha^2\beta - 1)(\alpha - 1)(\beta - 1) > 0$.

We observe that the inequality $T^5(0) < T^4(0)$ is equivalent to

$$(\alpha - 1)(\alpha^2\beta^2 - \alpha^2 - \beta) < 0.$$

Therefore, we get $T^5(0) < T^4(0)$ if $(\alpha,\beta) \in G_2$, and $T^5(0) \geq T^4(0)$ if $(\alpha,\beta) \in H_1$. Because $T^5(0) > \frac{1}{\alpha}$, we get $T^6(0) = 1 + \frac{\beta}{\alpha} - \beta T^5(0) < 1$ and hence

$$T^6(0) = 1 + \frac{\beta}{\alpha} - \beta - \frac{\beta^2}{\alpha} + \beta^2 - \alpha\beta^2 + \alpha^2\beta^2 + \alpha\beta^3 - \alpha^2\beta^3.$$

Therefore, for $(\alpha,\beta) \in G_2$, we get $T^4(0) < T^6(0)$ since this inequality is equivalent to $(\alpha - 1)(\beta - 1)(\alpha^2\beta^2 - \alpha^2 - \beta) < 0$. □

We use Lemma 1 to find $\Omega(T)$ for $(\alpha,\beta) \in H_1$. To this end set

$$K_1 = \Big(\frac{1}{\alpha}, 1\Big], \quad K_2 = \Big[T^2(0), \frac{1}{\alpha}\Big], \quad K_3 = [0, T^3(0)] \quad \text{and} \quad U = (T^3(0), T^2(0)).$$

We assume $(\alpha,\beta) \in G_1$. Then by Lemma 1 these four intervals are disjoint and nonempty. Furthermore, set $L_1 = K_1 \cup K_2 \cup K_3$ which is the disjoint union of two closed intervals. Again by Lemma 1, we get $T(L_1) \subset L_1$ and $T(U) \supset U$. Since T has slope $\alpha < -1$ on U, there is a fixed point P_1 in U and all other points in U are wandering.

Now assume $(\alpha,\beta) \in H_1 = G_1 \setminus G_2$. It remains to show that L_1 is topologically transitive. The first return map S to the interval $K_1 \cup K_2$ is T on K_1 and T^2 on K_2. Hence S is a tent map on the interval $[T^2(0), 1]$ with critical point $\frac{1}{\alpha}$ and $S(T^2(0)) = T^4(0)$. Since $(\alpha,\beta) \in H_1$, we have either $T^4(0) \leq \frac{1}{\alpha}$ or $T^4(0) > \frac{1}{\alpha}$ and $T^4(0) \leq T^5(0)$ by Lemma 1. In the second case we get $T^4(0) \leq \varrho$, where ϱ is the fixed point of $S = T$ in $K_1 = (\frac{1}{\alpha}, 1]$. Hence in both cases we have $T^4(0) \leq \varrho$. Now Proposition 1 implies that there is a dense orbit in $K_1 \cup K_2$ under S. And this implies then that there is a dense orbit in L_1 under T proving that L_1 is topologically transitive. We have also shown that $\Omega(T) = P_1 \cup L_1$.

3. The orbit of the point zero

In order to determine $\Omega(T)$ for $(\alpha,\beta) \in G_2$ we need further properties of the orbit of the point 0. We introduce the kneading sequence $\mathbf{e} = e_0e_1e_2\cdots \in \{0,1\}^{\mathbb{N}_0}$ of the transformation T. It is defined such that $T^j(0) \in M_{e_j}$ holds for all $j \geq 0$, where M_0 and M_1 are as in (1). We analyze the symbolic sequences which can occur as initial segments of the kneading sequence.

Let B be a block consisting of the symbols 0 and 1. We call the number of symbols in B the length of the block B. We define B^* as follows. If B ends with 1, then let B^* be the block B with this 1 replaced by 00. If B ends with 00, then let B^* be the block B with this 00 replaced by 1. In particular, we have $B^{**} = B$.

Set $B_1 = 1$ and for $n \geq 2$ set $B_n = B_{n-1}B^*_{n-1}$. We have then $B_2 = 100$, $B_3 = 10011$, $B_4 = 10011100100$ and so on. Lemma 1 implies that $\mathbf{e}$ begins with $010011 = 0B_2B_2^* = 0B_3$ if $(\alpha,\beta) \in G_2$.

For $n \geq 1$, let a_n be the number of zeros and b_n be the number of ones in the block B_n. Set $r_n = a_n + b_n$ which is the length of the block B_n. In particular, we have $a_1 = 0$ and $b_1 = r_1 = 1$. We connect these numbers with those defined in (2) and in (3).

Lemma 2. *For $n \geq 1$, we have $a_{n+1} = 2a_n - 2(-1)^n$, $b_{n+1} = 2b_n + (-1)^n$ and $r_{n+1} = 2r_n - (-1)^n$. Furthermore, $c_n = a_n + 2\delta_n$, $d_n = b_n + 1 - \delta_n$ and $s_n = r_n + \delta_n$.*

Proof. The recursion formulas for a_n, b_n and r_n follow from the definition $B_{n+1} = B_n B_n^*$ since B_n ends with 00 if n is even and with 1 if n is odd. The equation connecting a_n and c_n and that connecting b_n and d_n are then easily checked by induction using (2). Since $s_n = c_n + d_n - 1$ and $r_n = a_n + b_n$ hold for all n by definition, also the equation connecting s_n and r_n follows. □

Lemma 3. *For $n \geq 1$, we have $s_{n+1} = r_n + s_n$, $r_{n+1} = 2s_n - 1$, $r_{n+1} = r_n + 2r_{n-1}$ and $r_{n+1} - s_{n+1} = r_{n-1} - s_{n-1}$.*

Proof. These equations can be easily checked using $r_{n+1} = 2r_n - (-1)^n$ and $s_n = r_n + \delta_n$ which are contained in Lemma 2, and $s_{n+1} = 2s_n - \delta_n$ which is contained in (3). □

The next lemma investigates the orbit of the point 0. Set

$$T_0(x) = 1 - \alpha x \quad \text{and} \quad T_1(x) = 1 + \frac{\beta}{\alpha} - \beta x.$$

Then $T^j(0) = T_{e_{j-1}} \circ \cdots \circ T_{e_1} \circ T_{e_0}(0)$ for all $j \geq 1$. Let C be a block containing p zeros and q ones, so that $l = p + q$ is the length of C. If $\mathbf{e}$ begins with $0C1C$, then we have

$$T^{2l+2}(0) = (-\alpha)^p(-\beta)^{q+1}T^{l+1}(0) + \frac{\beta}{\alpha}(-\alpha)^p(-\beta)^q + T^{l+1}(0). \tag{6}$$

This can be shown by induction. Since $e_{l+1} = 1$ and hence $T^{l+1}(0) \in M_1$, we have

$$T^{l+2}(0) = T_1(T^{l+1}(0)) = 1 + \frac{\beta}{\alpha} - \beta T^{l+1}(0) = T(0) + \frac{\beta}{\alpha} - \beta T^{l+1}(0). \tag{7}$$

If $1 \leq m \leq l$ and

$$T^{l+m+1}(0) = T^m(0) + (-\alpha)^{p_m}(-\beta)^{q_m}\Big(\frac{\beta}{\alpha} - \beta T^{l+1}(0)\Big)$$

is already shown, where p_m is the number of zeros and q_m is the number of ones in $e_1 e_2 \dots e_{m-1}$, we get

$$T^{l+m+2}(0) = \begin{cases} 1-\alpha T^m(0)+(-\alpha)^{p_m+1}(-\beta)^{q_m}\Big(\frac{\beta}{\alpha} - \beta T^{l+1}(0)\Big) & \text{if } e_m = 0 \\ 1+\frac{\beta}{\alpha} - \beta T^m(0)+(-\alpha)^{p_m}(-\beta)^{q_m+1}\Big(\frac{\beta}{\alpha} - \beta T^{l+1}(0)\Big) & \text{if } e_m = 1. \end{cases}$$

In both cases we have $T^{l+m+2}(0) = T^{m+1}(0) + (-\alpha)^{p_{m+1}}(-\beta)^{q_{m+1}}(\frac{\beta}{\alpha} - \beta T^{l+1}(0))$. Hence (6) is shown by induction since $p_{l+1} = p$ and $q_{l+1} = q$.

Now suppose that $\mathbf{e}$ begins with $0C00C$. In this case we have

$$T^{2l+3}(0) = (-\alpha)^{p+2}(-\beta)^q T^{l+1}(0) + (-\alpha)^{p+1}(-\beta)^q + T^{l+1}(0). \tag{8}$$

This can be proved in the same way as (6), except that we have now

$$T^{l+2}(0) = T_0(T^{l+1}(0)) = 1 - \alpha T^{l+1}(0) \quad \text{and}$$

$$T^{l+3}(0) = T_0(T^{l+2}(0)) = 1 - \alpha T^{l+2}(0) = T(0) - \alpha + \alpha^2 T^{l+1}(0)$$

instead of (7). Then we can proceed as above and get (8).

In the following proof we use equations like (6) and (8). All these equations can be proved in a similar way.

Lemma 4. *Suppose $n \geq 2$. If $(\alpha, \beta) \in G_n$, then* **e** *begins with $0B_nB_n^*$ and*

$$T^{r_{n-1}+1}(0) < T^{r_n+r_{n-1}+1}(0) < T^{r_n+1}(0) < T^{r_{n+1}+1}(0) < T(0).$$

Furthermore, $T^j(0) \neq \frac{1}{\alpha}$ for $0 \leq j \leq r_{n+1}$. If $(\alpha, \beta) \in H_{n-1}$ and if $T^{s_{n-1}-1}(0)$ and $T^{r_n+s_{n-1}-1}(0)$ are on the same side of $\frac{1}{\alpha}$, then $T^{r_n+r_{n-1}+1}(0) \geq T^{r_n+1}(0)$.

Proof. We have $r_1 = 1$, $r_2 = 3$, $r_3 = 5$ and $s_1 = 2$. Hence for $n = 2$ the lemma is contained in Lemma 1. We proceed by induction. Suppose that $n \geq 3$ and that the lemma is already proved for $n - 1$ instead of n. We assume $(\alpha, \beta) \in G_{n-1}$ and consider two cases.

First we assume that $n - 1$ is even. Then the block B_{n-1} ends with 00. Let C be the block B_{n-1} with 00 removed. By induction hypothesis **e** begins with $0B_{n-1}B_{n-1}^* = 0C00C1$. We set $u = a_{n-1}$ and $v = b_{n-1}$. Then the block C contains $u-2$ zeros and v ones. Set $k = u+v$. This is the length r_{n-1} of the block B_{n-1} and C has length $k - 2$. We have $c_{n-1} = u$ and $d_{n-1} = v + 1$ by Lemma 2 and the recursions in Lemma 2 give $a_n + 2 = c_n = 2u$ and $b_n = d_n = 2v + 1$. It follows that $s_{n-1} = r_{n-1} = k$ and $s_n = r_n + 1 = 2k$. By induction hypothesis we have also

$$T^j(0) \neq \frac{1}{\alpha} \quad \text{for} \quad 0 \leq j \leq 2k - 1 \tag{9}$$

and $T^{k+1}(0) < T^{2k}(0) < T(0) = 1$. Since we have $e_{k+1}e_{k+2}\dots e_{2k-2} = C$ and $e_1e_2\dots e_{k-2} = C$, for $0 \leq j \leq k - 3$, we get that $T^{2k+j}(0)$ lies in the open interval with endpoints $T^{k+1+j}(0)$ and $T^{1+j}(0)$ which is contained either in M_0 or M_1. This implies $T^j(0) \neq \frac{1}{\alpha}$ for $2k \leq j \leq 3k - 3$ and $e_{2k}e_{2k+1}\dots e_{3k-3} = C$. Therefore, **e** begins with $0C00C1C$. We compute

$$\begin{aligned} T^{3k-2}(0) = {} & (-\alpha)^{2u-2}(-\beta)^{2v+1}T^{k-1}(0) + (-\alpha)^{2u-3}(-\beta)^{2v+1} \\ & + (-\alpha)^{u-2}(-\beta)^{v+1}T^{k-1}(0) + \frac{\beta}{\alpha}(-\alpha)^{u-2}(-\beta)^v + T^{k-1}(0). \end{aligned}$$

Since u is even and v is odd by Lemma 2, this implies

$$\frac{1}{\alpha} - T^{3k-2}(0) = \Big(\frac{1}{\alpha} - T^{k-1}(0)\Big)(1 + \alpha^{u-2}\beta^{v+1} - \alpha^{2u-2}\beta^{2v+1}).$$

Either we assume that $T^{s_{n-1}-1}(0) = T^{k-1}(0)$ and $T^{r_n+s_{n-1}-1}(0) = T^{3k-2}(0)$ are on the same side of $\frac{1}{\alpha}$, or $(\alpha, \beta) \in G_n$ which means $\alpha^{2u}\beta^{2v+1} - \alpha^2 - \beta < 0$. In both cases we get

$$1 + \alpha^{u-2}\beta^{v+1} - \alpha^{2u-2}\beta^{2v+1} > 0.$$

We always have $1+\alpha^{u-2}\beta^{v+1}-\alpha^{2u-2}\beta^{2v+1}<1$. Furthermore, $T^{k-1}(0)<\frac{1}{\alpha}$ holds because $e_{k-1}=0$ and (9). This gives then $T^{k-1}(0)<T^{3k-2}(0)<\frac{1}{\alpha}$. Therefore, $e_{3k-2}=0$ is shown. Applying T we have also $0<T^{3k-1}(0)<T^{k}(0)<\frac{1}{\alpha}$, where $T^{k}(0)<\frac{1}{\alpha}$ holds because $e_k=0$ and (9). This gives $e_{3k-1}=0$. Applying T again, we get $T^{k+1}(0)<T^{3k}(0)<1$ which means

$$T^{r_{n-1}+1}(0)<T^{r_n+r_{n-1}+1}(0)<T(0).$$

Now we know that $\mathbf{e}$ begins with $0C00C1C00$. We compute

$$T^{2k}(0)=(-\alpha)^{u-1}(-\beta)^{v+1}T^{k}(0)+\frac{\beta}{\alpha}T^{k}(0)+1 \quad\text{and}$$
$$T^{3k}(0)=(-\alpha)^{u}(-\beta)^{v}T^{2k}(0)-(-\alpha)^{u}(-\beta)^{v}-\alpha T^{k}(0)+1.$$

Since u is even and v is odd by Lemma 2, we get

$$T^{2k}(0)=T^{k}(0)\Big(\frac{\beta}{\alpha}-\alpha^{u-1}\beta^{v+1}\Big)+1 \quad\text{and}$$
$$T^{3k}(0)=T^{k}(0)(\alpha^{2u-1}\beta^{2v+1}-\alpha^{u-1}\beta^{v+1}-\alpha)+1.$$

If now $(\alpha,\beta)\in G_n$, then $\alpha^{2u}\beta^{2v+1}-\alpha^2-\beta<0$ holds. We get $T^{3k}(0)<T^{2k}(0)$ which means $T^{r_n+r_{n-1}+1}(0)<T^{r_n+1}(0)$. On the other hand, if $(\alpha,\beta)\in H_{n-1}$, we have $\alpha^{2u}\beta^{2v+1}-\alpha^2-\beta\geq 0$ and we get $T^{r_n+r_{n-1}+1}(0)\geq T^{r_n+1}(0)$.

From now on we assume $(\alpha,\beta)\in G_n$. We have shown $T^{k+1}(0)<T^{3k}(0)<T(0)$ above. Since $e_{k+1}e_{k+2}\dots e_{2k-2}=C$ and $e_1e_2\dots e_{k-2}=C$, for $0\leq j\leq k-3$, we get that $T^{3k+j}(0)$ lies in the open interval with endpoints $T^{k+1+j}(0)$ and $T^{1+j}(0)$ which is contained either in M_0 or M_1. Hence $T^{j}(0)\neq\frac{1}{\alpha}$ for $3k\leq j\leq 4k-3$. This implies also $e_{3k}e_{3k+1}\dots e_{4k-3}=C$. Therefore, $\mathbf{e}$ begins with $0C00C1C00C$. We set $D=C00C$. Then D contains $2u-2$ zeros and $2v$ ones. The length of D is $2k-2$ and $B_n=C00C1=D1$. Furthermore, $\mathbf{e}$ begins with $0D1D$. We compute

$$T^{4k-2}(0)=(-\alpha)^{2u-2}(-\beta)^{2v+1}T^{2k-1}(0)+\frac{\beta}{\alpha}(-\alpha)^{2u-2}(-\beta)^{2v}+T^{2k-1}(0).$$

This implies

$$\frac{1}{\alpha}-T^{4k-2}(0)=\Big(T^{2k-1}(0)-\frac{1}{\alpha}\Big)(\alpha^{2u-2}\beta^{2v+1}-1).$$

Since $e_{2k-1}=1$, we get $T^{2k-1}(0)>\frac{1}{\alpha}$ using (9). This gives $\frac{1}{\alpha}-T^{4k-2}(0)>0$. Since $(\alpha,\beta)\in G_n$, we have $\alpha^{2u-2}\beta^{2v+1}-1<\frac{\beta}{\alpha^2}$ and we get

$$0<\frac{1}{\alpha}-T^{4k-2}(0)<\Big(1-\frac{1}{\alpha}\Big)\frac{\beta}{\alpha^2}<\frac{1}{\alpha(\alpha+1)}.$$

The last inequality is equivalent to $\alpha^2\beta-\alpha^2-\beta<0$ which holds because $(\alpha,\beta)\in G_n\subset G_2$. We have shown $\frac{1}{1+\alpha}<T^{4k-2}(0)<\frac{1}{\alpha}$ which gives $e_{4k-2}=0$. Applying T we get $T^{4k-1}(0)<\frac{1}{1+\alpha}<\frac{1}{\alpha}$ which gives $e_{4k-1}=0$. By Lemma 2, we have $r_{n+1}=2r_n+1=4k-1$. Therefore, $T^{j}(0)\neq\frac{1}{\alpha}$ is shown for all $j\leq r_{n+1}$.

Now we know that $\mathbf{e}$ begins with $0D1D00=0B_nB_n^*$. We compute

$$T^{4k}(0)=(-\alpha)^{2u}(-\beta)^{2v}T^{2k}(0)-(-\alpha)^{2u}(-\beta)^{2v}-\alpha\Big(\frac{\alpha}{\beta}T^{2k}(0)-\frac{\alpha}{\beta}\Big)+1.$$

This implies

$$1 - T^{4k}(0) = (1 - T^{2k}(0))\Big(\alpha^{2u}\beta^{2v} - \frac{\alpha^2}{\beta}\Big).$$

We have $0 < \alpha^{2u}\beta^{2v} - \frac{\alpha^2}{\beta} < 1$ since $(\alpha, \beta) \in G_n$. Furthermore, $T^{2k}(0) < 1$ holds by the induction hypothesis. Hence we get $T^{2k}(0) < T^{4k}(0) < 1$ which means $T^{r_n+1}(0) < T^{r_{n+1}+1}(0) < T(0)$. The lemma is completely proved in the case where $n-1$ is even.

Now we assume that $n-1$ is odd. We proceed as above, but the details are different. In particular, the block B_{n-1} ends with 1. Let C be the block B_{n-1} with this 1 removed. By induction hypothesis $\mathbf{e}$ begins with $0B_{n-1}B_{n-1}^* = 0C1C00$. We set $u = a_{n-1}$ and $v = b_{n-1}$. Then the block C contains u zeros and $v-1$ ones. Set $k = u + v$. This is the length r_{n-1} of the block B_{n-1} and C has length $k-1$. We have $c_{n-1} = u + 2$ and $d_{n-1} = v$ by Lemma 2 and the recursions in Lemma 2 give $a_n = c_n = 2u + 2$ and $b_n + 1 = d_n = 2v$. It follows that $s_{n-1} - 1 = r_{n-1} = k$ and $r_n = s_n = 2k + 1$. By induction hypothesis, we have also

$$T^j(0) \neq \frac{1}{\alpha} \quad \text{for} \quad 0 \leq j \leq 2k+1 \tag{10}$$

and $T^{k+1}(0) < T^{2k+2}(0) < T(0) = 1$. Since we have $e_{k+1}e_{k+2}\dots e_{2k-1} = C$ and $e_1e_2\dots e_{k-1} = C$, for $1 \leq j \leq k-1$, we get that $T^{2k+1+j}(0)$ lies in the open interval with endpoints $T^{k+j}(0)$ and $T^j(0)$ which is contained either in M_0 or M_1. This implies $T^j(0) \neq \frac{1}{\alpha}$ for $2k+2 \leq j \leq 3k$ and $e_{2k+2}e_{2k+3}\dots e_{3k} = C$. Therefore, $\mathbf{e}$ begins with $0C1C00C$. We compute

$$\begin{aligned} T^{3k+1}(0) = {} & (-\alpha)^{2u+2}(-\beta)^{2v-1}T^k(0) + \frac{\beta}{\alpha}(-\alpha)^{2u+2}(-\beta)^{2v-2} \\ & + (-\alpha)^{u+2}(-\beta)^{v-1}T^k(0) + (-\alpha)^{u+1}(-\beta)^{v-1} + T^k(0). \end{aligned}$$

Since u is even and v is odd by Lemma 2, this implies

$$T^{3k+1}(0) - \frac{1}{\alpha} = \Big(T^k(0) - \frac{1}{\alpha}\Big)(1 + \alpha^{u+2}\beta^{v-1} - \alpha^{2u+2}\beta^{2v-1}).$$

Either we assume that $T^{s_{n-1}-1}(0) = T^k(0)$ and $T^{r_n+s_{n-1}-1}(0) = T^{3k+1}(0)$ are on the same side of $\frac{1}{\alpha}$, or $(\alpha, \beta) \in G_n$ which means $\alpha^{2u+2}\beta^{2v} - \alpha^2 - \beta < 0$. In both cases we get

$$1 + \alpha^{u+2}\beta^{v-1} - \alpha^{2u+2}\beta^{2v-1} > 0.$$

We always have $1 + \alpha^{u+2}\beta^{v-1} - \alpha^{2u+2}\beta^{2v-1} < 1$. Furthermore, we get $\frac{1}{\alpha} < T^k(0)$ using $e_k = 1$ and (10). This together implies $\frac{1}{\alpha} < T^{3k+1}(0) < T^k(0)$. Therefore, we have $e_{3k+1} = 1$. Applying T we have also $T^{k+1}(0) < T^{3k+2}(0) < 1$ which means

$$T^{r_{n-1}+1}(0) < T^{r_n+r_{n-1}+1}(0) < T(0).$$

Now we have shown that $\mathbf{e}$ begins with $0C1C00C1$. We compute

$$T^{2k+2}(0) = (-\alpha)^{u+2}(-\beta)^vT^k(0) + \frac{\beta}{\alpha}(-\alpha)^{u+2}(-\beta)^{v-1} + (-\alpha)^2T^k(0) - \alpha + 1,$$

$$T^{3k+2}(0) = (-\alpha)^u(-\beta)^vT^{2k+2}(0) - (-\alpha)^u(-\beta)^v - \beta T^k(0) + 1 + \frac{\beta}{\alpha}.$$

Since u is even and v is odd by Lemma 2, we get

$$1-T^{2k+2}(0)=\Big(T^k(0)-\frac{1}{\alpha}\Big)(\alpha^{u+2}\beta^v-\alpha^2)\quad\text{and}$$
$$1-T^{3k+2}(0)=\Big(T^k(0)-\frac{1}{\alpha}\Big)(\alpha^{u+2}\beta^v-\alpha^{2u+2}\beta^{2v}+\beta).$$

If now $(\alpha,\beta)\in G_n$, then $\alpha^{2u+2}\beta^{2v}-\alpha^2-\beta<0$ holds. We get $T^{3k+2}(0)<T^{2k+2}(0)$ which means $T^{r_n+r_{n-1}+1}(0)<T^{r_n+1}(0)$. On the other hand, if $(\alpha,\beta)\in H_{n-1}$, then $\alpha^{2u+2}\beta^{2v}-\alpha^2-\beta\geq 0$ and we get $T^{r_n+r_{n-1}+1}(0)\geq T^{r_n+1}(0)$.

From now on assume $(\alpha,\beta)\in G_n$. We have shown $T^{k+1}(0)<T^{3k+2}(0)<T(0)$ above. Since $e_{k+1}e_{k+2}\dots e_{2k-1}=C$ and $e_1e_2\dots e_{k-1}=C$, for $1\leq j\leq k-1$, we get that $T^{3k+1+j}(0)$ lies in the open interval with endpoints $T^{k+j}(0)$ and $T^j(0)$ which is contained either in M_0 or M_1. This gives $T^j(0)\neq\frac{1}{\alpha}$ for $3k+2\leq j\leq 4k$ and $e_{3k+2}e_{3k+3}\dots e_{4k}=C$. Hence $\mathbf{e}$ begins with $0C1C00C1C$. We set $D=C1C$. Then D contains $2u$ zeros and $2v-1$ ones. The length of D is $2k-1$ and we have $B_n=C1C00=D00$. Furthermore, $\mathbf{e}$ begins with $0D00D$. We compute

$$T^{4k+1}(0)=(-\alpha)^{2u+2}(-\beta)^{2v-1}T^{2k}(0)+(-\alpha)^{2u+1}(-\beta)^{2v-1}+T^{2k}(0).$$

This gives

$$T^{4k+1}(0)-\frac{1}{\alpha}=\left(\frac{1}{\alpha}-T^{2k}(0)\right)(\alpha^{2u+2}\beta^{2v-1}-1).$$

Since $e_{2k}=0$, we get $T^{2k}(0)<\frac{1}{\alpha}$ using (10). This implies $T^{4k+1}(0)>\frac{1}{\alpha}$ and $e_{4k+1}=1$. By Lemma 2 we have $r_{n+1}=2r_n-1=4k+1$. Therefore, $T^j(0)\neq\frac{1}{\alpha}$ is shown for all $j\leq r_{n+1}$.

Now we know that $\mathbf{e}$ begins with $0D00D1=0B_nB_n^*$. We compute

$$T^{4k+2}(0)=(-\alpha)^{2u}(-\beta)^{2v}T^{2k+2}(0)-(-\alpha)^{2u}(-\beta)^{2v}-\frac{\beta}{\alpha}\frac{1}{\alpha}(T^{2k+2}(0)-1)+1.$$

This implies

$$1-T^{4k+2}(0)=(1-T^{2k+2}(0))\Big(\alpha^{2u}\beta^{2v}-\frac{\beta}{\alpha^2}\Big).$$

Since $(\alpha,\beta)\in G_n$, we have $0<\alpha^{2u}\beta^{2v}-\frac{\beta}{\alpha^2}<1$, and $T^{2k+2}(0)<1$ holds by induction hypothesis. Hence we get $T^{2k+2}(0)<T^{4k+2}(0)<1$ which means $T^{r_n+1}(0)<T^{r_{n+1}+1}(0)<T(0)$. The lemma is completely proved also in the case where $n-1$ is odd. □

4. The nonwandering set

We define the intervals which are used to construct T-invariant sets. For $n\geq 1$, we define

$$K_{s_n-1}=\begin{cases}\Big[T^{s_n-1}(0),\dfrac{1}{\alpha}\Big], & \text{if}\quad T^{s_n-1}(0)\leq\frac{1}{\alpha},\\[2mm] \Big(\dfrac{1}{\alpha},T^{s_n-1}(0)\Big], & \text{if}\quad T^{s_n-1}(0)>\frac{1}{\alpha}.\end{cases}$$

For all $(\alpha,\beta)\in G$, we have $K_0=K_{s_0-1}=[0,\frac{1}{\alpha}]$, $K_1=K_{s_1-1}=(\frac{1}{\alpha},T(0)]$ and $K_2=K_{s_2-1}=[T^2(0),\frac{1}{\alpha}]$.

Suppose that $n \geq 2$ and $(\alpha, \beta) \in G_{n-1}$. Then $\mathbf{e}$ begins with $0B_{n-1}B_{n-1}^* = 0B_n$ and $T^j(0) \neq \frac{1}{\alpha}$ for $j \leq r_n$ by Lemmas 1 and 4. If n is even, then B_n ends with 00 and has length $r_n = s_n$. Hence $e_{s_n-1} = e_{s_n} = 0$ and we get $T^{s_n-1}(0) < \frac{1}{\alpha}$ and $T^{s_n}(0) < \frac{1}{\alpha}$. We have then $K_{s_n-1} = [T^{s_n-1}(0), \frac{1}{\alpha}] \subset M_0$. We define

$$K_{s_n} = T(K_{s_n-1}) = [0, T^{s_n}(0)] \subset M_0 \quad \text{and}$$
$$K_{r_n+1} = K_{s_n+1} = T(K_{s_n}) = [T^{s_n+1}(0), T(0)].$$

If n is odd, then B_n ends with 1 and has length $r_n = s_n - 1$. Hence $e_{s_n-1} = 1$ and we get $T^{s_n-1}(0) > \frac{1}{\alpha}$. We have then $K_{s_n-1} = (\frac{1}{\alpha}, T^{s_n-1}(0)] \subset M_1$ and we define

$$K_{r_n+1} = K_{s_n} = \overline{T(K_{s_n-1})} = [T^{s_n}(0), 1] = [T^{s_n}(0), T(0)].$$

Notice that in both cases, for even and odd n, we have $K_{r_n+1} = [T^{r_n+1}(0), T(0)]$.

Suppose now $(\alpha, \beta) \in G_n \subset G_{n-1}$. Then $\mathbf{e}$ begins with $0B_nB_n^*$ and we have $T^j(0) \neq \frac{1}{\alpha}$ for $j \leq r_{n+1}$ by Lemma 4. We continue to define the intervals K_j. The block B_n has length r_n and the initial segment which the blocks B_n and B_n^* have in common, has length $r_n - 2 + \delta_n$ which is equal to $s_n - 2$ by Lemma 2. Using this and Lemma 3 we get

$$e_j = e_{r_n+j} \quad \text{for} \quad 1 \leq j \leq s_n - 2 \quad \text{and} \quad e_{s_n-1} \neq e_{r_n+s_n-1} = e_{s_{n+1}-1}.$$

For $1 \leq j \leq s_n - 2$, we set $K_{r_n+j} = T^{j-1}(K_{r_n+1})$ which is a closed interval contained either in M_0 or in M_1 and has endpoints $T^{r_n+j}(0)$ and $T^j(0)$. Now K_j is defined for $j \leq s_{n+1} - 2$. Furthermore, $T(K_{s_{n+1}-2})$ has endpoints $T^{s_{n+1}-1}(0)$ and $T^{s_n-1}(0)$ which are on different sides of $\frac{1}{\alpha}$. Hence both $T(K_{s_{n+1}-2}) \cap M_0$ and $T(K_{s_{n+1}-2}) \cap M_1$ are nonempty. One of these intervals is $K_{s_{n+1}-1}$ and the other one is K_{s_n-1}. Hence the interval $T(K_{s_{n+1}-2})$ is the disjoint union of the intervals $K_{s_{n+1}-1}$ and K_{s_n-1}. Now we can continue to define intervals K_j for $j \geq s_{n+1}$ as in the previous paragraph.

If $(\alpha, \beta) \in G_n$, then the intervals K_j for $0 \leq j \leq r_{n+1} + 1$ are defined. The interval K_j is mapped monotonically onto K_{j+1} if $j \notin \{s_i - 2 : 1 \leq i \leq n+1\}$, and the interval K_{s_i-2} is mapped monotonically onto $K_{s_i-1} \cup K_{s_{i-1}-1}$ for $1 \leq i \leq n+1$.

Set $L_0 = [0, 1]$ and $L_n = \bigcup_{j=s_n-1}^{r_{n+1}} K_j$. We can prove now the following results.

Proposition 2. *Suppose that $n \geq 2$ and $(\alpha, \beta) \in G_n$.*

(a) The intervals K_j for $s_n - 1 \leq j \leq r_{n+1}$ are disjoint and K_{s_n-1} and $K_{s_{n+1}-1}$ have the common endpoint $\frac{1}{\alpha}$.
(b) L_n is T-invariant and $L_n \subset L_{n-1}$.
(c) $L_{n-1} \setminus L_n$ is the union of disjoint open intervals U_j with $1 \leq j \leq r_{n-1}$ and we have $T(U_j) = U_{j+1}$ for $1 \leq j \leq r_{n-1} - 1$ and $T(U_{r_{n-1}}) \supset U_1$.

Proof. We have $r_2 = s_2 = 3$ and $r_3 = s_3 - 1 = 5$. For $(\alpha, \beta) \in G_2$, we have $K_2 = K_{s_2-1} = [T^2(0), \frac{1}{\alpha}]$, $K_3 = [0, T^3(0)]$, $K_4 = [T^4(0), T(0)]$ and $K_5 = K_{s_3-1} = K_{r_3} = (\frac{1}{\alpha}, T^5(0)]$. Hence for $n = 2$ we get (a) from Lemma 1.

We proceed by induction. Suppose $n \geq 3$ and $(\alpha, \beta) \in G_n \subset G_{n-1}$. We assume that (a) is already shown for $n - 1$ instead of n, this means that the intervals K_j

for $s_{n-1} - 1 \le j \le r_n$ are disjoint. In the following we write $\uplus$ for the union of disjoint sets.

We start with $K_{r_{n-1}+1} = [T^{r_{n-1}+1}(0), T(0)] \subset M_1$. By Lemma 4 we have

$$T^{r_{n-1}+1}(0) < T^{r_n+r_{n-1}+1}(0) < T^{r_n+1}(0) < T(0).$$

It follows that the intervals

$$K_{r_n+r_{n-1}+1} = [T^{r_{n-1}+1}(0), T^{r_n+r_{n-1}+1}(0)] \quad \text{and} \quad K_{r_n+1} = [T^{r_n+1}(0), T(0)]$$

are disjoint and the nonempty open interval $U_1 = (T^{r_n+r_{n-1}+1}(0), T^{r_n+1}(0))$ lies between them. Therefore, we have

$$K_{r_n+r_{n-1}+1} \uplus U_1 \uplus K_{r_n+1} = K_{r_{n-1}+1}.$$

We set $U_j = T^{j-1}(U_1)$ for $j \ge 2$. Furthermore, for $r_{n-1}+2 \le j \le s_n - 2$ we have that $K_j = T(K_{j-1})$ is contained either in M_0 or M_1. Since $U_1 \subset K_{r_{n-1}+1}$, the sets U_j for $1 \le j \le s_n - r_{n-1} - 1$ are intervals and we get

$$K_{r_n+j} \uplus U_{j-r_{n-1}} \uplus K_{r_n-r_{n-1}+j} = K_j \quad \text{for} \quad r_{n-1}+1 \le j \le s_n - 2 \tag{11}$$

and $T(K_{r_n+s_n-2}) \uplus U_{s_n-r_{n-1}-1} \uplus K_{r_n-r_{n-1}+s_n-1} = T(K_{s_n-2})$. By the first equation of Lemma 3 this means

$$T(K_{s_{n+1}-2}) \uplus U_{s_{n-1}-1} \uplus K_{r_n+s_{n-1}-1} = T(K_{s_n-2}).$$

We have $T(K_{s_{n+1}-2}) = K_{s_{n+1}-1} \uplus K_{s_n-1}$ and $T(K_{s_n-2}) = K_{s_n-1} \uplus K_{s_{n-1}-1}$. Therefore, we get

$$K_{s_{n+1}-1} \uplus U_{s_{n-1}-1} \uplus K_{r_n+s_{n-1}-1} = K_{s_{n-1}-1}.$$

Since $T(K_j) = K_{j+1}$ holds for $s_{n-1} - 1 \le j \le r_{n-1}$ (notice that $r_{n-1} = s_{n-1} - 1$ or $r_{n-1} = s_{n-1}$), the sets U_j for $s_{n-1} = s_n - r_{n-1} \le j \le r_{n-1} + 1$ are intervals and

$$K_{s_{n+1}-s_{n-1}+j} \uplus U_j \uplus K_{r_n+j} = K_j \quad \text{for} \quad s_{n-1} - 1 \le j \le r_{n-1}. \tag{12}$$

By induction hypothesis, the intervals K_j for $s_{n-1} - 1 \le j \le r_n$ are disjoint. By (11) and (12), each of the intervals K_j for $s_{n-1} - 1 \le j \le s_n - 2$ contains three disjoint intervals. Hence the intervals on the left hand sides of (11) and (12) are disjoint and they are also disjoint from K_j for $s_n - 1 \le j \le r_n$ (notice that $r_n = s_n - 1$ or $r_n = s_n$). The intervals on the left hand sides of (11) and (12) are U_j for $1 \le j \le r_{n-1}$ and

$$\begin{array}{ll} K_{r_n+j} & \text{for} \;\; s_{n-1} - 1 \le j \le s_n - 2 = s_{n+1} - 2 - r_n \\ K_{r_n+l} & \text{for} \;\; 1 \le l \le s_n - 2 - r_{n-1} = s_{n-1} - 2 \\ K_{s_{n+1}-2+l} & \text{for} \;\; 1 \le l \le r_{n-1} - s_{n-1} + 2 = r_{n+1} - s_{n+1} + 2 \end{array}$$

where we have used the equations of Lemma 3. This list contains the intervals K_j for $r_n + 1 \le j \le r_{n+1}$. Therefore, these intervals are disjoint and they are also disjoint from the intervals K_j for $s_n - 1 \le j \le r_n$. By definition the intervals $K_{s_{n+1}-1}$ and K_{s_n-1} have the common endpoint $\frac{1}{\alpha}$. Hence (a) is shown by induction.

The intervals U_j are disjoint. We have $T(U_j) = U_{j+1}$ for $1 \le j \le r_{n-1} - 1$ and

$$\bigcup_{j=s_{n-1}-1}^{r_n} K_j \setminus \bigcup_{j=s_n-1}^{r_{n+1}} K_j = \bigcup_{j=s_{n-1}-1}^{s_n-2} K_j \setminus \bigcup_{j=r_n+1}^{r_{n+1}} K_j = \bigcup_{j=1}^{r_{n-1}} U_j \tag{13}$$

by (11) and (12). Since r_{n-1} is odd by Lemma 2 and $r_n + 2r_{n-1} = r_{n+1}$ holds by Lemma 3, we get

$$T(U_{r_{n-1}}) = T^{r_{n-1}}(U_1) = (T^{r_n+r_{n-1}+1}(0), T^{r_{n+1}+1}(0)).$$

We have $T^{r_n+1}(0) < T^{r_{n+1}+1}(0)$ by Lemma 4 and therefore,

$$T(U_{r_{n-1}}) \supset U_1 = (T^{r_n+r_{n-1}+1}(0), T^{r_n+1}(0)).$$

We show (b). We get $L_n \subset L_{n-1}$ from Lemma 1 if $n = 2$, and from (11) and (12) if $n \ge 3$. We show $T(L_n) \subset L_n$. For $s_n - 1 \le j \le s_{n+1} - 3$, we have $T(K_j) = K_{j+1}$. Furthermore, $T(K_{s_{n+1}-2}) = K_{s_{n+1}-1} \cup K_{s_n-1}$. If $n+1$ is even, then

$$s_{n+1} = r_{n+1}, \quad T(K_{s_{n+1}-1}) = K_{s_{n+1}} = K_{r_{n+1}} \quad \text{and} \quad T(K_{r_{n+1}}) = K_{r_{n+1}+1}.$$

If $n+1$ is odd, then

$$s_{n+1} = r_{n+1} + 1 \quad \text{and} \quad T(K_{s_{n+1}-1}) \subset K_{s_{n+1}} = K_{r_{n+1}+1}.$$

Since $T^{r_n+1}(0) < T^{r_{n+1}+1}(0)$ holds by Lemma 4, we have also

$$K_{r_{n+1}+1} = [T^{r_{n+1}+1}(0), T(0)] \subset [T^{r_n+1}(0), T(0)] = K_{r_n+1}$$

and $T(L_n) \subset L_n$ is shown.

We show (c). If $n = 2$, we have $r_{n-1} = r_1 = 1$ and $U_1 = (T^5(0), T^4(0))$. In this case $T(U_1) \supset U_1$ and $L_1 \setminus L_2 = U_1$ follow from Lemma 1. For $n \ge 3$, let U_j for $1 \le j \le r_{n-1}$ be as above. We have shown $T(U_j) = U_{j+1}$ for $1 \le j \le r_{n-1} - 1$ and $T(U_{r_{n-1}}) \supset U_1$. Furthermore, we have $L_{n-1} \setminus L_n = \bigcup_{j=1}^{r_{n-1}} U_j$ by (13). □

Finally we are able to determine the nonwandering set $\Omega(T)$.

Theorem 2. *For $n \ge 1$ and $(\alpha, \beta) \in H_n$, we have $\Omega(T) = L_n \cup \bigcup_{k=1}^n P_k$, where P_k is a periodic orbit of period $s_k - s_{k-1}$. The set L_n is topologically transitive and the disjoint union of s_n closed intervals.*

Proof. For $n = 1$, this is already shown at the end of Section 2. Therefore, we suppose $n \ge 2$. We first consider L_n. We have $L_n = \bigcup_{j=s_n-1}^{r_{n+1}} K_j$ and the intervals in this union are disjoint by Proposition 2. Since two of these intervals have a common endpoint, L_n is the disjoint union of $r_{n+1} - s_n + 1 = s_n$ closed intervals.

For $r_n + 2 \le j \le s_{n+1} - 2$, the interval K_{j-1} is mapped monotonically onto the interval K_j. The interval $K_{s_{n+1}-2}$ is mapped monotonically onto $K_{s_{n+1}-1} \cup K_{s_n-1}$ which contains $\frac{1}{\alpha}$ in its interior. Let δ be the inverse image of $\frac{1}{\alpha}$ in the interval $K_{r_n+1} = [T^{r_n+1}(0), T(0)]$ under the composition of these maps. The interval K_{s_n-1} is mapped monotonically onto K_{r_n+1} and $K_{s_{n+1}-1}$ is mapped monotonically onto $K_{r_{n+1}+1} = [T^{r_{n+1}+1}(0), T(0)]$, one under T and the other one under T^2. Since

$(\alpha, \beta) \in H_n \subset G_n$, we have $K_{r_{n+1}+1} \subset K_{r_n+1}$ by Lemma 4. The first return map S of T to the interval K_{r_n+1} is therefore $S = T^{r_n}$ on $[\delta, T(0)]$ which is linear with

$$S(\delta) = T(0) \quad \text{and} \quad S(T(0)) = T^{r_n+1}(0),$$

and $S = T^{r_{n+1}-r_n}$ on $[T^{r_n+1}(0), \delta]$ which is linear with

$$S(\delta) = T(0) \quad \text{and} \quad S(T^{r_n+1}(0)) = T^{r_{n+1}+1}(0).$$

Hence S is a tent map and has the fixed point ϱ in $(\delta, T(0))$. Since $(\alpha, \beta) \in H_n$, by Lemma 4 we have $T^{r_{n+1}+1}(0) \le T^{r_{n+1}+r_n+1}(0)$, if $T^{r_{n+1}+s_n-1}(0)$ and $T^{s_n-1}(0)$ are on the same side of $\frac{1}{\alpha}$ which means that $T^{r_{n+1}+1}(0)$ and $T(0)$ are on the same side of δ. Therefore, we have either $T^{r_{n+1}+1}(0) \le \delta$ or $\delta < T^{r_{n+1}+1}(0) \le S(T^{r_{n+1}+1}(0))$. In both cases we get $T^{r_{n+1}+1}(0) \le \varrho$. The tent map S fulfills the assumption of Proposition 1 and has therefore a dense orbit. This implies that there is also a dense orbit under T in L_n and hence L_n is topologically transitive.

For $1 \le k \le n$ we have $(\alpha, \beta) \in H_n \subset G_k$ and hence L_k is T-invariant and a finite union of intervals by Proposition 2 and Lemma 1. Again by Proposition 2 we have $[0, 1] = L_0 \supset L_1 \supset \ldots \supset L_n$. This gives

$$[0, 1] = L_n \cup \bigcup_{k=1}^{n} (L_{k-1} \setminus L_k).$$

We have shown at the end of Section 2 that $L_0 \setminus L_1$ contains a fixed point P_1 and wandering points otherwise. For $k \ge 2$, it follows from Proposition 2 that $L_{k-1} \setminus L_k$ is the union of disjoint open intervals U_j with $1 \le j \le r_{k-1}$ satisfying $T(U_j) = U_{j+1}$ for $1 \le j \le r_{k-1} - 1$ and $T(U_{r_{k-1}}) \supset U_1$. Furthermore, $\frac{1}{\alpha} \notin U_j$ for $1 \le j \le r_{k-1}$ because we have $\frac{1}{\alpha} \in K_{s_k-1} \cup K_{s_{k+1}-1} \subset L_k$. Since r_{k-1} is odd by Lemma 2, there is a periodic orbit P_k of period $r_{k-1} = s_k - s_{k-1}$ in $L_{k-1} \setminus L_k$ and all other points in $L_{k-1} \setminus L_k$ are wandering since T is expanding. Therefore, we get $\Omega(T) = L_n \cup \bigcup_{k=1}^{n} P_k$. □

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